ZEROES OF GAUSSIAN ANALYTIC FUNCTIONS WITH TRANSLATION-INVARIANT DISTRIBUTION

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Abstract. We study zeroes of Gaussian analytic functions in a strip in the complex plane, with translation-invariant distribution. We prove that the horizontal limiting measure of the zeroes exists almost surely, and that it is non-random if and only if the spectral measure is continuous (or degenerate). In this case, the limiting measure is computed in terms of the spectral measure. We compare the behavior with Gaussian analytic function with symmetry around the real axis. These results extend a work by Norbert Wiener.

Keywords: zeroes of Gaussian analytic functions, translation-invariance, ergodicity.

1. Introduction

Following Wiener, we study zeroes of Gaussian analytic functions with translation-invariant distribution, defined on a strip in the complex plane. Under certain assumptions on the spectral measure, Wiener proved that the zeroes obey the law of large numbers, and computed their horizontal density (limiting measure). This result appears in his classical treatise with Paley [17, chapter X]. Wiener's proof is quite intricate; this may be why it attracted little attention.

In this work, we simplify Wiener's arguments and remove unnecessary assumptions on the spectral measure. We incorporate the result into a theorem that guarantees the existence of the horizontal limiting measure in question, and asserts it is not random if and only if the spectral measure is continuous or consists of a single atom. Then we prove a counterpart of this theorem for a natural class of Gaussian analytic functions which have a symmetry with respect to the real axis.

For this purpose, we developed a general Edelman-Kostlan-type formula for computing the average zero-counting measure of zeroes of a symmetric Gaussian analytic function in some domain (see Theorem 3 below). This result extends those of Shepp and Vanderbei [19], Prosen [16] and Macdonald [13].

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1.1. **Gaussian Analytic Functions.** We deal with two classes of random Gaussian analytic functions.

Definition 1.1. *Let $D \subset \mathbb{C}$ be a domain, and let $\{\phi_n\}_{n\in\mathbb{N}}$ be analytic functions in D such that the series $\sum_n |\phi_n(z)|^2$ converges uniformly on compact subsets of D.*

(1) *Let a_n be independent standard complex Gaussian random variables ($a_n \sim \mathcal{N}_{\mathbb{C}}(0,1)$). The random series $\sum_n a_n \phi_n(z)$ is called a Gaussian Analytic Function (GAF, for short).*
(2) *Let b_n be independent standard* real *Gaussian variables ($b_n \sim \mathcal{N}_{\mathbb{R}}(0,1)$). If the domain D and the functions ϕ_n are symmetric w.r.t. the real axis (the latter means that $\phi(\overline{z}) = \overline{\phi(z)}$, $z \in D$) then the random series $\sum_n b_n \phi_n(z)$ is called a symmetric Gaussian Analytic Function.*

Our assumptions on $\{\phi_n\}$ ensure that the sums above a.s. converge to an analytic function in D [5, Chapter 2]. Throughout the paper we assume that there is no $z_0 \in D$ such that $\phi_n(z_0) = 0$ for all $n \in \mathbb{N}$ (hence the function f has no deterministic zeroes).

The covariance kernel of $f(z)$ is defined by

$$K(z,w) = \mathbb{E}(f(z)\overline{f(w)}) = \sum_n \phi_n(z)\overline{\phi_n(w)}\,. \tag{1}$$

The function $K(z,w)$ is positive definite, analytic in z, anti-analytic in w, and obeys the law $K(z,w) = \overline{K(w,z)}$. It turns out that every such function $K(z,w)$ of two variables $z, w \in D$ uniquely defines a GAF in D.

If in addition $K(x,y)$ is real whenever $x, y \in D \cap \mathbb{R}$, then $K(z,w)$ also uniquely defines a symmetric GAF with this kernel. We stress that a GAF and a symmetric GAF with the same kernel are different random processes.

1.2. **Stationarity.** We assume our domain is the Δ-strip $D = D_\Delta = \{|\mathrm{Im}\, z| < \Delta\}$ with $0 < \Delta \le \infty$.

Definition 1.2. *A GAF or a symmetric GAF in a strip D_Δ is called* stationary *if it is distribution-invariant with respect to all horizontal shifts, i.e., for any $t \in \mathbb{R}$, any $n \in \mathbb{N}$, and any $z_1, \dots, z_n \in D$, the random n-tuples*

$$\big(f(z_1), \dots, f(z_n)\big) \qquad \text{and} \qquad \big(f(z_1+t), \dots, f(z_n+t)\big)$$

have the same distribution.

If $f(z)$ is stationary in the Δ-strip, then for any $x, y \in \mathbb{R}$ the covariance $\mathbb{E}(f(x)\overline{f(y)})$ depends on $(x-y)$ only, so that $K(x,y) = r(x-y)$ for some real-analytic function $r : \mathbb{R} \to \mathbb{C}$. From this we deduce

$$K(z,w) = r(z - \bar{w})$$

(both functions are analytic in z, anti-analytic in w, and coincide for $z, w \in \mathbb{R}$), and so $r(t)$ has an analytic continuation to the 2Δ-strip $D_{2\Delta}$.

Since $r(t)$ is continuous and positive-definite, it is the Fourier transform of a positive measure ρ (Bochner's Theorem):

$$r(t) = \int_{\mathbb{R}} e^{2\pi i t\lambda} d\rho(\lambda).$$

The measure ρ is called the spectral measure of the process $f(z)$.

Since $r(t)$ has an analytical extension to the 2Δ-strip, $\rho(\lambda)$ has a finite exponential moment [12, Chapter 2]:

$$\text{for each } \Delta_1 < \Delta, \quad \int_{-\infty}^{\infty} e^{2\pi \cdot 2\Delta_1 |\lambda|} d\rho(\lambda) < \infty\,. \tag{2}$$

In fact, condition (2) is also sufficient for $r(t)$ to have an analytic extension to the 2Δ-strip. Therefore, beginning with a finite positive measure ρ obeying (2), we can construct a kernel by

$$K(z,w) = \int_{\mathbb{R}} e^{2\pi i (z-\bar{w})\lambda} d\rho(\lambda). \tag{3}$$

which defines in its turn a stationary GAF in the Δ-strip.

What measures could be spectral measures of a symmetric GAF? As we mentioned earlier, a kernel $K(z,w)$ defines a symmetric GAF if and only if it is real for $z, w \in \mathbb{R}$; By relation (3) this is equivalent to ρ being symmetric with respect to the origin.

Finally, we mention that a random GAF or symmetric GAF may be explicitly constructed, as follows, from its spectral measure ρ. If $\{\psi_n(z)\}_n$ comprise an orthonormal basis in $L^2_\rho(\mathbb{R})$, then their Fourier transforms

$$\phi_n(z) = \widehat{\psi_n}(z) = \int_{\mathbb{R}} e^{2\pi i z\lambda} \psi_n(\lambda) d\rho(\lambda)$$

comprise a basis in the Hilbert space $\mathcal{F}\{L^2_\rho(\mathbb{R})\}$ (the Fourier image of $L^2_\rho(\mathbb{R})$ with the scalar product transferred from $L^2_\rho(\mathbb{R})$). One easily checks that

$$r(z - \overline{w}) = \mathbb{E}(f(z)\overline{f(w)}) = \sum_n \phi_n(z)\overline{\phi_n(w)}\,.$$

Therefore, when used in Definition 1.1, the basis $\{\phi_n\}$ will us a random function with the desired kernel.

2. Results and Discussion

2.1. **Main Theorem.** It will be convenient to introduce some notation:

Notation 1. (zero-set, zero-counting measure) Let $D \subset \mathbb{C}$ be a region, and f a holomorphic function in D. Denote the zero-set of f (counted with multiplicities) by Z_f, and the zero-counting measure by n_f, i.e.,

$$\forall \phi \in C_0^\infty(D), \qquad \int_D \phi(z) dn_f(z) = \sum_{z \in Z_f} \phi(z)\,.$$

We use the abbreviation $n_f(B) = \int_B dn_f(z)$ for the number of zeroes in a Borel subset $B \subset D$.

Notation 2. Let $y \in (-\Delta, \Delta)$. For a stationary GAF or symmetric-GAF in D_Δ with kernel $K(z, w)$, denote

$$\psi(y) = K(iy, iy) = \int_{-\infty}^{\infty} e^{-4\pi y \lambda} d\rho(\lambda)\,.$$

In the case of a GAF, define the function

$$L(y) = \frac{d}{dy}\left(\frac{\psi'(y)}{4\pi\psi(y)}\right) = -\frac{d}{dy}\left(\frac{\int_{-\infty}^{\infty} \lambda e^{-4\pi y\lambda} d\rho(\lambda)}{\int_{-\infty}^{\infty} e^{-4\pi y \lambda} d\rho(\lambda)}\right). \tag{4}$$

In the case of a symmetric-GAF, define for $y \neq 0$ the function
(5)

$$S(y) = \frac{d}{dy}\left(\frac{\psi'(y)}{4\pi\sqrt{\psi(y)^2 - \psi(0)^2}}\right) = -\frac{d}{dy}\left(\frac{\int_{-\infty}^{\infty} \lambda e^{-4\pi y \lambda} d\rho(\lambda)}{\sqrt{\left(\int_{-\infty}^{\infty} e^{-4\pi y\lambda} d\rho(\lambda)\right)^2 - \left(\int_{-\infty}^{\infty} d\rho(\lambda)\right)^2}}\right),$$

and the positive number

$$R = \frac{1}{4\pi}\sqrt{\frac{\psi''(0)}{\psi(0)}} = 2\sqrt{\frac{\int_{-\infty}^{\infty} \lambda^2 d\rho(\lambda)}{\int_{-\infty}^{\infty} d\rho(\lambda)}}. \tag{6}$$

Finally, a stationary GAF is *degenerate* if its spectral measure ρ_f consists of exactly one atom. Similarly a stationary symmetric GAF is degenerate if ρ_f consists of two symmetric atoms (i.e., $\rho_f = c(\delta_q + \delta_{-q})$ for some $c, q > 0$).

The following theorem is our main result. Denote by m_1 the linear Lebesgue measure.

Theorem 1. *Let f be a stationary non-degenerate GAF or symmetric GAF in the strip D_Δ with $0 < \Delta \leq \infty$. Denote by $\nu_{f,T}$ the non-negative locally-finite random measure on $(-\Delta, \Delta)$ defined by*

$$\nu_{f,T}(Y) = \frac{1}{T} n_f([0, T) \times Y), \;\; Y \subset (-\Delta, \Delta).$$

Then:

(i) *Almost surely, the measures* $\nu_{f,T}$ *converge weakly to a measure* ν_f *when* $T \to \infty$.

(ii) *The measure* ν_f *is not random (i.e.* $\operatorname{var} \nu_f = 0$*) if and only if the spectral measure* ρ_f *has no atoms.*

(iii) *If the measure* ν_f *is not random, then:*

$$\begin{aligned} \nu_f &= L\, m_1, && \text{if } f \text{ is a GAF,} \\ \nu_f &= S\, m_1 + R\, \delta_0, && \text{if } f \text{ is a symmetric-GAF,} \end{aligned}$$

where δ_0 *is the unit point measure at the origin.*

The measure ν_f is referred to as "the horizontal limiting measure of the zeroes of f", or simply "the limiting measure". In the discussion and examples that follow, we assume the normalization $\psi(0) = \int_{\mathbb{R}} d\rho(\lambda) = 1$.

Remark 2.1. The limiting measure ν_f might have atoms. Generally speaking, the weak convergence in the theorem guarantees that $\nu_{f,T}([a,b))$ converges to $\nu_f([a,b))$ for all $a, b \in (\Delta, \Delta)$ with a possible exception of an at most countable set, which corresponds to atoms of the limiting measure ν_f. Yet, due to stationarity, in our case the limit exists *on all intervals*. We prove, as an example, the following result:

Proposition 2.1. *Almost surely, for any* $a, b \in (-\Delta, \Delta)$*, we have:*

$$\lim_{T\to\infty} \nu_{f,T}([a,b)) = \nu_f([a,b)).$$

The proof is included in appendix A. Notice that in particular for any $a \in (-\Delta, \Delta)$, a is an atom of ν_f if and only if it is an atom of $\nu_{f,T}$ for large enough T.

Remark 2.2. The part of the theorem pertaining to GAFs extends the aforementioned Wiener's theorem. In his work, Wiener assumed that the spectral measure ρ has the L^2-density $d\rho(\lambda) = |\phi(\lambda)|^2 d\lambda$, that satisfies convergence conditions:

For any $|y| < \Delta$,

$$\int_{-\infty}^{\infty} (1+x^2)^2\, |\widehat{\phi}(x+iy)|^2 dx < \infty,$$

and

$$\int_{-\infty}^{\infty} (1+x^2)\, |(\widehat{\phi})'(x+iy)|^2 dx < \infty.$$

As above, $\widehat{\phi}$ is the Fourier transform of ϕ. Under these assumptions, Wiener proved that the limiting measure ν_f exists and equals Lm_1, where L is defined by (4).

Remark 2.3. ***(atomic spectral measure)***
Consider a spectral measure consisting of two atoms:

$$\rho = \frac{1}{2}(\delta_{-q} + \delta_q)\,.$$

The corresponding GAF is $f(z) = (\zeta_1 e^{-2\pi i q z} + \zeta_2 e^{2\pi i q z})/\sqrt{2}$, where $\zeta_1, \zeta_2 \sim \mathcal{N}_{\mathbb{C}}(0,1)$, independently. The zeroes of such a function are

$$z_k = \frac{1}{4\pi q}\left[\arg(\frac{\zeta_2}{\zeta_1}) + 2\pi k - i\log\left|\frac{\zeta_2}{\zeta_1}\right|\right], \qquad k \in \mathbb{N}\,.$$

We see that all zeroes lie on the same (random) horizontal line, equally spaced upon it. The height of this horizontal line is a non-degenerate random variable, and so in this example ν_f is indeed random.

For symmetric GAFs, the spectral measure above is degenerate (all zeroes of the corresponding function are real). We mention that it is possible to construct a random analytic function with continuous spectrum, for which a given asymptotic proportion of zeroes lie on the real line. For this, choose a continuous symmetric spectral measure, sufficiently close (in the weak sense) to the degenerated measure $\delta_q + \delta_{-q}$.

Remark 2.4. ***(behavior near the boundary and near the real line.)*** We observe that $S(y)$ and $L(y)$ have the same asymptotic behavior as y approaches the boundary $\pm\Delta$. Therefore, zeroes of a GAF and of a symmetric GAF with the same kernel behave similarly near the boundary of the domain of definition.

For a symmetric GAF, we observe a "contraction" of the zeroes to the real line: there are zeroes on the line itself, but they are scarce as we approach it from below or above (see figure 3.1 below). This is confirmed by a straightforward computation, which shows that $S(y) = O(y)$, as $y \to 0$.

2.2. **Expected Zero-Counting Measures.** In part (iii) of the theorem, the limit $\nu_f(a,b)$ is actually the expectation $\mathbb{E}n_f([0,1]\times[a,b])$. In order to calculate this quantity in the GAF case, we use the following classical formula, which appeared in Edelman and Kostlan's joint work on random polynomials [6]. Several proofs of this formula are known ([5], chapter 2).

Theorem 2. (Edelman-Kostlan formula) *For a GAF f with covariance kernel $K(z,w)$, the expected zero-counting measure is given by*

$$\mathbb{E}(n_f) = \frac{1}{4\pi} \triangle \log K(z,z). \tag{7}$$

This should be understood as equality of measures in the following sense: for any compactly supported $h \in C^\infty(D)$,

$$\mathbb{E}\int_D h(z)dn_f(z) = \frac{1}{4\pi}\int h(z)\ \triangle \log K(z,z) dm_2(z).$$

Here and throughout this paper, m_2 denotes the planar Lebesgue measure.

The proofs of this formula depend inherently on the fact that $f(z)$ is a *complex* Gaussian random variable for all z, which fails for the symmetric GAF. To that end, we prove the following result, that extends previous results by Shepp and Vanderbei [19], Prosen [16] and Macdonald [13].

Theorem 3. *For a symmetric GAF f on some region with covariance kernel $K(z,w)$, the expected zero-counting measure is given by*

$$\mathbb{E}(n_f) = \frac{1}{4\pi} \triangle \log\left(K(z,z) + \sqrt{K(z,z)^2 - |K(z,\overline{z})|^2}\right), \tag{8}$$

where the Laplacian is taken in the distribution sense.

Notice that stationarity is not assumed in the last two theorems. Moreover, this formula combines information about real and complex zeroes.

3. Examples

3.1. **Paley-Wiener Process (Sinc-kernel Process).** Consider the spectrum

$$d\rho_a(\lambda) = \frac{1}{2a}\chi_{[-a,a]}(\lambda)d\lambda\,, \qquad a>0.$$

Condition (2) holds for any $\Delta > 0$, so the sample function f is entire. The kernel is:

$$K(z,w) = \frac{\sin(2\pi a(z-\overline{w}))}{2\pi a(z-\overline{w})} = r(z-\overline{w})$$

A base for construction of the GAF (in the sense of definition 1.1) is

$$\phi_n(z) = \frac{\sin(2\pi a z)}{2\pi a z - n\pi}, \qquad n\in\mathbb{Z}.$$

This example yields a surprising construction of a random series of simple fractions with known poles and stationary zeroes: Take for instance $a=1$. Our function is

$$f(z) = \sum a_n \frac{\sin(2\pi z)}{2\pi z - n\pi}$$

where $\{a_n\}$ are independent Gaussian random variables. Almost surely, $Z_f \cap \frac{1}{2}\mathbb{Z} = \emptyset$, so we may divide by $\sin(2\pi z)/\pi$ and get the random series

$$g(z) = \sum \frac{a_n}{2z-n}\,.$$

The poles of g are known (and lie on a one-dimensional lattice), but its zeroes are a random set invariant to *all* horizontal shifts!

Using Theorem 1 for ρ_a, we get that the zero-counting measure has the following density of zeroes:

$$L_a\left(\frac{y}{4\pi a}\right) = 4\pi a^2 \frac{d}{dy}\left(\coth y - \frac{1}{y}\right).$$

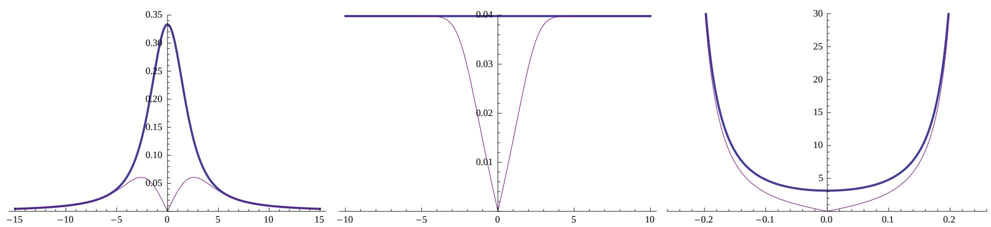
Similarly, the symmetric GAF with the same spectral measure has the continuous density of zeroes

$$S_a\left(\frac{y}{4\pi a}\right) = 4\pi a^2 \, \frac{d}{dy}\left(\frac{\cosh y - \frac{\sinh y}{y}}{\sqrt{\sinh^2 y - y^2}}\right)$$

plus an atom at $y = 0$, of size

$$R = \frac{a}{\sqrt{3}}.$$

Figure 1(a) represents the graphs of the continuous densities for the parameter $a = \frac{1}{4\pi}$.



(a) Paley-Wiener (b) Fock-Bargmann (c) exponential spectrum

FIGURE 1. Horizontal density of zeroes for GAF and symmetric GAF models with the same kernel. In each model, the lower graph represents the continuous component of the mean zero-counting measure for the symmetric GAF (the atomic part is an atom at $y = 0$, which is not graphed). The upper graph represents the continuous (and only) part of this measure for the appropriate GAF.

3.2. **Fock-Bargmann Space (Gaussian Spectrum).**

Set

$$d\rho_a(\lambda) = \frac{1}{a\sqrt{\pi}} e^{-\lambda^2/a^2} d\lambda\,, \qquad a > 0.$$

Once again, f is entire. The Fourier transform of the measure is

$$r(z) = \frac{1}{a\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\lambda^2/a^2} e^{2\pi i\lambda z} d\lambda = e^{-a^2\pi^2 z^2},$$

therefore the covariance kernel is:

$$K(z, w) = e^{-a^2\pi^2(z-\overline{w})^2}.$$

This space has an orthonormal basis of the form $\frac{(bz)^n}{\sqrt{n!}} e^{-cz^2}$, where $b = \sqrt{2}\frac{a}{\pi}$ and $c = -\frac{a^2}{\pi^2}$.

In this model, the density of zeroes is constant:

$$L_a\left(\frac{y}{2\pi a}\right) = 2\pi a^2.$$

This is the only model with Lebesgue measure as expected counting measure of zeroes. For more information about this model and why the distribution of zeroes determines the GAF, see [22], [15], [14] or [5, Chapters 2.3, 2.5].

However, for the real coefficients case the continuous part of the limiting measure has density

$$S_a\left(\frac{y}{2\pi a}\right) = 2\pi a^2 \frac{d}{dy}\left(\frac{e^{y^2}}{\sqrt{e^{2y^2}-1}}\right)$$

and the atom at $y = 0$ is of size $\sqrt{2}a$.

Both continuous densities are graphed in 1(b) for the parameter $a = \frac{1}{4\pi}$.

3.3. **Exponential Spectrum.** Consider a symmetric measure with exponential decay, for instance

$$d\rho(\lambda) = \operatorname{sech}(\pi\lambda)d\lambda = \frac{1}{\cosh(\pi\lambda)}d\lambda\,.$$

Here $r(z) = \operatorname{sech}(\pi z)$ as well. This model is valid in the strip $-\frac{1}{4} < \operatorname{Im}(z) < \frac{1}{4}$. Here

$$L(y) = \frac{\pi}{\cos^2(2\pi y)}.$$

For the symmetric GAF in this model, we have

$$S(y) = \frac{\pi|\sin(2\pi y)|}{\cos^2(2\pi y)}.$$

We see that the zeroes concentrate near the boundaries of the region of convergence (figure 1(c)).

4. Proof of Theorem 3 - Zero-Counting Measure for a symmetric GAF

In this section we prove Theorem 3. Similar formulas were proved in specific cases. Our proof follows Macdonald [13], who has considered random polynomials (also in the multi-dimensional case). A novelty is in the extension of his result to arbitrary symmetric GAFs.

Recall that for any analytic function f (not necessarily random) in a domain D we have

$$n_f = \frac{1}{2\pi}\triangle \log|f|.$$

This is understood in the distribution sense.

Using this for our random f, we would like to take expectation of both sides, to get:

$$\begin{aligned}\mathbb{E}\left[\int_X h(z)dn_f(z)\right] = \mathbb{E}\left[\frac{1}{2\pi}\int_X \triangle h(z)\log|f(z)|dm_2(z)\right] = \\ \frac{1}{2\pi}\int_X \triangle h(z)\mathbb{E}\left[\log|f(z)|\right]dm_2(z),\end{aligned} \tag{9}$$

where m_2 denotes the Lebesgue measure in $\mathbb{C}$. The last equality is justified by Fubini's Theorem, as we show at the end of this section. Thus we can conclude that (in the weak sense):

$$\mathbb{E}(n_f) = \frac{1}{2\pi} \triangle \mathbb{E} \log |f|. \tag{10}$$

Let us return to our setup: f is a random function generated by a basis $\phi_k(z)$ of holomorphic functions, each real on $\mathbb{R}$, and such that the sum $\sum_k |\phi_k(z)|^2$ converges locally-uniformly. Denote $\phi_k(z) = u_k(z) + iv_k(z)$ where u_k, v_k are real functions. Our random function is decomposed thus:

$$f(z) = \sum b_k \phi_k(z) = \sum b_k u_k(z) + i \sum b_k v_k(z) = u(z) + iv(z),$$

where $b_k \sim \mathcal{N}_{\mathbb{R}}(0,1)$ are *real* Gaussian standard variables. $(u(z), v(z))$ have a joint Gaussian distribution, with mean (0,0) and covariance matrix

$$\Sigma = \begin{pmatrix} \sum u_k^2 & \sum u_k v_k \\ \sum u_k v_k & \sum v_k^2 \end{pmatrix}.$$

Lemma 4.1. *The above matrix Σ has two positive eigenvalues $\lambda_2 \geq \lambda_1$ obeying:*

$$\lambda_{2,1} = \frac{K(z,z) \pm |K(z,\overline{z})|}{2}$$

where $K(z,w) = \sum \phi_k(z)\overline{\phi_k(w)} = \sum \phi_k(z)\phi_k(\overline{w})$.

Proof. For any complex number $\phi = u + iv$, we have:

$$u^2 = \frac{1}{2}\left(|\phi|^2 + \mathrm{Re}(\phi^2)\right), \quad v^2 = \frac{1}{2}\left(|\phi|^2 - \mathrm{Re}(\phi^2)\right), \quad uv = \frac{1}{2}\mathrm{Im}(\phi^2).$$

Applying this, we can rewrite Σ as

$$\Sigma = \begin{pmatrix} \frac{1}{2}\left(\sum |\phi_k|^2 + \mathrm{Re}\sum \phi_k^2\right) & \frac{1}{2}\mathrm{Im}\sum \phi_k^2 \\ \frac{1}{2}\mathrm{Im}\sum \phi_k^2 & \frac{1}{2}\left(\sum |\phi_k|^2 - \mathrm{Re}\sum \phi_k^2\right) \end{pmatrix},$$

and then calculate its determinant and trace:

$$\begin{aligned} \lambda_1\lambda_2 = \det \Sigma &= \frac{1}{4}\left((\sum |\phi_k|^2)^2 - (\mathrm{Re}\sum(\phi_k^2))^2 - (\mathrm{Im}\sum(\phi_k^2))^2\right) \\ &= \frac{1}{4}\left(K(z,z)^2 - |K(z,\overline{z})|\right), \end{aligned} \tag{11}$$

$$\lambda_1 + \lambda_2 = \mathrm{trace}\ \Sigma = \sum |\phi_k|^2 = K(z,z).$$

The lemma follows. □

Using the law of bi-normal distribution, we get:

$$\begin{aligned} \mathbb{E}[\log|f(z)|] &= \frac{1}{2\pi\sqrt{\det\Sigma}} \iint_{\mathbb{R}^2} \log(\sqrt{x^2+y^2}) e^{-\frac{1}{2}(x,y)\Sigma^{-1}(x,y)^T} dx\, dy = \\ &\frac{1}{2\pi\sqrt{\det\Sigma}} \iint_{\mathbb{R}^2} \log(\sqrt{x^2+y^2}) e^{-\frac{1}{2}(\lambda_1^{-1}x^2 + \lambda_2^{-1}y^2)} dx\, dy. \end{aligned} \tag{12}$$

Applying to the last integral the change of variables $x = u\sqrt{\lambda_1},\ y = w\sqrt{\lambda_2}$, with Jacobian $\sqrt{\lambda_1\lambda_2} = \sqrt{\det\Sigma}$, we have

$$\mathbb{E}[\log|f(z)|] = \frac{1}{2\pi}\iint_{\mathbb{R}^2} \log(\sqrt{\lambda_1 u^2 + \lambda_2 w^2})e^{-(u^2+w^2)/2} du\, dw.$$

Now, changing to polar coordinates $u = r\cos\theta,\ w = r\sin\theta$, we get:

$$\mathbb{E}[\log|f(z)|] = \frac{1}{2\pi}\int_0^{2\pi} \log(\sqrt{\lambda_1\cos^2\theta + \lambda_2\sin^2\theta})\, d\theta + C,$$

where C is a constant which does not depend on the point z (i.e., is independent of λ_1 and λ_2). In the following, we write C for any such constant (which may be different each time we use this symbol). These constants will vanish when we apply Laplacian (recall (10)).

So, the integral we should compute is:

$$\begin{aligned}&\int_0^{2\pi} \log\left|\sqrt{\lambda_1}\cos\theta + i\sqrt{\lambda_2}\sin\theta\right| \frac{d\theta}{2\pi}\\ &= \log(\sqrt{\lambda_1}+\sqrt{\lambda_2}) + \int_0^{2\pi}\log\left|e^{2i\theta} + \frac{\sqrt{\lambda_1}-\sqrt{\lambda_2}}{\sqrt{\lambda_1}+\sqrt{\lambda_2}}\right| \frac{d\theta}{2\pi} + C.\end{aligned}$$

The remaining integral is computed easily by Jensen's formula for the function $g(z) = z^2 + c$, where $c = \frac{\sqrt{\lambda_1}-\sqrt{\lambda_2}}{\sqrt{\lambda_1}+\sqrt{\lambda_2}} < 1$. Indeed, it has two zeroes in the unit circle, denoted a_1 and a_2, and so:

$$\int_0^{2\pi} \log\left|e^{2i\theta} + c\right| \frac{d\theta}{2\pi} = \log|g(0)| - \log|a_1| - \log|a_2| = \log|c| - 2\log\sqrt{|c|} = 0.$$

Recalling (10) and using the relations (11), we arrive at

$$\begin{aligned}\mathbb{E}(dn_f) &= \frac{\triangle}{2\pi}\frac{1}{2}\log\left(\lambda_1+\lambda_2+\sqrt{4\lambda_1\lambda_2}\right)\\ &= \frac{1}{4\pi}\triangle\log\left(K(z,z)+\sqrt{K(z,z)^2 - |K(z,\overline{z})|^2}\right).\end{aligned}$$

4.1. **Justification of** (9)**.** We must show that the following integral converges:

$$\frac{1}{2\pi}\int_X |\triangle h(z)| \cdot \mathbb{E}|\log|f(z)||\,dm_2(z).$$

It is enough to prove that $\mathbb{E}|\log|f(z)||$ is bounded on a compact subset S of the plane. $f(z)$ is a 2-dimensional real Gaussian variable with parameters noted above, so we get

$$\mathbb{E}|\log|f(z)|| = \frac{1}{2\pi\sqrt{\det\Sigma}}\iint_{\mathbb{R}^2} |\log(\sqrt{x^2+y^2})|e^{-\frac{1}{2}(\lambda_1^{-1}x^2+\lambda_2^{-1}y^2)}dxdy.$$

As before, λ_1, λ_2 are the eigenvalues of Σ, dependent on z. By another change of variables ($x = u\sqrt{\lambda_1},\ y = w\sqrt{\lambda_2}$) we get:

$$\mathbb{E}|\log|f(z)|| = \frac{1}{4\pi}\iint_{\mathbb{R}^2} |\log(\lambda_1 u^2 + \lambda_2 w^2)|e^{-(u^2+w^2)/2}du\, dw.$$

Fix z, and assume $\lambda_1 \le \lambda_2$. Let us split the integral into two domains: $\Omega_+ = \{(u,w) \in \mathbb{R}^2 : \log(\lambda_1 u^2 + \lambda_2 w^2) \ge 0\}$ and $\Omega_- = \{(u,w) \in \mathbb{R}^2 : \log(\lambda_1 u^2 + \lambda_2 w^2) < 0\}$.

Then, on Ω_+ we estimate $0 < \log(\lambda_1 u^2 + \lambda_2 w^2) < \log(\lambda_2) + \log(u^2+w^2)$. From here clearly the integral on Ω_+ is bounded by $C_0 + C_1 \log \lambda_2$. By lemma 4.1, $\lambda_2 = \frac{1}{2}(K(z,z) + |K(z,\overline{z})|)$ is a continuous function of z, and therefore is bounded on our compact set S.

For $(u,w) \in \Omega_-$ notice that $0 > \log(\lambda_1 u^2 + \lambda_2 w^2) > \log(\lambda_1 u^2)$, therefore:

$$\iint_{\Omega_-} |\log(\lambda_1 u^2 + \lambda_2 w^2)| e^{-(u^2+w^2)/2} du\, dw \le$$
$$\iint_{\Omega_-} (|\log(\lambda_1) + \log(u^2)|) e^{-(u^2+w^2)/2} du\, dw \le C_0 + C_1 |\log \lambda_1|.$$

Denote $m = \min\{\lambda_1(z) : z \in S\}$. If $m = 0$, this leads to $K(z_0, z_0) = 0$ for some $z_0 \in K$, but this means z_0 is a deterministic zero. Therefore $m > 0$ and $|\log \lambda_1|$ is bounded from above.

5. Proof of Theorem 1 - Horizontal Limiting Measure

5.1. **Preliminaries.** We present the probability space of our interest, equipped with a measure-preserving transformation. We explain the notion of ergodicity in this setup.

The probability space Ω is a countable product of copies of $\mathbb{C}$, with $\mathbb{P}$ being the product of complex Gaussian measures (one on each copy). These copies represent the random coefficients in the construction of f: each $\omega = \{a_n\}_n \in \Omega$ corresponds to a function $f_\omega(z) = \sum a_n \phi_n(z)$. $\mathcal{F}_f$ is the Borel σ-algebra generated by the basic sets $\{\omega \in \Omega : f_\omega(z) \in B(w,r)\}$, where $z \in D, r > 0$. Here $B(w,r) = \{p \in \mathbb{C} : |p - w| < r\}$. The group of automorphisms S_t shall be defined via the correspondence $\omega \leftrightarrow f_\omega$:

$$f_{S_t \omega}(z) = f_\omega(z+t).$$

The map S_t is measure-preserving, since we assumed that f is stationary. Thus, we will say the random process $f(z)$ is ergodic, if any measurable set $A \in \mathcal{F}_f$ which is invariant to all translations ($S_t A = A$, $\forall t \in \mathbb{R}$) is in fact trivial ($\mathbb{P}A \in \{0,1\}$).

In a similar way, one can define when is the zero-set Z_f ergodic (it is itself a random point-process in the plane). The space Ω, the measure $\mathbb{P}$ on it and the automorphisms $\{S_t\}$ are just as before. Now, the σ-algebra $\mathcal{F}_{Z_f}$ is generated by the basic sets $\{\omega \in \Omega : Z_{f_\omega} \cap B(z,r) \ne \emptyset\}$ with $z \in D, r > 0, B(z,r) \subset D$.

Corollary 5.1. *Ergodicity of f implies ergodicity of Z_f.*

Proof. It is enough to prove $\mathcal{F}_{Z_f} \subset \mathcal{F}_f$. Let A be a countable dense set in $\mathbb{C}$. Basic sets of $\mathcal{F}_{Z_f}$ can be written as

$$\{Z_{f_\omega} \cap B(z,r) \ne \emptyset\} = \bigcup_{m\in\mathbb{N}} \bigcap_{n \in \mathbb{N}} \bigcup_{p \in A \cap B(z, r - \frac{1}{m})} \left\{ f_\omega(p) \in B\left(0, \frac{1}{n}\right)\right\},$$

which is indeed in $\mathcal{F}_f$. □

We will use the following classical result:

Theorem 4. (Fomin, Grenander, Maruyama) *A stationary GAF (symmetric or not) is ergodic w.r.t. horizontal shifts $\{f(z) \to f(z+t)\}_{t\in\mathbb{R}}$ if and only if its spectral measure ρ has no atoms.*

This theorem was originally proved for real processes over $\mathbb{R}$ (see for instance Grenander [7]), but small modifications extend it to a strip in the complex plane for both types of functions (GAFs and symmetric GAFs).

5.2. **Existence of the horizontal limiting measure (statement (i)).** As above, for $T \geq 1$, let ν_T be the random locally-finite measure on $(-\Delta, \Delta)$ defined by:

$$\nu_T(Y) = \nu_{f,T}(Y) := \frac{n_f([0,T) \times Y)}{T}\,, \qquad Y \subset (-\Delta, \Delta). \tag{13}$$

In this section we show that a.s. the measures ν_T converge weakly as T tends to infinity. First, we assume that T tends to infinity along positive integers.

In this case, we use the subscript N instead of T. By a known theorem in distribution theory (see for instance, [8, section 2.1]), a sequence of measures ν_N converges weakly to some measure if and only if the sequence of real numbers $\nu_N(h)$ is convergent for every $h \in C_0^\infty(-\Delta, \Delta)$. It suffices to check whether $\nu_N(h)$ is convergent for all $h \in M$, where $M \subset C_0^\infty(-\Delta, \Delta)$ is a dense set of test-functions, and we may choose M to be countable. Given a test-function $h \in M$, denote by A_h the event that $\nu_N(h)$ is a convergent sequence of numbers. To prove our claim it suffices to show $\mathbb{P}(A_h) = 1$ for every $h \in M$. Note that $\nu_N(h) = \frac{1}{N}(X_1 + X_2 + \cdots + X_N)$, where

$$X_k = X_k(h) = \int 1\!\!1_{[k,k+1)}(x) h(y)\, dn_f(x,y) \tag{14}$$

is a stationary sequence of random variables.

The random variables X_k are integrable. This follows at once from an Offord-type large deviations estimate [5, Theorem 3.2.1]:

Theorem 5 (Offord-type estimate)**.** *Let f be a GAF on a domain D. Then for any compact set $K \subset D$, the number $n_f(K)$ of zeroes of f on K has exponential tail: there exist positive constants C and c depending on the covariance function of f and on K such that, for each $\lambda \geq 1$,*

$$\mathbb{P}\big\{n_f(K) > \lambda\big\} < Ce^{-c\lambda}\,.$$

Therefore, we can apply the Birkhoff theorem [4, chapter 7]. It yields that the limit $\frac{1}{N}(X_1 + X_2 + \cdots + X_N)$ almost surely exists, and so $\mathbb{P}(A_h) = 1$. This completes the proof of the weak convergence of the sequence ν_N.

Now, we consider the general case in statement (i). Let $T \geq 1$, and let $N = [T]$ be the integer part of T. Then

$$\nu_T(h) = \frac{N}{T}\,\nu_N(h) + \underbrace{\frac{1}{T}\int \mathbb{1}_{[N,T)}(x)h(y)\,dn_f(x,y)}_{=:R_T(h)}\,.$$

We show that a.s. the second term on the right-hand side converges to zero for all bounded compactly supported test functions h. It suffices to prove this for all bounded test functions supported by an interval $[-\Delta_1, \Delta_1]$ with an arbitrary $0 < \Delta_1 < \Delta$. We have

$$|R_T(h)| \leq \frac{\|h\|_\infty}{T}\, n_f\big([N, N+1) \times [-\Delta_1, \Delta_1]\big)\,.$$

Employing the Offord-type estimate with $K = [0,1] \times [-\Delta_1, \Delta_1]$ and using translation-invariance of the zero distribution of f, we see that for each $\varepsilon > 0$,

$$\mathbb{P}\big\{\, n_f([N, N+1]\times[-\Delta_1,\Delta_1]) \geq \varepsilon T \,\big\} = \mathbb{P}\big\{\, n_f([0,1]\times[-\Delta_1,\Delta_1]) \geq \varepsilon T \,\big\} < Ce^{-c\varepsilon N}\,.$$

Hence, for each $M \in \mathbb{N}$,

$$\mathbb{P}\big\{\limsup_{T\to\infty} |R_T(h)| \geq \varepsilon\,\|h\|_\infty\big\} \leq \sum_{M}^{\infty} Ce^{-c\varepsilon N} = C(1-e^{-c\varepsilon})^{-1}e^{-c\varepsilon M}\,,$$

and we conclude that a.s.

$$\lim_{T\to\infty} R_T(h) = 0$$

for all smooth compactly supported test functions h. This completes the proof of statement (i) in Theorem 1. $\square$

5.3. **Non-random limiting measure (statement (iia),(iii)).** Here we will prove that if the spectral measure ρ_f has no atoms, then the horizontal mean zero-counting measure ν_f is not random, which is half of statement (ii). We then compute the limit ν_f, which is statement (iii).

Assume the spectral measure ρ is continuous. By Theorem 4 (Fomin-Maruyama-Grenander) we get that f is ergodic, and by corollary 5.1 so is Z_f. Using the notation introduced in the proof of statement (i), we get that for any smooth test function h, the stationary sequence of random variables $X_k(h)$ introduced in (14) is ergodic. In this case the Birkhoff ergodic theorem asserts that, a.s.,

$$\lim_{N\to\infty} \nu_N(h) = \lim_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} X_k(h) = \mathbb{E}X_0(h)\,.$$

Therefore, the horizontal mean zero-counting measure is non-random, and equals

$$\mathbb{E}X_0(h) = \mathbb{E}\int \mathbb{1}_{[0,1)}(x)h(y)\,dn_f(x,y) = \int \mathbb{1}_{[0,1)}(x)h(y)\,\mathbb{E}dn_f(x,y)\,,$$

where $\mathbb{E}dn_f(x,y)$ is the mean zero-counting measure. For a GAF, we compute it directly by Edelman-Kostlan formula (7); while for a symmetric GAF we use the formula (8) (in Theorem 3). As before, denote $\psi(y) = \int_{-\infty}^{\infty} e^{-4\pi y\lambda} d\rho(\lambda)$. Note that

$$K(z,z) = \int e^{2\pi i\cdot 2yt} d\rho(t) = \psi(y), \text{ and}$$
$$K(z,\overline{z}) = \int e^{2\pi i(z-\overline{z})t} d\rho(t) = \int d\rho(t) = \psi(0),$$

where $z = x+iy$. Putting this into (8), we get the first intensity of zeroes:

$$\mathbb{E}n_f = \frac{1}{4\pi}\frac{d^2}{dy^2}\log\left(\psi(y) + \sqrt{\psi(y)^2 - \psi(0)^2}\right) = \frac{1}{4\pi}\frac{d}{dy}\frac{\psi'(y)}{\sqrt{\psi(y)^2-\psi(0)^2}}.$$

For any $y \neq 0$, this is a derivative in the functional sense, which equals $S(y)$. At $y = 0$, the function is not defined; but the limits

$$\lim_{y\to 0+}\frac{\psi'(y)}{4\pi\sqrt{\psi(y)^2-\psi(0)^2}} = -\lim_{y\to 0-}\frac{\psi'(y)}{4\pi\sqrt{\psi(y)^2-\psi(0)^2}} = A$$

exist. This follows from $\frac{\psi'(y)}{\sqrt{\psi(y)^2-\psi(0)^2}}$ being an odd function, increasing in $y \in (0,\Delta)$. So, in order to compute $\mathbb{E}n_f$ we take the required derivative in the distribution sense, which yields the continuous point-wise derivative $S(y)$ (for $y \neq 0$) plus an atom of size $2A$ at $y = 0$.

In order to compute A let us write this limit again, and apply L'Hôpital's rule:

$$4\pi A = \lim_{y\to 0+}\frac{\psi'(y)}{\sqrt{\psi(y)^2-\psi(0)^2}} = \lim_{y\to 0+}\frac{\psi''(y)\cdot\sqrt{\psi(y)^2-\psi(0)^2}}{\psi(y)\cdot\psi'(y)} = (4\pi)^2\mathcal{E}_2\frac{1}{4\pi A},$$

where $\mathcal{E}_2 = \dfrac{\int_{-\infty}^{\infty}\lambda^2 d\rho(\lambda)}{\int_{-\infty}^{\infty} d\rho(\lambda)}$ is the ratio between the second and the zero moments of the spectral measure. We conclude that $A = \sqrt{\mathcal{E}_2}$, and therefore the atom has twice this size.

5.4. **Random limiting measure (statement (iib)).** In this section we prove the second half of (ii). We present the proof for symmetric GAFs, since it is slightly more involved. We assume that the spectral measure has the form

$$\rho_f = c\delta_q + c\delta_{-q} + \mu,$$

where μ is a non-trivial measure. Our goal is to show that the horizontal mean zero-counting measure of some segment $\nu_f(a,b)$ is a non-constant random variable. We may assume that $c = 1$ and $q = 1$ (if $q = 0$ the analysis is easier).

Since $L^2_\rho(\mathbb{R})$ is the direct sum of $L^2_{\delta_1+\delta_{-1}}(\mathbb{R})$ and $L^2_\mu(\mathbb{R})$, a union of any orthonormal bases in these subspaces is an orthonormal basis in $L^2_\rho(\mathbb{R})$. By the remark at the end of section 1.2, after applying Fourier transform on this union we get a basis $\phi_n(z)$ from which a GAF with spectral measure ρ can be constructed. This gives the representation

$$f(z) = g(z) + \alpha\cos(2\pi z) + \beta\sin(2\pi z),$$

where $g(z)$ is a symmetric GAF with spectral measure μ and $\alpha,\ \beta \sim \mathcal{N}_{\mathbb{R}}(0,1)$ are real Gaussians, independent of each other and of g. We write for short $\eta(z) = \eta_{\alpha,\beta}(z) = \alpha\cos(2\pi z) + \beta\sin(2\pi z)$.

Fix $a, b \in (-\Delta, \Delta)$. Denote the number of zeroes of f in $[0,T]\times[a,b)$ by $N_T(g,\alpha,\beta) = \#\{z \in [0,T]\times[a,b) : g(z) = -\eta_{\alpha,\beta}(z)\}$.

Assume to the contrary that there is some constant C (depending on a and b) such that

$$\text{a.s. in } \alpha,\beta,\ \exists \lim_{T\to\infty} \frac{N_T(g,\alpha,\beta)}{T} = C. \tag{15}$$

Here we denote by $\mathbb{P}_g$ and $\mathbb{E}_g$ the probability and expectation (respectively) conditioned on α, β. We claim that:

$$\mathbb{E}_g \lim_{T\to\infty} \frac{N_T(g,\alpha,\beta)}{T} = \lim_{T\to\infty} \frac{\mathbb{E}_g N_T(g,\alpha,\beta)}{T}. \tag{16}$$

This exchange is justified by the dominated convergence principle, as seen by the following Offord-type estimate:

Proposition 5.1. *Let g be a symmetric stationary GAF on a horizontal strip, α and β are fixed complex numbers. There exist positive constants C and c such that:*

$$\sup_{T\geq 1} \mathbb{P}_g\left(\frac{N_T(g,\alpha,\beta)}{T} > s\right) < Ce^{-cs},$$

This fact is proved in appendix B below.

Next we claim that the right-hand side of (16) is just $\mathbb{E}_g N_1(g,\alpha,\beta)$. To see this, notice that for integer T, $N_T(g,\alpha,\beta)$ is the sum of T identically distributed random variables, all distributed like $N_1(g,\alpha,\beta)$. This follows immediately from the stationarity of g and 1-periodicity of $\eta_{\alpha,\beta}(z)$. Therefore, for integer T,

$$\frac{1}{T}\,\mathbb{E}_g N_T(g,\alpha,\beta) = \mathbb{E}_g N_1(g,\alpha,\beta).$$

For non-integer T, denote $M = \lfloor T \rfloor$. Since $\mathbb{E}_g N_{[M,T]} := \mathbb{E}\#\{z \in [M,T]\times[a,b) : g(z) = -\eta_{\alpha,\beta}(z)\} \leq \mathbb{E}_g N_1 < \infty$, it follows that for non-integer T,

$$\lim_{T\to\infty} \frac{\mathbb{E}_g N_T(g,\alpha,\beta)}{T} = \lim_{T\to\infty}\left(\frac{\mathbb{E}_g N_M}{M}\cdot\frac{M}{T} + \frac{\mathbb{E}_g N_{[M,T]}}{T}\right) = \mathbb{E}_g N_1(g,\alpha,\beta). \tag{17}$$

Combining (15), (16) and (17) we have:

$$\text{a.s. in } \alpha,\beta,\ \mathbb{E}_g N_1(g,\alpha,\beta) = C. \tag{18}$$

We divide the rest of our argument into three claims.

Claim 5.1. *$\mathbb{E}_g N_1(g,\alpha,\beta)$ is continuous in $(\alpha,\beta)\in\mathbb{R}^2$.*

Claim 5.2. *For any compact set $K \subset D$, let $N(g, \alpha, \beta; K)$ be the number of solutions to $g(z) = -\eta_{\alpha,\beta}(z)$ with $z \in K$. Then*

$$\mathbb{E}_g N(g, \alpha, \beta; K) \to n_{\cos(2\pi z)}(K) \quad \text{as} \quad |\alpha| \to \infty.$$

Here $n_{\cos(2\pi z)}$ is the zero-counting measure of $\cos(2\pi z)$.

Relying on the last claim and (18), we get that

$$\mathbb{E}_g N_1(g, \alpha, \beta) = 2\delta_0([a, b)) \tag{19}$$

for almost all α, β. Since $\mathbb{E}_g N_1(g, \alpha, \beta)$ is continuous in α, β, (19) is true *for all* α, β, and in particular for $(\alpha, \beta) = (0, 0)$. The following claim asserts this happens only for one family of symmetric GAFs:

Claim 5.3. *If for $-\Delta < a < 0 < b < \Delta$,*

$$\mathbb{E}_g N_1(g, 0, 0) = \mathbb{E} n_g([0, 1) \times [a, b)) = 2\delta_0([a, b)),$$

then the spectral measure of g is $\frac{1}{2}(\delta_1 + \delta_{-1})$, up to a constant multiplier.

From this last claim it follows that the spectral measure of f consists only of symmetric atoms at ± 1, which contradicts our assumption.

It remains now to prove the claims. In the course of their proof, we justify the exchange of limits and expectations by the following

Proposition 5.2 (Offord-type estimate for sine-like levels)**.** *Let g be a symmetric GAF on a domain D, and let α and β be fixed complex numbers. Then for any compact $K \subset D$, the number $N(g, \alpha, \beta; K)$ of solutions to $g(z) = -\eta_{\alpha,\beta}(z)$ with $z \in K$ has exponential tail: There exist positive constants C and c such that*

$$\mathbb{P}(N(g, \alpha, \beta; K) > s) \le Ce^{-cs}.$$

The proof of the last proposition is similar to that of Proposition 5.1, and we omit it.

Proof of Claim 5.1. Fix $-\Delta < \alpha_0 < \beta_0 < \Delta$. It is clear that almost surely in g, $N_1(g, \alpha, \beta_0)$ approaches $N_1(g, \alpha_0, \beta_0)$ as α approaches α_0 (the event of having a solution to $g(z) = -\eta_{\alpha_0,\beta_0}$ on the boundary of $[0, 1] \times [a, b]$ is negligible).

By Proposition 5.2, we may pass to the limit:

$$\lim_{\alpha \to \alpha_0} \mathbb{E}_g N_1(g, \alpha, \beta_0) = \lim_{\alpha \to \alpha_0} \int \mathbb{P}_g(N_1(g, \alpha, \beta_0) > s) ds =$$

$$= \int \lim_{\alpha \to \alpha_0} \mathbb{P}_g(N_1(g, \alpha, \beta_0) > s) ds = \int \mathbb{P}_g(N_1(g, \alpha_0, \beta_0) > s) ds = \mathbb{E}_g(N_1(g, \zeta_0)).$$

$\square$

Proof of Claim 5.2. Fix β and g. For any $\alpha \neq 0$, the zeroes of

$$h_\alpha(z) = \frac{g(z) + \beta \sin(2\pi z)}{\alpha} + \cos(2\pi z)$$

and of $f(z) = g(z) + \eta_{\alpha,\beta}(z)$ are identical. Now notice that $h_\alpha(z)$ converges locally uniformly to $\cos(2\pi z)$ as $\alpha \to \infty$ (i.e., uniformly on any compact set). By Hurwitz's Theorem, this implies that the zero-counting measures also converge locally uniformly, in the sense that for any compact $K \subset D$,

$$\lim_{\alpha\to\infty} n_{h_\alpha}(K) = n_{\cos(2\pi z)}(K).$$

By the bound in Proposition 5.2, this almost sure convergence in g yields moment convergence:

$$\mathbb{E}_g n_{h_\alpha}(K) \to n_{\cos(2\pi z)}(K), \text{ as } \alpha \to \infty.$$

□

Proof of Claim 5.3. Suppose the spectral measure is normalized, so that $\psi(0) = \int_{\mathbb{R}} d\rho(\lambda) = 1$ (else, multiply it by a constant). The premise and Theorem 1 give two conditions on $\psi(y) = K(iy, iy)$:

$$\frac{\psi'(y)}{\sqrt{\psi(y)^2 - 1}} = c\,, \qquad\qquad R = 2\sqrt{\int_{\mathbb{R}} \lambda^2 d\rho(\lambda)} = 2\,,$$

for some constant $c \in \mathbb{R}$. Solving the left-hand side ordinary differential equation, and using $\psi(0) = 1$, we get $\psi(y) = \cosh(cy)$. Since ψ is a Laplace transform of ρ, we get $\rho = \frac{1}{2}(\delta_{c/2\pi} + \delta_{-c/2\pi})$. But the right-hand side equation is satisfied only if $c = 2\pi$. □

Appendix A. Convergence on all intervals (proof of Proposition 2.1)

In this section we prove Proposition 2.1. We use the notations developed in section 5.1. For any point in the probability space $\omega \in \Omega$, let ν_N^ω be the sequence of measures introduced in (13), for integer $T = N$ (the non-integer case follows just as in the proof of part (i) of Theorem 1, and will not be discussed).

Define the set

$$C = \{\omega \in \Omega : (\nu_N^\omega)_N \text{ converges weakly}\}$$

Notice that by part (i) of Theorem 1, $\mathbb{P}(C) = 1$. For convenience of notation we denote the limit $\nu_f = \nu^\omega$. From general measure theory, one can deduce that almost surely, $\nu_N([a, b))$ converges to $\nu^\omega([a, b))$ for all a, b out of a countable exceptional set. This exceptional set is the set of atoms of ν_f (which might be random). We thus turn to define

$$A = \{\omega \in \Omega : \lim_{N\to\infty} \nu_N^\omega\{a\} = \nu^\omega\{a\}, \text{ for each atom } a \text{ of } \nu^\omega\} \subset C.$$

Claim A.1. *A is measurable with respect to $\mathcal{F}_f$.*

The proof of this claim will be presented in the end of this section. Our next goal would be to prove:

Claim A.2. $\mathbb{P}(A) = 1$.

Our main tool will be the Ergodic Decomposition Theorem (proved, for instance, in [1, chapter 2.2.8]):

Theorem 6 (Ergodic Decomposition). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a standard Borel-space, equipped with a measure preserving transformation $S : \Omega \to \Omega$. Then the set $E^S(\Omega)$ of ergodic probability measures on Ω is not empty, and there exists a map $\beta : \Omega \to E^S(\Omega)$ such that for any measurable set $A \in \mathcal{F}$ the following holds:*

1. *the map* $\left\{ \begin{array}{c} \Omega \to [0,1] \\ \omega \mapsto \beta_\omega(A) \end{array} \right.$ *is measurable.*
2. $\mathbb{P}(A) = \int_\Omega \beta_\omega(A) d$
   $Pro(\omega)$.

Proof of Claim A.2. The stationary system $(\Omega, \mathcal{F}_{Z_f}, \mathbb{P}, S)$ defined in section 5.1 and the set A defined above meet the requirements of the Ergodic Decomposition Theorem. Therefore, in order to prove our claim it is enough to show that

$$\forall \gamma \in E^S(\Omega), \gamma(A) = 1.$$

Fix an S-ergodic measure γ. Since A is an S-invariant set, we get $\gamma(A) \in \{0, 1\}$. Moreover, the event $\{\nu^\omega$ has an atom in the interval $I\}$ is also invariant, for any interval $I \subset (-\Delta, \Delta)$. Therefore, γ-a.s. the limiting measure ν^ω has atoms at some known points $(a_n)_{n\in\mathbb{N}} \subset (-\Delta, \Delta)$.

For a certain atom $a = a_n$, define the stationary sequence:

$$X_k(a) = n_f([k, k+1) \times \{a\}),$$

and notice that

$$\nu_N\{a\} = \frac{1}{N} \sum_{k=0}^{N-1} X_k(a).$$

As γ is ergodic, we have by Birkohff's ergodic Theorem:

$$\gamma\text{-a.s. } \nu_N^\omega\{a\} \text{ converges to } \mathbb{E}_\gamma n_f([0,1) \times \{a\}) = \nu^\omega\{a\}, \text{ as } N \to \infty$$

Since there are at most countably many atoms, we get $\gamma(A) = 1$. □

We now know that $\mathbb{P}$-a.s., the sequence ν_N is weak convergent and converges on any atom of the limiting measure (to the desired limit). A general claim from measure theory will assure us that in this case, ν_N converge on any interval:

Claim A.3. *Suppose $(\nu_N)_N$ is a weak-converging sequence of measures on some interval I, and let ν be the limiting measure. If $\lim_{N\to\infty} \nu_N\{a\} = \nu\{a\}$ for every atom a of ν, then $\lim_{N\to\infty} \nu_N(J) = \nu(J)$ for* every *interval $J \subset I$.*

Proof. We demonstrate the case $J = [a, b)$, where ν has no atom at b (other cases are similar).

Given $\epsilon > 0$, one can construct piecewise linear functions ϕ^+, $\phi^- \in C(I)$ such that:

$$\forall x,\ \phi^-(x) \le 1\!\!1_{(a,b)} \le 1\!\!1_{[a,b)}(x) \le \phi^+(x), \tag{20}$$

and additionally

$$0 < \nu(\phi^+) - \big(\nu(\phi^-) + \nu\{a\}\big) < \epsilon.$$

(For instance, for large enough parameter n, take ϕ^+ supported on $[a - \frac{1}{n}, b]$, equals 1 on $[a, b - \frac{1}{n}]$ and linear otherwise; ϕ^- supported on $[a, b]$, equals 1 on $[a + \frac{1}{n}, b - \frac{1}{n}]$, and linear otherwise).

By applying the measure ν_N to relation (20), we get:

$$\nu_N(\phi^-) + \nu_N\{a\} \le \nu_N([a, b)) \le \nu_N(\phi^+)$$

But, from our assumptions, for large enough N we have

$$\nu(\phi^-) + \nu\{a\} - \epsilon \le \nu_N([a, b)) \le \nu(\phi^+) + \epsilon$$

As the difference between those bounds does not exceed 3ϵ, we see the limit $\lim_{N\to\infty} \nu_N([a, b))$ exists. Since $\nu(\phi^+)$ is as close as we want to $\nu([a, b))$, we are done.

□

It remains only to prove the measurability of A.

Proof of Claim A.1. We first investigate some underlying objects. Denote by $P = P(-\Delta, \Delta)$ the space of all locally finite measures on $(-\Delta, \Delta)$ induced with the Lévy - Prokhorov metric (for which convergence in metric is equivalent to weak convergence):

$$\pi(\mu, \nu) := \inf\{\epsilon > 0 \,|\, \forall Y \in \mathcal{B}\, \mu(Y) \le \nu(Y^\epsilon) + \epsilon \text{ and } \nu(Y) \le \mu(Y^\epsilon) + \epsilon\},$$

where $\mathcal{B}$ is the sigma-algebra of Borel subsets of $(-\Delta, \Delta)$, and $Y^\epsilon = \cup_{p\in Y} B(p, \epsilon)$ is an ϵ-neighborhood of Y.

We claim that the map

$$\omega \mapsto \nu_1^\omega(\cdot) = n_{f_\omega}([0, 1) \times \cdot) \in P$$

is measurable; in fact, it is continuous (A small change of the coefficients $\omega = (a_1, a_2, \dots)$ in l^2 sense will yield a small change in the counting measure of zeroes ν_1^ω in Lévy - Prokhorov sense).

Now consider the space $X = P^{\mathbb{N}}$ of sequences of measures with the product topology. Notice that the map $\Omega \to X$ defined by $\omega \mapsto \{\nu_N^\omega\}$ is measurable, as each coordinate is measurable; Moreover, its image lies almost surely in the (measurable subset) of weak converging sequences. The map $C \to P$ which takes a weak converging sequence $(\nu_N) \in C$ to its limit $\nu \in P$ is also measurable. We arrive at

Observation 1. *Any measurable set $M \in P$ induces a measurable set $\widetilde{M} = \{\omega : \nu^\omega \in M\} \subset C \subset \Omega$.*

Consider the event:

$$B = \{\omega \in \Omega : \text{ the limiting measure } \nu^\omega \text{ has at least one atom}\} \subset C \subset X$$

By the last observation, B is measurable w.r.t. $\mathcal{F}$.

We construct a measurable function $h : B \to (-\Delta, \Delta)^{\mathbb{N}}$ which maps $\omega \in B$ to a list of all atoms of the limiting measure ν^ω, as follows. Let $h_1 : B \to (-\Delta, \Delta)$ be the map which maps some $\omega \in B$ to the largest atom among those of ν^ω (if some (finite) number of atoms share this property, return the left-most one). Again by observation 1, h_1 is a measurable map. In a similar manner we construct h_2, which gives the second (left-most) largest atom; and so forth. Our list of atoms is simply $h = (h_1, h_2, \dots)$. We notice that

$$A = \cap_{i\in\mathbb{N}}\{\omega : (\nu_N^\omega\{h_i\omega\})_N \text{ is a convergent sequence of numbers}\} =: \cap_{i\in\mathbb{N}} E_i.$$

All that remains is to prove measurability of E_1.

Indeed, the map $H : X \times (-\Delta, \Delta) \to \{0, 1\}$ which matches $(\{\nu_N\}, a)$ to the indicator of the event $\{(\nu_N\{a\})_N$ is a convergent sequence$\}$ is measurable; by composition of measurable maps $1\!\!1_{E_1} = H((\nu_N^\omega), h_1\omega)$ is a measurable function, as anticipated. □

Appendix B. Exponential Decay of Some Tail Events - Offord-Type Estimates

In the course of the proof of the main theorem, we used several times exponential estimates on certain probabilities: Theorem 5, Propositions 5.2 and 5.1, and similar propositions for GAFs (which were not stated explicitly). Such estimates are sometimes referred to as "Offord-type large deviations estimates". We demonstrate the proof of Proposition 5.1, as the rest are similar. We adopt the proof of Sodin [20], presented also in [5, chapter 7].

We first present our key-lemma, which deals with 2-dimensional Gaussian random variables.

Lemma B.1. *If $\eta \sim \mathcal{N}_{\mathbb{R}^2}(\mu, \Sigma)$, and E is an event in the probability space with $\mathbb{P}(E) = p$, then:*

$$|\mathbb{E}(\chi_E \log|\eta|)| \le p\left[-(1+\frac{1}{2\lambda_1})\log p + \frac{p}{4\lambda_1} + \frac{1}{2}\log(\mathit{trace}\ \Sigma + |\mu|^2)\right],$$

where λ_1 is the biggest eigenvalue of Σ.

Proof. Upper bound: by Jensen's inequality,

$$\frac{1}{p}\mathbb{E}(\chi_E \log|\eta|^2) \le \log\left(\frac{\mathbb{E}(|\eta|^2\chi_E)}{p}\right) \le \log \mathbb{E}|\eta|^2 - \log p.$$

If $\eta = u + iv$, then

$$\mathbb{E}|\eta|^2 = \mathbb{E}u^2 + Ev^2 = \text{var } u + (\mathbb{E}u)^2 + \text{var v} + (\mathbb{E}v)^2 = \text{trace } \Sigma + |\mu|^2$$

Putting this in the previous equation, we get:

$$\mathbb{E}(\chi_E \log|\eta|) \le \frac{p}{2}[\log(\text{trace } \Sigma + |\mu|^2) - \log p]. \tag{21}$$

Lower bound:

$$\begin{aligned}\mathbb{E}(\chi_E \log|\eta|) &\geq -\mathbb{E}(\chi_E \log^-|\eta|)\\ &= -\mathbb{E}(\log^-|\eta|\chi_{E\cap\{|\eta|<p\}}) - \mathbb{E}(\log^-|\eta|\chi_{E\cap\{|\eta|>p\}})\end{aligned}$$

The second term may be bounded below by

$$-\mathbb{E}(\log^-|\eta|\chi_{E\cap\{|\eta|>p\}}) \geq p\log p \tag{22}$$

For the first term, we begin with some general manipulations:

$$\begin{aligned}-\mathbb{E}(\log^-|\eta|\chi_{E\cap\{|\eta|<p\}}) \geq -\mathbb{E}(\log^-|\eta|\chi_{\{|\eta|<p\}}) &= -\mathbb{E}\left[\chi_{|\eta|\leq p}\int_0^1 \chi_{s>|\eta|}\frac{ds}{s}\right]\\ &= -\int_0^1 \mathbb{P}[|\eta| < \min(p,s)]\frac{ds}{s}\end{aligned}$$

Let us therefore bound from above the probability $\mathbb{P}(|\eta| < R)$. Denote by λ_1, λ_2 the eigenvalues of Σ, where $\lambda_1 \geq \lambda_2 \geq 0$.

$$\begin{aligned}\mathbb{P}(|\eta| < R) &= \frac{1}{2\pi\sqrt{|\Sigma|}}\int_{|x|<R}\exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right)dm_2(x)\\ &\leq \frac{1}{2\pi\sqrt{|\Sigma|}}\int_{|x|<R}\exp\left(-\frac{1}{2}x^T\Sigma^{-1}x\right)dm_2(x)\\ &\leq \frac{1}{2\pi}\int_{|y|<R}\exp\left(-\frac{1}{2}(\lambda_1^{-1}y_1+\lambda_2^{-1}y_2)\right)dm_2(y)\\ &\leq \int_0^{R/\sqrt{\lambda_1}} e^{-\frac{1}{2}r^2}rdr = 1-e^{-\frac{R^2}{2\lambda_1}}\end{aligned}$$

We have used the changes of variables $y = Ux$ where U is the orthogonal matrix diagonalizing Σ, and later $y_i = \sqrt{\lambda_i}w_i$ for $i = 1, 2$.

Continuing, we have:

$$\begin{aligned}-\mathbb{E}(\log^-|\eta|\chi_{E\cap\{|\eta|<p\}}) &\geq -\int_0^p \frac{1-e^{-s^2/2\lambda_1}}{s}ds - \int_p^1 \frac{1-e^{-p^2/2\lambda_1}}{s}ds\\ &= \frac{1}{2}\int_0^{p^2/2\lambda_1} e^{-t}\log(2\lambda_1 t)dt\\ &\geq \frac{1}{2}\int_0^{p^2/2\lambda_1}\log(t)dt + \frac{p^2}{4\lambda_1}\log(2\lambda_1) = \frac{p^2}{2\lambda_1}(\log p - \frac{1}{2})\end{aligned}$$

Therefore our lower bound is

$$\mathbb{E}(\chi_E\log|\eta|) \geq \frac{p^2}{2\lambda_1}(\log p - \frac{1}{2}) - p\log p \geq -\frac{p^2}{4\lambda_1} + (1-\frac{1}{2\lambda_1})p\log p$$

Combining the two bounds we get the desired result. $\square$

We now turn to the proof of Proposition 5.1. Take $\phi(z) = \phi_T(z)$ a real C^2 function, whose support is $[-\frac{1}{2}, T+\frac{1}{2}]\times[a', b']$ with $-\Delta < a' < a < b < b' < \Delta$, and which takes the value 1 on $[0,T]\times[a,b)$. We may build such $\phi_T(z)$ that will obey also the bound $\|\Delta\phi\|_{L^1} < 10(T+b-a)$. Assume α and β are fixed for now, and fix also $s > 0$. We are interested in dominating the probability of the event $A_T = \{N_T > sT\}$. Write $p = p_T = \mathbb{P}(A_T)$.

We have

$$N_T < \frac{1}{2\pi} \int \Delta\phi_T(z) \log|f(z)| dm_2(z) \,,$$

and therefore,

$$\begin{aligned} sT \cdot p \le \mathbb{E}_g(\chi_{A_T} N_T) &\le \mathbb{E}_g \left(\chi_{A_T} \frac{1}{2\pi} \int \Delta\phi(z) \log|f(z)| dm_2(z) \right) \\ &= \frac{1}{2\pi} \int \Delta\phi \, \mathbb{E}_g \left(\chi_{A_T} \log|f(z)| \right) dm_2(z) \\ &\le \frac{1}{2\pi} \|\Delta\phi\|_{L^1} \sup_{z \in D} \mathbb{E}_g \left(\chi_{A_T} \log|f(z)| \right) \end{aligned}$$

Before we continue, let us justify the exchange of expectation and integral. Recall $f(z) = g(z) + \eta_{\alpha,\beta}(z)$; so in order to use Fubini's theorem we need

(23)

$$\int_D \mathbb{E}_g |\Delta\phi(z) \cdot \log|g(z) + \eta_{\alpha,\beta}(z)|| dm_2(z) = \int_D |\Delta\phi(z)| \, \mathbb{E}_g |\log|g(z) + \eta_{\alpha,\beta}(z)|| < \infty$$

For each $z \in D$, $f(z) = g(z) + \eta_{\alpha,\beta}(z)$ is a 2 dimensional Gaussian random variable, with mean $\mu(z) = \eta_{\alpha,\beta}(z)$, and the same covariance matrix $\Sigma(z)$ as the 2 dimensional Gaussian r.v. $g(z)$. By lemma 4.1, we see that both $\mu(z)$ and $\Sigma(z)$ depend continuously on the paremeter z. So, the function $\mathbb{E}_g |\log|g(z) + \eta_{\alpha,\beta}(z)|$ is bounded above for $z \in \text{support}(\phi)$, which ends this argument.

Notice further, that in our stationary case $\lambda_1(z)$, $\lambda_2(z)$, the eigenvalues of $\Sigma(z)$, depend on y only, where $z = x + iy$. Therefore they have lower and upper bounds on $\mathbb{R} \times [a', b']$. Notice that also $\mu(z)$, being a trigonometric function, has such bounds. By applying lemma B.1 with $\eta(z) = g(z) + \eta_{\alpha,\beta}(z)$, we get:

$$\sup_{z \in \mathbb{R} \times [a', b']} \mathbb{E}(\chi_{A_T} \log|g(z) + \zeta|) < p(c_1 - c_2 \log p).$$

where c_1, c_2 are positive constants (c_1 depending on α, β, the horizontal lines a, b, and the kernel of g). Putting all this together, we get:

$$sT \cdot p \le \frac{5}{\pi}(T + b - a) p (c_1 - c_2 \log p),$$

which leads to the exponential bound we strived for:

$$\exists c, C > 0 \text{ such that } p_T = \mathbb{P}_g(N_T > T\,s) \le C e^{-cs}, \ \forall T \ge 1\,.$$

□

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