

May 22, 2024

INFINITESIMAL ANALYSIS 88-503 HOMEWORK SET 2

Due Date: 2 june '24

Most of the problems are similar to arguments that appeared in the lecture. Try to solve the problems on your own.

1. Consider a nonempty subset $A \subseteq \mathbb{N}$. Prove that there exists an ultrafilter F on \mathbb{N} with $A \in F$.
2. Prove that an ultrafilter on a finite set is necessarily principal.
3. Let F be a nonprincipal ultrafilter on \mathbb{N} . Define an equivalence relation \equiv on $\mathbb{R}^{\mathbb{N}}$ as follows. Set $\langle r_n \rangle \equiv \langle s_n \rangle$ if and only if the set $\{n \in \mathbb{N} : r_n = s_n\} \in F$. Prove that

$$\langle 1, \frac{1}{2}, \frac{1}{3}, \dots \rangle \not\equiv \langle 0, 0, 0, \dots \rangle.$$

4. Consider sequences $r = \langle r_n \rangle$ and $s = \langle s_n \rangle$. Denote the set $\{n \in \mathbb{N} : r_n = s_n\}$ by $[[r = s]]$. Let $t = \langle t_n \rangle$ be another sequence. Prove that

$$[[r = s]] \cap [[s = t]] \subseteq [[r = t]].$$

5. Let ε be a sequence so that $[\varepsilon] \in {}^*\mathbb{R}$ is a positive infinitesimal. Prove that $\frac{1}{[\varepsilon]}$ is infinite.