VARIANCE OF THE NUMBER OF ZEROES OF SHIFT-INVARIANT GAUSSIAN ANALYTIC FUNCTIONS

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Abstract. Following Wiener, we consider the zeroes of Gaussian analytic functions in a strip in the complex plane, with translation-invariant distribution. We show that the variance of the number of zeroes in a long horizontal rectangle $[-T, T] \times [a, b]$ is asymptotically between cT and CT^2, with positive constants c and C. We also supply with conditions (in terms of the spectral measure) under which the variance grows asymptotically linearly with T, as a quadratic function of T, or has intermediate growth. The results are compared with known results for real stationary Gaussian processes and other models.

1. Introduction

The distribution of zeroes of random functions for various ensembles has been a subject of deep study, with motivations from analysis, mathematical physics, engineering, random matrix theory and probability. One prominent example is zeroes of real stationary Gaussian functions (SGFs), which were studied extensively in the mid-20th century (e.g. [5]) with the development of the theory of random signals. In the last two decades, zeroes of complex Gaussian Analytic Functions (GAFs) have drawn much attention, as they are one of the few natural examples of point processes with local repulsion and suppressed fluctuations - features used to model electrons (see [16]). As a first step towards analyzing these models, elegant formulas were given for the expected number of zeroes in a region (See [19, 30] for the real case, and [8] for the complex case). However, in order to understand the behavior of a random system, one must estimate the fluctuations around the mean. This is generally a hard task, which received much attention with only partial success. We will briefly survey some of the literature in Section 1.1.

In this paper we consider the zeroes of complex *Gaussian Analytic Functions*, whose distribution is invariant under real shifts (i.e., *stationary GAFs*). Being on the one hand complex analytic, while on the other hand stationary in the real sense, these random functions have common properties with both models mentioned above. Stationary GAFs were studied by Paley and Wiener in the last chapter of their celebrated treatise [28, Ch. X], where they

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prove a "law of large numbers" for the zeroes in large rectangles under certain spectral conditions. This result was extended in a preceding work [10], where unnecessary conditions were removed, and a formula for the expected number of zeroes was developed.

At present we study the variance $V(T)$ of the number of zeroes of a stationary GAF in $[-T,T]\times[a,b]$, giving a detailed description of its possible asymptotic growth. We identify which processes demonstrate quadratic growth of $V(T)$ (Theorem 1), and supply sufficient and necessary conditions for $V(T)$ to be asymptotically linear (Theorem 2 and 3 respectively). The case of linear variance is of particular interest, as this indicates that at large scale the zero process is "nearly independent". In Theorem 2 we also prove that $V(T)$ is always at least linear in T, which indicates that the zeroes are never "super-concentrated" around their mean. The last result is probably the most difficult and interesting part of our work.

Our methods require various tools from harmonic analysis, classical analysis and probability. In order to acquire our results we derive an asymptotic formula for $V(T)$ (Proposition 2). This formula consists of a series of non-negative terms involving the spectral measure, and is therefore relatively easy to analyze. Ideas from this paper were already used for studying the winding number of Gaussian functions from $\mathbb{R}$ to $\mathbb{C}$, in a recent work with Buckley [2].

1.1. **Related works.** In this section we discuss related models of zeros of random functions, with emphasis on the work done to determine fluctuations and large-scale behavior.

As mentioned above, the direct real analogue of our setting is the *zero-set of real Gaussian stationary processes*. Though expectation of the number of zeroes was known, the variance remained untackled for many years. Cramer and Leadbetter [5] obtained an asymptotic formula for the variance in 1967, but the rate of growth could not be inferred from it. In 1976 Cuzick [6] was able to show that, under some technical assumptions, if the growth of the variance is linear then the number of zeroes in $[-T,T]$ satisfies a Central Limit Theorem (CLT). It was only fifteen years later that Slud [32] obtained accessible conditions (in terms of the covariance kernel) for this assumption to hold. To do so Slud used primarily a sophisticated method for stochastic integration which he developed jointly with Chambers [4]. Unfortunately, this method involves many computations specifically tailored to tackle the real case, which do not generalize to the complex setting easily. Consequently, while some of our results for the complex case are very similar to those of Slud (see Remark 1.2 below), the methods are different and, very likely, may be more widely applied.

Another model which remained popular since the 1960's is *zeroes of Gaussian trigonometric polynomials* of large degree N in the interval $[0,2\pi]$. It was only in 2011 that Granville and Wigman [14], using similar methods to those of Cuzick and Slud, were able to show that the variance of this number is linear in N and a CLT holds. Their main observation was that, under a proper scaling limit, this model becomes stationary. On the sphere, a basic wave model is *random spherical harmonics*, also known as *arithmetic random waves*. In this model a Gaussian measure is endowed on the space of eigenfunctions belonging to the N-th eigenvalue of the Laplacian on the sphere (which equals $N(N+1)$), and the resulting nodal line is studied. Wigman [36] showed that, while the expected length of a nodal line is of order N, the variance is surprisingly of order $\log N$ due to unexpected cancellations.

One of the most well-studied models is *random polynomials with i.i.d. coefficients*, due to its simplicity and strong relations with mathematical physics and random matrix theory (see [12]). Kac [19] showed that for Gaussian coefficients the average number of real roots is asymptotically logarithmic in the degree. This stimulated many works to estimate the variance, which was finally retrieved by Maslova [23]. Several other notable advances about the distribution of zeroes were made quite recently, e.g. concerning universality (see Kabluchko-Zaporozhets [18], Nguyen-Nguyen-Vu [27], Söze [34, 35]), large deviations (see Zeitouni-Zelditch [37], Ghosh-Zeitouni [13]) and critical points (see Hanin [15]).

A natural extension of random polynomials are *Gaussian analytic functions* (GAFs, mentioned above). This has been a flourishing topic in recent years, as is accounted by the recent monograph [16] and the survey papers [24, 25]. For the *planar GAF*, a special GAF whose zeroes are invariant under all planar isometries, fluctuations of zeroes were studied by Sodin-Tsirelson [33] and Nazarov-Sodin [26]. They proved linear growth of the variance and a CLT for the zeroes in large balls (as the radius approaches infinity). Interestingly, the variance of smooth statistics of the zeroes *decays* as the radius grows, a difference which reflects high oscillations of random zeroes near the boundary. The results of the current paper establish this phenomena for a much larger family of GAFs, namely, that for "well-behaved" stationary GAFs the fluctuations of the zeroes are caused only by fluctuations near the boundary (see conditions in Theorem 2). However, as Theorems 1 and 3 show, for other stationary GAFS the fluctuations may be much larger, reflecting a strong dependency between distant zeroes.

Lastly, we mention that mean and fluctuations of zeroes of Gaussian functions on more general manifolds were studied in a series of papers by Bleher, Shiffman and Zelditch, see [3, 31] and references within.

1.2. **Definitions.** A *Gaussian Analytic Function (GAF)* in the strip $D = D_\Delta = \{z : |\text{Im} z| < \Delta\}$ is a random variable taking values in the space of analytic functions on D, so that for every $n \in \mathbb{N}$ and every $z_1, \dots, z_n \in D$ the vector $(f(z_1), \dots, f(z_n))$ has a mean zero complex Gaussian distribution.

A GAF in D is called *stationary*, if it is distribution-invariant with respect to all horizontal shifts, i.e., for any $t \in \mathbb{R}$, any $n \in \mathbb{N}$, and any $z_1, \dots, z_n \in D$, the random n -tuples

$$\big(f(z_1), \dots, f(z_n)\big) \qquad \text{and} \qquad \big(f(z_1 + t), \dots, f(z_n + t)\big)$$

have the same distribution.

For a stationary GAF in D_Δ, the covariance kernel

$$K(z, w) = \mathbb{E}\{f(z)\overline{f(w)}\}$$

may be written as

$$K(z,w) = r(z-\overline{w}), \quad z,w\in D_\Delta.$$ [1]

For $t\in\mathbb{R}$, the function $r(t)$ is positive-definite and continuous, and so it is the Fourier transform of some positive measure ρ on the real line:

$$r(t) = \mathcal{F}[\rho](t) = \int_{\mathbb{R}} e^{-2\pi i t\lambda} d\rho(\lambda).$$

Moreover, since $r(t)$ has an analytic continuation to the strip $D_{2\Delta}$, ρ must have a finite exponential moment:

$$\text{for each } |\Delta_1|<\Delta,\ \int_{-\infty}^{\infty} e^{2\pi\cdot 2\Delta_1|\lambda|} d\rho(\lambda) < \infty\,. \tag{1}$$

The measure ρ is called the *spectral measure* of f. A stationary GAF is *degenerate* if its spectral measure consists of exactly one atom.

For a holomorphic function f in a domain D, we denote by Z_f the zero-set of f (counted with multiplicities), and by n_f the *zero-counting measure*, i.e.,

$$\forall\varphi\in C_0(D): \qquad \int_D \varphi(z)dn_f(z) = \sum_{z\in Z_f}\varphi(z),$$

where $C_0(D)$ is the set of compactly supported continuous functions on D. We use the abbreviation $n_f(B) = \int_B dn_f(z)$ for the number of zeroes in a Borel subset $B\subset D$.

1.3. **Results.** First, we present a previous result which will serve as our starting point. This result can be viewed as a "law of large numbers" for the zeroes of stationary functions.

Theorem A. [10, Theorem 1] *Let f be a stationary non-degenerate GAF in the strip D_Δ, where $0<\Delta\le\infty$. Let $\nu_{f,T}$ be the non-negative locally-finite random measure on $(-\Delta,\Delta)$ defined by*

$$\nu_{f,T}(Y) = \frac{1}{2T} n_f([-T,T]\times Y),\ \ Y\subset(-\Delta,\Delta)\ \textit{measurable.}$$

Then:

(i) *Almost surely, the measures $\nu_{f,T}$ converge weakly and on every interval to a measure ν_f when $T\to\infty$.*

(ii) *The measure ν_f is not random (i.e.* $\operatorname{var}\nu_f=0$*) if and only if the spectral measure ρ_f has no atoms.*

(iii) *If the measure ν_f is not random, then $\nu_f(Y)=\mathbb{E}n_f([0,1]\times Y)$ and it has density:*

$$L(y) = \frac{d}{dy}\left(\frac{\int_{-\infty}^{\infty}\lambda e^{4\pi y\lambda}d\rho(\lambda)}{\int_{-\infty}^{\infty}e^{4\pi y\lambda}d\rho(\lambda)}\right) = \frac{1}{4\pi}\frac{d^2}{dy^2}\log\left(r(2iy)\right).$$

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[1] To see this, first observe that for any $x,y\in\mathbb{R}$ the covariance $\mathbb{E}(f(x)\overline{f(y)})$ depends only on $(x-y)$, so that $K(x,y)=r(x-y)$ for some real-analytic function $r:\mathbb{R}\to\mathbb{C}$. Now note that the functions $K(z,w)$ and $r(z-\bar{w})$ are both analytic in z, anti-analytic in w, and coincide for $z,w\in\mathbb{R}$, thus must be equal on the entire strip $D_{2\Delta}$.

In the above and in what follows, the term “density” means the Radon-Nikodym derivative w.r.t. the Lebesgue measure on $\mathbb{R}$.

A natural question is, how big are the fluctuations of the number of zeroes in a long rectangle? More rigorously, define

$$R_T^{a,b} = [-T,T) \times [a,b], \;\; V_f^{a,b}(T) = \text{var}\left[n_f(R_T^{a,b})\right],$$

where for a random variable X the variance is defined by

$$\text{var}\,(X) = \mathbb{E}\,(X - \mathbb{E}X)^2 .$$

We are interested in the asymptotic behavior of $V_f^{a,b}(T)$ as T approaches infinity. The next theorems show that $V_f^{a,b}(T)$ is asymptotically bounded between cT and CT^2 for some $c, C > 0$, and give conditions under which each of the bounds is achieved. We begin by stating the upper bound result, a relatively easy consequence of Theorem A.

Theorem 1. *Let f be a non-degenerate stationary GAF in a strip D_Δ. Then for all $-\Delta < a < b < \Delta$ the limit*

$$L_2 = L_2(a,b) := \lim_{T\to\infty} \frac{V_f^{a,b}(T)}{T^2} \in [0,\infty)$$

exists. This limit is positive if and only if the spectral measure of f has a non-zero discrete component.

The lower bound result, which is our main result, is stated in the following theorem.

Theorem 2. *Let f be a non-degenerate stationary GAF in a strip D_Δ. Then for all $-\Delta < a < b < \Delta$ the limit*

$$L_1 = L_1(a,b) := \lim_{T\to\infty} \frac{V_f^{a,b}(T)}{T} \in (0,\infty]$$

exists. Moreover, the limit $L_1(a,b)$ is finite if ρ is absolutely continuous with density $d\rho(\lambda) = p(\lambda)d\lambda$, such that

$$(1+\lambda^2)e^{2\pi\cdot 2y\lambda}p(\lambda) \in L^2(\mathbb{R}), \;\; \textit{for } y \in \{a,b\}. \tag{2}$$

Several remarks are due before continuing.

Remark 1.1. Another form of condition (2) is the following: For $y \in \{2a, 2b\}$,

$$\int_{\mathbb{R}} |r(x+iy)|^2 dx, \;\; \int_{\mathbb{R}} |r''(x+iy)|^2 dx < \infty.$$

This implies also that $\int_{\mathbb{R}} |r'(x+iy)|^2 dx < \infty$. Moreover, since the set $\{c : \; e^{2\pi\cdot c\lambda}p(\lambda) \in L^2(\mathbb{R})\}$ is convex, it implies the same condition for all $y \in [2a, 2b]$.

Remark 1.2. It is interesting to note that condition (2) is precisely the condition that Slud gave in [32] for linear variance in the case of real (non-analytic) stationary Gaussian processes (with $a = b = 0$). Nonetheless, no direct implication between the results is known and the methods to obtain them are quite remote.

As for the first part of Theorem 2, we would expect an analogue to hold for real stationary Gaussian functions; that is, that under mild conditions, the variance of the number of zeroes in $[-T, T]$ is always at least linear in T. To the best of our knowledge, this is yet unknown.

Remark 1.3. In case condition (2) holds, we shall give an expression for the limit L_1 as a convergent series of the form:

$$\lim_{T\to\infty} \frac{V_f^{a,b}(T)}{2T} = \sum_{k\geq 1} \int_{\mathbb{R}} \left(p^{*k}(\lambda)\right)^2 w_k^{a,b}(\lambda) d\lambda.$$

Here p^{*k} denotes the k-fold convolution of p, and $w_k^{a,b}(\lambda)$ is a positive function which can be computed explicitly in terms of a, b, k and $r(2ia)$, $r(2ib)$ (and no other reliance on p).

The next theorem deals with conditions under which $L_1(a, b)$ is infinite, i.e. the variance is super-linear.

Theorem 3. *Let f be a non-degenerate stationary GAF in a strip D_Δ.*

(i) *Suppose $J \subset (-\Delta, \Delta)$ is a closed interval such that for every $y \in J$, the function $\lambda \mapsto (1 + \lambda^2)e^{2\pi\cdot 2y\lambda}p(\lambda)$ does not belong to $L^2(\mathbb{R})$. Then for every $\alpha \in J$ the set $\{\beta \in J : \ L_1(\alpha, \beta) < \infty\}$ is finite.*

(ii) *The limit $L_1(a, b)$ is infinite for particular a, b if either ρ does not have density, or, if it has density p and for any two points $\lambda_1, \lambda_2 \in \mathbb{R}$ there exists intervals I_1, I_2 such that I_j contains λ_j $(j = 1, 2)$ and*

$$(1 + \lambda)e^{2\pi\cdot 2y\lambda}p(\lambda) \notin L^2(\mathbb{R} \setminus (I_1 \cup I_2)), \tag{3}$$

for at least one of the values $y = a$ or $y = b$.

Remark 1.4. There is a gap between the conditions given for linear variance (in Theorem 2) and those for super-linear variance (in Theorem 3). For instance, the theorems do not decide about all the suitable pairs (a, b) in case the spectral measure has density $\frac{1}{\sqrt{|\lambda|}} 1\!\!1_{[-1,1]}(\lambda)$. On the other hand, we are ensured to have super-linear variance in case ρ has a singular part. If ρ has density $p \in L^1(\mathbb{R})$ which is bounded on any compact set, then $(1 + \lambda^2)p(\lambda) \in L^2(\mathbb{R})$ implies asymptotically linear variance, and $(1+\lambda)p(\lambda) \notin L^2(\mathbb{R})$ implies asymptotically super-linear variance.

Remark 1.5. Minor changes to the developments in this paper may be made in order to prove analogous results regarding the increment of the argument of a stationary GAF f along a horizontal line. Namely, let $V^{a,a}(T)$ denote the variance of the increment of the argument of f along the line $[0, T] \times \{a\}$ (for some $-\Delta < a < \Delta$). Then:

- the limit $L_2(a) = \lim_{T\to\infty} \frac{V^{a,a}(T)}{T^2}$ exists, belongs to $[0, \infty)$, and is positive if and only if the spectral measure contains an atom.
- the limit $L_1(a) = \lim_{T\to\infty} \frac{V^{a,a}(T)}{T}$ exists, belongs to $(0, \infty]$, and is finite if ρ has density $p(\lambda)$ such that $(1 + \lambda^2)e^{2\pi\cdot 2a\lambda}p(\lambda) \in L^2(\mathbb{R})$. Moreover, $L_1(a)$ is infinite if for any

$\lambda_0 \in \mathbb{R}$ there is an interval I containing λ_0 such that the measure $(1+\lambda)e^{2\pi\cdot 2a\lambda}d\rho(\lambda)$ restricted to $\mathbb{R}\setminus I$ is not in $L^2(\mathbb{R})$.

In our recent work with Buckley [2] we extend these statements to hold for a differentiable (not necessarily analytic) Gaussian process from $\mathbb{R}$ to $\mathbb{C}$. Also notice that the first item is essentially proved in this paper (Claim 3 below).

The rest of the paper is organized as follows: Theorem 1 concerning quadratic growth of variance is proved in Section 2, and is mainly a consequence of Theorem A. In Section 3 we develop an asymptotic formula for $V_f^{a,b}(T)/T$ (Proposition 2 below), which will be used to prove Theorems 2 and 3 in Sections 4 and 5 respectively. Appendices A, B and C include proofs of some technical lemmas.

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2. Theorem 1: Quadratic Variance

Recall the notation $R_T = R_T^{a,b} = [-T,T)\times[a,b]$. From Theorem A we know that

$$\lim_{T\to\infty}\frac{n_f(R_T)}{T} = Z,$$

where Z is some random variable and the limit is in the almost sure sense. Moreover, $\operatorname{var} Z > 0$ if and only if the spectral measure of f contains an atom. The theorem now follows

$$\lim_{T\to\infty}\frac{\operatorname{var}\,(n_f(R_T))}{T^2} = \lim_{T\to\infty}\operatorname{var}\left(\frac{n_f(R_T)}{T}\right) = \operatorname{var}\left(\lim_{T\to\infty}\frac{n_f(R_T)}{T}\right) = \operatorname{var} Z.$$

The exchange of limit and variance in the middle equality is justified by the following proposition, which is proved in Appendix A.

Proposition 1. *Let f be a stationary GAF in D_Δ, and $-\Delta < a < b < \Delta$. Denote $X_T = n_f(R_T^{a,b})/T$. Then there exist $C, c > 0$ (depending on ρ, a and b) so that:*

$$\sup_{T\geq 1}\mathbb{P}(X_T > s) < Ce^{-cs}.$$

3. An Asymptotic Formula for the Variance

This section is devoted to the derivation of a formula for the variance $V_f^{a,b}(T) = \operatorname{var} n_f([-T,T]\times[a,b])$ where T is large. We prove the following:

Proposition 2. *Let f be a stationary GAF in D_Δ with spectral measure ρ. Suppose ρ has no discrete component. Then for any $-\Delta < a < b < \Delta$, and any $T\in\mathbb{R}$, the series*

$$v^{a,b}(T) = \frac{1}{4\pi^2}\sum_{k\geq 1}\frac{1}{k^2}\int_{\mathbb{R}}\int_{\mathbb{R}} T\operatorname{sinc}^2\left(2\pi T(\lambda-\tau)\right)h_k^{a,b}(\lambda+\tau)d\rho^{*k}(\lambda)d\rho^{*k}(\tau)$$

converges, and

$$\lim_{T\to\infty}\left(\frac{V^{a,b}(T)}{2T}-v^{a,b}(T)\right)=0.$$

Here ρ^{*k} *is the* k*-fold convolution of* ρ*,* $\operatorname{sinc}(x)=\frac{\sin x}{x}$*, and*

$$h_k^{a,b}(\lambda)=\left(l_k^a(\lambda)e^{2\pi a\lambda}-l_k^b(\lambda)e^{2\pi b\lambda}\right)^2,$$

where for $y\in(-\Delta,\Delta), k\in\mathbb{N}$ *we write*

$$l_k^y(\lambda)=e^{-2\pi\lambda y}\frac{\partial}{\partial y}\left(\frac{e^{2\pi\lambda y}}{r^k(2iy)}\right)=\frac{\partial}{\partial y}\left(\frac{1}{r^k(2iy)}\right)+\frac{2\pi}{r^k(2iy)}\lambda.$$

We begin with some definitions and facts that will be needed along the proof.

3.1. Preliminaries.

3.1.1. *Tools from Harmonic Analysis.*

In this section we discuss some operations on measures and their relation to the Fourier transform.

Denote by $\mathcal{M}(\mathbb{R})$ the space of all finite measures on $\mathbb{R}$, similarly $\mathcal{M}^+(\mathbb{R})$ denotes all finite non-negative measures on $\mathbb{R}$. For two measure $\mu,\nu\in\mathcal{M}(\mathbb{R})$ the *convolution* $\mu*\nu\in\mathcal{M}(\mathbb{R})$ is a measure defined by:

$$\forall\varphi\in C_0(\mathbb{R}):\ (\mu*\nu)(\varphi)=\iint\varphi(\lambda+\tau)d\mu(\lambda)d\nu(\tau).$$

When both measures have density, this definition agrees with the standard convolution of functions. We write μ^{*k} for the iterated convolution of μ with itself k times.

for a measure $\mu\in\mathcal{M}^+(\mathbb{R})$ having exponential moments up to 2Δ (i.e., obeying condition (1)), and a number $y\in(-2\Delta,2\Delta)$, we define the *exponentially rescaled measure* $\mu_y\in\mathcal{M}^+(\mathbb{R})$ by

$$\forall\varphi\in C_0(\mathbb{R}):\ \mu_y(\varphi)=\mu(e^{2\pi y\lambda}\varphi(\lambda))=\int_{\mathbb{R}}e^{2\pi y\lambda}\varphi(\lambda)d\mu(\lambda)$$

Observation. *For any* $\mu,\nu\in\mathcal{M}(\mathbb{R})$ *and any* $|y|<2\Delta$*,*

$$(\mu*\nu)_y=\mu_y*\nu_y.$$

Proof. for any test function $\varphi\in C_0(\mathbb{R})$ we have:

$$\begin{aligned}\int\varphi\ d(\mu_y*\nu_y)&=\iint\varphi(\lambda+\tau)\ d\mu_y(\lambda)d\nu_y(\tau)\\&=\iint\varphi(\lambda+\tau)e^{2\pi y(\lambda+\tau)}\ d\mu(\lambda)d\nu(\tau)=\int\varphi\ d(\mu*\nu)_y\end{aligned}$$

□

Corollary. *If* $\rho\in\mathcal{M}^+(\mathbb{R})$ *is such that* (1)*, then for any* $|y|<2\Delta$ *and* $k\in\mathbb{N}$ *we have* $(\rho_y)^{*k}=(\rho^{*k})_y$*, so there will be no ambiguity in the notation* ρ_y^{*k}*.*

Next, we define for $\mu \in \mathcal{M}(\mathbb{R})$ the *flipped measure* $\text{flip}\{\mu\} \in \mathcal{M}(\mathbb{R})$ by:

$$\text{flip}\{\mu\}(I) = \mu(-I) \text{ for any interval } I \subset \mathbb{R},$$

and the *cross-correlation* of measures $\mu, \nu \in \mathcal{M}(\mathbb{R})$ by:

$$\mu \star \nu := \mu * \text{flip}\{\nu\}.$$

An alternative definition, via actions on test-functions, would be:

$$\forall \varphi \in C_0(\mathbb{R}) : \ (\mu \star \nu)(\varphi) = \iint \varphi(\lambda - \tau) d\mu(\lambda) d\nu(\tau).$$

Notice that the cross-correlation operator is bi-linear, but not commutative. In all following expressions, convolution precedes cross-correlation.

We are now ready to prove a lemma relating these notions to the Fourier transform.

Lemma 3.1. *Suppose $\rho \in \mathcal{M}^+(\mathbb{R})$ obeys (1), and $r = \mathcal{F}[\rho]$. Then, for any $|y| < 2\Delta$, $x \in \mathbb{R}$ and $k \in \mathbb{N}$:*

$$|r(x+iy)|^{2k} = \mathcal{F}[\rho_y^{*k} \star \rho_y^{*k}](x).$$

This measure acts on a test-function $\varphi \in C_0(\mathbb{R})$ in the following way:

$$(\rho_y^{*k} \star \rho_y^{*k})(\varphi) = \iint \varphi(\lambda - \tau) e^{2\pi y(\lambda+\tau)} d\rho^{*k}(\lambda) d\rho^{*k}(\tau).$$

Proof. Fix $k \in \mathbb{N}$. Since $r(z) = \mathcal{F}[\rho](z)$, by a standard property of Fourier transform one has $r^k(z) = \mathcal{F}[\rho^{*k}](z)$. Writing $z = x + iy$, this reads

$$r^k(x+iy) = \int_{\mathbb{R}} e^{-2\pi i x\lambda} e^{2\pi y\lambda} d\rho^{*k}(\lambda).$$

This implies:

- $r^k(x+iy) = \mathcal{F}[\rho_y^{*k}](x)$
- $\overline{r^k(x+iy)} = \mathcal{F}[\rho_y^{*k}](-x) = \mathcal{F}[\text{flip}\{\rho_y^{*k}\}](x)$,

which leads to

$$|r(x+iy)|^{2k} = \mathcal{F}[\rho_y^{*k} * \text{flip}\{\rho_y^{*k}\}](x) = \mathcal{F}[\rho_y^{*k} \star \rho_y^{*k}](x).$$

□

Also useful to us will be the following special case of Parseval's identity for measures (see Katznelson [20, VI.2.2]):

Lemma 3.2. *For any finite measure γ on $\mathbb{R}$,*

$$\int_{-2T}^{2T} \left(1 - \frac{|x|}{2T}\right) \mathcal{F}[\gamma](x) dx = \int_{\mathbb{R}} 2T \text{sinc}^2(2\pi T\xi) d\gamma(\xi).$$

where $\text{sinc}(\xi) = \frac{\sin \xi}{\xi}$ *and $\mathcal{F}[\gamma]$ is the Fourier transform of γ.*

3.1.2. *Properties of a "normalized" covariance function.* Here we summarize some properties of a normalized version of the covariance function, namely $\frac{|r(x+ia+ib)|^2}{r(2ia)\,r(2ib)}$, which will be used later in our proofs. In the following, when we do not specify the variables we mean the statements holds on all the domain of definition. We use the subscript notation for partial derivatives (such as q_a for $\frac{\partial}{\partial a}q$).

Lemma 3.3. *The function*

$$q(x,a,b) := \frac{|r(x+ia+ib)|^2}{r(2ia)\,r(2ib)},$$

is well-defined, infinitely differentiable on $\mathbb{R}\times(-\Delta,\Delta)^2$, and satisfies the following properties:

1. $q(x,y_1,y_2)\in[0,1]$.
   $q(x,y_1,y_2)=1$ *if and only if* $(x=0$ *and* $y_1=y_2)$.
2. $\sup_{x\in\mathbb{R}} q(x,y_1,y_2)<1$ *for any* $y_1\neq y_2$ *in* $(-\Delta,\Delta)$.
3. *For fixed y_1 and y_2 let $g_{y_1,y_2}(x)$ be one of the functions q, q_a, q_b, q_{ab} evaluated on the line $\{(x,y_1,y_2): x\in\mathbb{R}\}$. Then $g_{y_1,y_2}\in L^\infty(\mathbb{R})$. If condition (2) holds, then for any $y_1,y_2\in[a,b]$ we have also $g_{y_1,y_2}\in L^1(\mathbb{R})\cap C_0(\mathbb{R})$ (i.e., is integrable and tends to zero as $x\to\pm\infty$).*
4. $q_a(0,t,t)=0$, *for any* $t\in(-\Delta,\Delta)$.

Proof. Since $r(2iy)>0$ for all $y\in\mathbb{R}$, the function q is indeed well-defined; differentiability follows from that of $r(z)$.

For item 1, notice that

$$q(x,a,b) = \frac{\left(\int e^{2\pi(a+b)\lambda}e^{-2\pi ix\lambda}d\rho(\lambda)\right)^2}{\int e^{2\pi\cdot 2a\lambda}d\rho(\lambda)\int e^{2\pi\cdot 2a\lambda}d\rho(\lambda)}$$

and so, by Cauchy-Schwarz, is in $[0,1]$. Equality $q(x,a,b)=1$ holds only if the function $\lambda\mapsto e^{2\pi\cdot a\lambda}e^{-2\pi ix\lambda}$ is a constant times the function $\lambda\mapsto e^{2\pi\cdot b\lambda}$, ρ-a.e., but, if ρ is non-atomic, this is impossible unless $x=0$ and $a=b$.

Further, we notice that

$$|r(x+ia+ib)| = \left|\int e^{2\pi(a+b)\lambda}e^{-2\pi ix\lambda}d\rho(\lambda)\right| \le \int e^{2\pi(a+b)\lambda}d\rho(\lambda) = r(ia+ib),$$

so that $q(x,a,b)\le q(0,a,b)<1$ (the right-most inequality is by item 1). Taking the supremum yields item 2.

For item 3, notice any one of the functions q,q_a,q_b,q_{ab} is the sum of terms of the form

$$C(a,b)\; r^{(j)}(x+ia+ib)\; r^{(m)}(-x+ia+ib), \tag{4}$$

where $0\le j,m\le 2$ are integers. It is enough therefore to explain why $r^{(j)}(x+ia+ib)$ is bounded and approaches zero as $x\to\pm\infty$, for any integer $0\le j\le 2$. Recall that

$$r^{(j)}(x+iy) = c_j\mathcal{F}[\lambda^j e^{2\pi y\lambda}\,d\rho(\lambda)](x),$$

where c_j is some constant. As a function of x, this is a Fourier transform of a non-atomic measure, therefore has the desired properties.

If condition (2) holds, then $d\rho(\lambda) = p(\lambda)d\lambda$, and the function $\lambda \mapsto \lambda^j e^{2\pi(y_1+y_2)\lambda} p(\lambda)$ is in $L^2(\mathbb{R})$. Then, its Fourier transform $r^{(j)}(x + iy_1 + iy_2)$ is also in $L^2(\mathbb{R})$, and each term of the form (4) is in $L^1(\mathbb{R})$, as anticipated.

For item 4, notice that for all $x \in \mathbb{R}$ and all $a, b \in (-\Delta, \Delta)$ we have the symmetry $q(x, a, b) = q(x, b, a)$, and therefore for all $t \in \mathbb{R}$: $q_a(x, t, t) = q_b(x, t, t)$. On the other hand, for all $t \in (-\Delta, \Delta)$ it holds that $q(0, t, t) = 1$, so taking derivative by t we get $q_a(0, t, t) \cdot 1 + q_b(0, t, t) \cdot 1 = 0$. This proves the result. $\square$

We are now ready to begin the proof of Proposition 2.

3.2. **Integrals on significant edges.**

The boundary of the rectangle $R_T = [-T, T] \times [a, b]$ is composed of four segments $\partial R_T = \bigcup_{1\le i\le 4} I_j$ with induced orientation from the counter-clockwise orientation of ∂R_T, where $I_1 = [-T, T] \times \{a\}$ and $I_3 = [T, -T] \times \{b\}$. By the argument principle,

$$n_f(R_T) = \sum_{1\le i\le 4} \frac{1}{2\pi} \triangle_i^T \arg f,$$

where $\triangle_i^T \arg f$ is the increment of the argument of f along the segment I_i (a.s. f has no zeroes on the boundary of the rectangle R_T [2]).

Then, by the argument principle,

$$V_f^{a,b}(T) = \text{var}\,[n_f(R_T)] = \frac{1}{4\pi^2} \sum_{1\le i,j\le 4} \text{cov}\,\left(\triangle_i^T \arg f,\ \triangle_j^T \arg f\right), \tag{5}$$

where

$$\text{cov}\,(X, Y) = \mathbb{E}[XY] - \mathbb{E}X \cdot \mathbb{E}Y.$$

Our first claim is that asymptotically when T is large, the terms involving the (short) vertical segments are negligible in this sum.

Claim 1. *As $T \to \infty$, one has:*

$$V_f^{a,b}(T) = \frac{1}{4\pi^2} \sum_{i,j\in\{1,3\}} \text{cov}\,\left(\triangle_i^T \arg f,\ \triangle_j^T \arg f\right) + O\left(1 + \sqrt{\text{var}\,(\triangle_1^T \arg f)} + \sqrt{\text{var}\,(\triangle_3^T \arg f)}\right).$$

---

[2] To see this, first notice that the distribution of $n_f(I_j)$ for $j = 2, 4$ (the number of zeroes in a "short" vertical segments) does not depend on T. If it were not a.s. zero, then $\mathbb{E}n_f(I_2) > 0$. Now for any finite set of points $\{t_j\}_{j=1}^N \subset [0, 1]$, we have $\mathbb{E}n_f([0, 1] \times [a, b]) \ge \sum_{j=1}^N \mathbb{E}n_f(\{t_j\} \times [a, b]) = N\mathbb{E}n_f(I_2)$, yielding $\mathbb{E}n_f([0, 1] \times [a, b]) = \infty$ - which is false. For $j = 1, 3$, recall that since there are no atoms in the spectral measure, f is ergodic with respect to horizontal shifts (this is Fomin-Grenander-Maruyama Theorem, see explanation and references within [10]). This implies that each horizontal line (such as $L_a = \mathbb{R} \times \{a\}$) either a.s. contains a zero or a.s. contains no zeroes. If the former holds, then also $\mathbb{E}n_f([0, 1] \times \{a\}) > 0$, and the measure ν_f from Theorem A has an atom at a - contradiction to part (iii) of that Theorem.

Proof. We demonstrate how to bound one of the terms in (5) involving a "short" vertical segment (corresponding, say, to $i=2$). By stationarity, $\mathrm{var}\,(\triangle_2^T \arg f) = \mathrm{var}\,(\triangle_2^0 \arg f) =: c^2$. Applying the Cauchy-Schwarz inequality, we have:

$$\begin{aligned}\mathrm{cov}\,\left(\triangle_1^T \arg f,\ \triangle_2^T \arg f\right) &\le \sqrt{\mathrm{var}\,(\triangle_1^T \arg f)}\sqrt{\mathrm{var}\,(\triangle_2^T \arg f)}\\ &= c\cdot\sqrt{\mathrm{var}\,(\triangle_1^T \arg f)}.\end{aligned}$$

□

Let us now give an alternative formulation of Claim 1. Using Cauchy-Riemann equations, we have:

$$\begin{aligned}\triangle_1^T \arg f &= \int_{-T}^{T}\left(\frac{\partial}{\partial x}\arg f(x+ia)\right)dx = -\int_{-T}^{T}\frac{\partial}{\partial a}\log|f(x+ia)|\,dx =: -X^a(T)\\ \triangle_3^T \arg f &= -\int_{-T}^{T}\left(\frac{\partial}{\partial x}\arg f(x+ib)\right)dx = \int_{-T}^{T}\frac{\partial}{\partial b}\log|f(x+ib)|\,dx = X^b(T)\end{aligned}$$

Denoting $C^{a,b}(T) = \mathrm{cov}\,(X^a(T), X^b(T))$ we may rewrite Claim 1 as

$$V_f^{a,b}(T) = \frac{1}{4\pi^2}\left(C^{a,a}(T) - 2C^{a,b}(T) + C^{b,b}(T)\right) + O\left(1 + \sqrt{C^{a,a}(T)} + \sqrt{C^{b,b}(T)}\right),$$

or alternatively:

Claim 1a. *As $T\to\infty$, we have:*

$$\frac{V^{a,b}(T)}{2T} = \frac{C^{a,a}(T) - 2C^{a,b}(T) + C^{b,b}(T)}{4\pi^2\cdot 2T} + O\left(\frac{1+\sqrt{C^{a,a}(T)}+\sqrt{C^{b,b}(T)}}{T}\right).$$

where

$$\begin{aligned}C^{a,b}(T) =& \mathbb{E}\left\{\int_{-T}^{T}dt\int_{-T}^{T}ds\,\left(\frac{\partial}{\partial a}\log|f(t+ia)|\,\frac{\partial}{\partial b}\log|f(s+ib)|\right)\right\}\\ &- \mathbb{E}\left\{\int_{-T}^{T}\frac{\partial}{\partial a}\log|f(t+ia)|\,dt\right\}\mathbb{E}\left\{\int_{-T}^{T}\frac{\partial}{\partial b}\log|f(s+ib)|\,ds\right\}\end{aligned} \tag{6}$$

3.3. **Changing order of operations.** Our goal now is to prove the following:

Claim 2.

$$C^{a,b}(T) = \int_{-T}^{T}\int_{-T}^{T}\frac{\partial^2}{\partial a\,\partial b}\mathrm{cov}\,\left(\log|f(t+ia)|,\ \log|f(s+ib)|\right)dt\ ds. \tag{7}$$

The meaning of this formula for $C^{a,a}(T)$ is as follows: on the RHS, first take the mixed partial derivative (as if $a\neq b$), then substitute $b=a$ and integrate by t and s.

The proof is an application of the following two lemmas, which are proved in Appendix B. In both, we assume f is a stationary GAF in D_Δ, and $a,b\in(-\Delta,\Delta)$ (not necessarily different).

Lemma 3.4. *For any $T>0$ the following integrals are finite:*

(A-I) $$\int_{-T}^{T} \mathbb{E}\left|\frac{f'(t+ia)}{f(t+ia)}\right| dt < \infty.$$

(A-II) $$\int_{-T}^{T}\int_{-T}^{T} \mathbb{E}\left|\frac{f'(t+ia)}{f(t+ia)}\frac{f'(s+ib)}{f(s+ib)}\right| dt\ ds < \infty.$$

Lemma 3.5. *For almost all $t, s \in [-T, T]^2$,*

(B-I) $$\mathbb{E}\left[\frac{\partial}{\partial a}\log|f(t+ia)|\right] = \frac{\partial}{\partial a}\mathbb{E}\Big[\log|f(t+ia)|\Big]$$

(B-II) $$\mathbb{E}\left[\frac{\partial^2}{\partial a\ \partial b}\log|f(t+ia)|\log|f(s+ib)|\right] = \frac{\partial^2}{\partial a\ \partial b}\mathbb{E}\Big[\log|f(t+ia)|\log|f(s+ib)|\Big].$$

Proof of Claim 2. Recall the definition of $C^{a,b}(T)$ in (6), and notice that

(8) $$\left|\frac{\partial}{\partial a}\log|f(x+ia)|\right| \le \left|\frac{f'(x+ia)}{f(x+ia)}\right|.$$

Step 1: Exchange integrals and expectation: By (8) and Lemma 3.4, we may apply Fubini's theorem to get:

$$C^{a,b}(T) = \int_{-T}^{T}\int_{-T}^{T} \Big[\mathbb{E}\frac{\partial^2}{\partial a\ \partial b}\{\log|f(t+ia)|\ \log|f(s+ib)|\} - \mathbb{E}\frac{\partial}{\partial a}\log|f(t+ia)|\,\mathbb{E}\frac{\partial}{\partial b}\log|f(s+ib)|\ \Big] dt\ ds.$$

Step 2: Exchange the order of expectation and derivative inside the integral by t and s. This is justified directly by Lemma 3.5. We arrive at the desired form. ☐

3.4. **The error term.** Next, we show that the error term in Claim 1a approaches zero as T tends to infinity.

Claim 3. *If ρ contains no atoms, then for any $a \in (-\Delta, \Delta)$:*

$$\lim_{T\to\infty}\frac{C^{a,a}(T)}{T^2} = 0.$$

Proof. Since ρ has no atoms, f is an ergodic process (this is the classical Fomin-Grenander-Maruyama theorem, see [10, Theorem 4] and references therein). Thus, by the ergodic theorem,

(9) $$\lim_{T\to\infty}\frac{1}{T}X^a(T) = \mathbb{E}X^a(1), \quad \text{almost surely and in } L^1.$$

Recall $X^a(T)$ has finite second moment (this is precisely relation (A-II)). Therefore, the convergence in (9) is also in the L^2 sense (this is an easy adaptation of the proof for L^1 convergence, see [7, Exercise 7.2.1]). We conclude that:

$$\lim_{T\to\infty}\frac{1}{T^2}\mathrm{var}\left(X^a(T)\right) = \lim_{T\to\infty}\frac{1}{T^2}C^{a,a}(T) = 0.$$

☐

Claims 1a , 2 and 3 give together:

Corollary.

$$\frac{V^{a,b}(T)}{2T} = \frac{C^{a,a}(T) - 2C^{a,b}(T) + C^{b,b}(T)}{4\pi^2 \cdot 2T} + o(1), \quad T \to \infty, \tag{10}$$

where $C^{a,b}(T)$ *is given by* (7).

3.5. **A formula in terms of the covariance function.** Our goal now is to replace (7) by a simpler formula, using the covariance function. This is done in the next claim.

Claim 4.

$$\frac{C^{a,b}(T)}{2T} = \frac{1}{2}\int_{-2T}^{2T}\left(1 - \frac{|x|}{2T}\right)\sum_{k\geq 1}\frac{\partial^2}{\partial a\ \partial b}\frac{q(x,a,b)^k}{k^2}\ dx, \tag{11}$$

where

$$q(x,a,b) := \frac{|r(x+ia+ib)|^2}{r(2ia)\,r(2ib)}. \tag{12}$$

For the proof, we will need the following Lemma, which is a direct consequence of a lemma by Nazarov and Sodin [26, Lemma 2.2] (see also [16, Lemma 3.5.2]).

Lemma 3.6. *If* ξ *and* η *are centered jointly complex Gaussian random variables, then:*

$$\operatorname{cov}\left(\log|\xi|, \log|\eta|\right) = \frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k^2}\left(\frac{|\mathbb{E}(\xi\overline{\eta})|^2}{\mathbb{E}|\xi|^2\mathbb{E}|\eta|^2}\right)^k.$$

Proof of Claim 4. Taking $\xi = f(t+ia)$ and $\eta = f(s+ib)$, we have due to stationarity:

$$\frac{|\mathbb{E}(\xi\overline{\eta})|^2}{\mathbb{E}|\xi|^2\mathbb{E}|\eta|^2} = \frac{|\mathbb{E}(f(t+ia)\overline{f(s+ib)})|^2}{\mathbb{E}|f(t+ia)|^2\,\mathbb{E}|f(s+ib)|^2} = \frac{|r(t-s+ia+ib)|^2}{r(2ia)\,r(2ib)} = q(t-s,a,b).$$

Therefore, by Lemma 3.6 equation (7) becomes:

$$\begin{aligned} C^{a,b}(T) &= \frac{1}{4}\int_{-T}^{T}\int_{-T}^{T}\left\{\frac{\partial^2}{\partial a\ \partial b}\sum_{k=1}^{\infty}\frac{1}{k^2}q(t-s,a,b)^k\right\}dt\,ds \\ &= \frac{1}{2}\int_{-2T}^{2T}\left(1-\frac{|x|}{2T}\right)\left\{\frac{\partial^2}{\partial a\ \partial b}\sum_{k\geq 1}\frac{q(x,a,b)^k}{k^2}\right\}\ dx. \end{aligned}$$

In the last equality we used that $\int_{-T}^{T}\int_{-T}^{T}Q(t-s)dt\,ds = 2\int_{-2T}^{2T}(2T-|x|)Q(x)dx$ for any $Q \in L^1([-2T,2T]))$, which can be proved by a simple change of variables.

All that remains in order to get (11), is to prove that

$$\forall x \neq 0, \quad \frac{\partial^2}{\partial a\ \partial b}\sum_{k\geq 1}\frac{q(x,a,b)^k}{k^2} = \sum_{k\geq 1}\frac{\partial^2}{\partial a\ \partial b}\frac{q(x,a,b)^k}{k^2} \tag{13}$$

Fix $x \neq 0$. For shortness, we do not write the variables (x,a,b), and use again the subscript notation for partial derivatives. We compute:

$$S_k^{a,b}(x) := \frac{\partial^2}{\partial a\, \partial b}\left\{q^k\right\} = \begin{cases} q_{ab}, & k=1 \\ k(k-1)q^{k-2}q_a q_b, + kq^{k-1}q_{ab} & k>1. \end{cases}$$

Therefore,

$$\left|\frac{S_k^{a,b}(x)}{k^2}\right| \le q^{k-2}|q_a q_b| + \frac{1}{k}q^{k-1}|q_{ab}|.$$

By part 1 of Lemma 3.3, $q(x,a,b) < 1$ (notice this holds also if $a = b$). We deduce that $\sum \left|\frac{S_k^{a,b}}{k^2}\right| < \infty$. i.e., the RHS of (13) converges in absolute value. By standard arguments, this is enough to prove equality (13). Claim 4 follows. □

3.6. **A formula in terms of the spectral measure.** In this section, we finally prove Proposition 2, by carefully moving to the spectral representation of the formula we had at hand.

Proof of Proposition 2. Using Lemma 3.1 and the definition of q in (12), we get

$$q(x,a,b)^k = \frac{\mathcal{F}[\rho_{a+b}^{*k} \star \rho_{a+b}^{*k}](x)}{r^k(2ia)\, r^k(2ib)}$$

Define

$$S_k^{a,b}(x) := \frac{\partial^2}{\partial a\, \partial b}\left\{q(x,a,b)^k\right\} = \frac{\partial^2}{\partial a\, \partial b}\left\{\frac{\mathcal{F}[\rho_{a+b}^{*k} \star \rho_{a+b}^{*k}](x)}{r^k(2ia)\, r^k(2ib)}\right\}. \tag{14}$$

Observation.

$$S_k^{a,b}(x) = \mathcal{F}\left[l_k^a(\lambda)\rho_{a+b}^{*k} \star l_k^b(\lambda)\rho_{a+b}^{*k}\right](x).$$

where $l_k^a(\lambda), l_k^b(\lambda)$ are linear functions in λ, given by

$$l_k^a(\lambda) = \frac{\partial}{\partial a}\left(\frac{1}{r^k(2ia)}\right) + \frac{2\pi}{r^k(2ia)}\lambda = \frac{2}{r^k(2ia)}\left(-ik\frac{r'(2ia)}{r(2ia)} + \pi\lambda\right).$$

Proof. Recall that

$$\mathcal{F}[\rho_{a+b}^{*k} \star \rho_{a+b}^{*k}](x) = \iint e^{-ix(\lambda-\tau)} e^{2\pi(a+b)(\lambda+\tau)} d\rho^{*k}(\lambda) d\rho^{*k}(\tau),$$

and notice we may differentiate by a and b under the integral, as the result would be continuous and integrable w.r.t. ρ^{*k}. From here, the proof is a straightforward computation. □

Using this observation, we rewrite equation (11) as follows:

$$\frac{C^{a,b}(T)}{2T} = \int_{-2T}^{2T}\left(1 - \frac{|x|}{2T}\right)\sum_{k\ge1}\frac{1}{2k^2}\frac{\partial^2}{\partial a\, \partial b}\left\{\frac{\mathcal{F}[\rho_{a+b}^{*k} \star \rho_{a+b}^{*k}](x)}{r^k(2ia) r^k(2ib)}\right\} dx. \tag{15}$$

Futher, using Lemma 3.2 (Parseval's identity), we have for fixed k,

$$\int_{-2T}^{2T}\left(1-\frac{|x|}{2T}\right)S_k^{a,b}(x)\ dx=\int_{-2T}^{2T}\left(1-\frac{|x|}{2T}\right)\mathcal{F}\left[l_k^a(\lambda)\rho_{a+b}^{*k}\star l_k^b(\lambda)\rho_{a+b}^{*k}\right](x)$$

$$=\int_{\mathbb{R}}\int_{\mathbb{R}}2T\mathrm{sinc}^2(2\pi T(\lambda-\tau))\ l_k^a(\lambda)l_k^b(\tau)e^{2\pi(a+b)(\lambda+\tau)}d\rho^{*k}(\lambda)d\rho^{*k}(\tau). \tag{16}$$

Plugging (15) into (10), we get:

$$\frac{V_f^{a,b}(T)}{2T}=\frac{1}{4\pi^2}\int_{-2T}^{2T}\left(1-\frac{|x|}{2T}\right)\sum_{k\geq1}\frac{1}{2k^2}(S_k^{a,a}(x)-2S_k^{a,b}(x)+S_k^{b,b}(x))dx+o(1)$$

$$=\frac{1}{4\pi^2}\sum_{k\geq1}\frac{1}{2k^2}\int_{-2T}^{2T}\left(1-\frac{|x|}{2T}\right)\ (S_k^{a,a}(x)-2S_k^{a,b}(x)+S_k^{b,b}(x))dx+o(1) \tag{17}$$

$$=\frac{1}{8\pi^2}\sum_{k\geq1}\frac{1}{k^2}\int_{\mathbb{R}}\int_{\mathbb{R}}T\mathrm{sinc}^2(2\pi T(\lambda-\tau))h_k^{a,b}(\lambda+\tau)d\rho^{*k}(\lambda)d\rho^{*k}(\tau)+o(1),$$

where

$$h_k^{a,b}(\lambda)=\left(l_k^a(\lambda)e^{2\pi a\lambda}-l_k^b(\lambda)e^{2\pi b\lambda}\right)^2. \tag{18}$$

The exchange of sum and integral in the second equality of (17) is justified by the monotone convergence theorem, as each term in the series is non-negative. The last equality follows from (16). Equation 17 establishes Proposition 2. □

4. Theorem 2: Linear and Intermediate Variance

We dedicate Section 4.1 to prove some facts which will be needed along the proof. Afterwards, we prove the existence of the limit L_1 and its positivity in Section 4.2. In Section 4.3 we prove L_1 is finite under condition (2).

4.1. **Preparation.**

4.1.1. *Tools from Analysis.* First we present two observations about convolutions and integration. We omit their proofs as they are straightforward.

Observation 4.1. *If $Q:\mathbb{R}\to[0,\infty)$ is integrable on $\mathbb{R}$, then*

$$\lim_{T\to\infty}\int_{-T}^{T}\left(1-\frac{|x|}{T}\right)Q(x)dx=\int_{\mathbb{R}}Q.$$

Observation 4.2. *For any $\psi_1,\psi_2\in C_0(\mathbb{R})$ and $\mu\in\mathcal{M}^+(\mathbb{R})$,*

$$\int\psi_1\,d(\mu*\psi_2)=\int\ (\psi_1*\mathrm{flip}\{\psi_2\})\,d\mu.$$

The following lemma shall play a key-role later on in our proof.

Lemma 4.1. *Let $\mu\in\mathcal{M}^+(\mathbb{R})$ ($\mu\not\equiv0$). Then the following limit exists (finite or infinite):*

$$\lim_{\varepsilon\to0+}\int_{\mathbb{R}}\frac{1}{2\varepsilon}\mu\left(\tau-\varepsilon,\tau+\varepsilon\right)d\mu(\tau)=\int_{\mathbb{R}}|\mathcal{F}[\mu]|^2(x)dx.$$

Proof. Denote $\varphi_\varepsilon = \frac{1}{2\varepsilon}1\!\!1_{(-\varepsilon,\varepsilon)}$ for $\varepsilon > 0$. Rewriting the integral and using Parseval's identity, we get:

$$\begin{aligned} I_\mu(\varepsilon) &:= \frac{1}{2\varepsilon}\int_{\mathbb{R}} \mu\left(\tau-\varepsilon,\tau+\varepsilon\right) d\mu(\tau) \\ &= \int_{\mathbb{R}} (\mu * \varphi_\varepsilon)(\tau) d\mu(\tau) \\ &= \int_{\mathbb{R}} (\mathcal{F}[\mu]\cdot\mathcal{F}[\varphi_\varepsilon])(x)\mathcal{F}[\mu](-x)dx \\ &= \int_{\mathbb{R}} \mathrm{sinc}(2\pi\varepsilon x)|\mathcal{F}[\mu]|^2(x)dx \end{aligned}$$

Since $|\mathrm{sinc}(2\pi\varepsilon x)| \le 1$, we have the upper bound $I_\mu(\varepsilon) \le \int_{\mathbb{R}} |\mathcal{F}[\mu]|^2(x)dx$. Using the Observation 4.2 and the fact that $\varphi_\varepsilon * \varphi_\varepsilon \le 2\varphi_{2\varepsilon}$ we get:

$$\int_{\mathbb{R}} (\mu * \varphi_\varepsilon)\ d(\mu*\varphi_\varepsilon) = \int_{\mathbb{R}} \mu * (\varphi_\varepsilon * \varphi_\varepsilon) d\mu \le 2\int_{\mathbb{R}} \mu * \varphi_{2\varepsilon}\ d\mu = 2I_\mu(2\varepsilon)$$

On the other hand,

$$\begin{aligned} \int_{\mathbb{R}} (\mu * \varphi_\varepsilon)\ d(\mu*\varphi_\varepsilon) &= \int |\mathcal{F}[\mu*\varphi_\varepsilon]|^2 = \int |\mathcal{F}[\mu]|^2 \cdot \mathrm{sinc}^2(2\pi\varepsilon x)\ dx \\ &\ge \int_K |\mathcal{F}[\mu]|^2 \cdot \mathrm{sinc}^2(2\pi\varepsilon x)\ dx \end{aligned}$$

for any compact set $K \subset \mathbb{R}$. Since the limit $\lim\limits_{\varepsilon\to 0+} \mathrm{sinc}(2\pi\varepsilon x) = 1$ is uniform in $x \in K$, the last expression approaches $\int_K |\mathcal{F}[\mu]|^2$ as $\varepsilon \to 0+$. Thus, by choosing K and then $\varepsilon > 0$ properly, the lower bound may be made arbitrarily close to $\int_{\mathbb{R}} |\mathcal{F}[\mu]|^2$. This concludes the proof. ☐

4.1.2. *Lower bounds on $h_1^{a,b}$.* The main goal of this subsection is to give lower bounds on $h_1^{a,b}$ (defined in (18)). We begin with a simple claim.

Claim 5. *The function $h_1^{a,b}$ has exactly two real zeroes.*

Proof. By the form of $h_1^{a,b}$, $h_1^{a,b}(\lambda) = 0$ if and only if

$$e^{2\pi(b-a)\lambda} = \frac{l_1^a(\lambda)}{l_1^b(\lambda)} = \frac{\frac{1}{r(2ia)}\left(\pi\lambda - i\frac{r'}{r}(2ia)\right)}{\frac{1}{r(2ib)}\left(\pi\lambda - i\frac{r'}{r}(2ib)\right)} = C\cdot\frac{\lambda-\psi(a)}{\lambda-\psi(b)},$$

where $C > 0$ is a positive constant and $\psi(y) = \frac{1}{2\pi}\frac{d}{dy}\left[\log r(2iy)\right]$. Since $y \mapsto \log r(2iy)$ is a convex function, for $a < b$ we have $\psi(a) < \psi(b)$. Therefore, $\lambda \mapsto C\ \frac{\lambda-\psi(a)}{\lambda-\psi(b)}$ is a strictly decreasing function, with a pole at $\psi(b)$ and with the same positive limit at $\pm\infty$. Thus, it crosses exactly twice the increasing exponential function $e^{2\pi(b-a)\lambda}$. ☐

The next claim will enable us to bound $h_1^{a,b}$ from below, on most of the real line. Denote by $z_1, z_2 \in \mathbb{R}$ ($z_1 < z_2$) the two real zeroes of $h_1^{a,b}$ whose existence is guaranteed by Claim 5. We also use the notation $B(x,\delta)$ for the interval of radius $\delta > 0$ around $x \in \mathbb{R}$.

Claim 6. *For all $\delta > 0$, there exists $c_\delta > 0$ such that for all $\lambda \in \mathbb{R} \setminus (B(z_1, \delta) \cup B(z_2, \delta))$:*

$$h_1^{a,b}(\lambda) > c_\delta (1 + \lambda^2) \max(e^{2a\cdot 2\pi\lambda}, e^{2b\cdot 2\pi\lambda}).$$

Proof. Since the function $\frac{h_1^{a,b}(\lambda)}{(1+\lambda^2)e^{2a\cdot 2\pi\lambda}} = \left(\frac{l_1^a(\lambda) - l_1^b(\lambda)e^{2\pi(b-a)\lambda}}{\sqrt{1+\lambda^2}}\right)^2$ approaches strictly positive limits as $|\lambda| \to \infty$, there exist $M_a, c_a > 0$ such that

$$\forall |\lambda| \geq M_a : h_1^{a,b}(\lambda) \geq c_a (1 + \lambda^2) e^{2a\cdot 2\pi\lambda}.$$

Similarly, there exist some $M_b, c_b > 0$ such that $\forall |\lambda| \geq M_b : h_1^{a,b}(\lambda) \geq c_b(1 + \lambda^2)e^{2b\cdot 2\pi\lambda}$. Take $M = \max(M_a, M_b)$. Since $h(\lambda)$ attains a positive minimum on $[-M, M] \setminus (B(z_1, \delta) \cup B(z_2, \delta))$, there exists some $c > 0$ such that for all λ in this set, $h(\lambda) \geq c(1 + \lambda^2) \max(e^{2a\cdot 2\pi\lambda}, e^{2b\cdot 2\pi\lambda})$. Choosing $c_\delta = \min(c, c_a, c_b)$ will yield the result. □

The next claim is a slight modification of the previous one, in order to fit our specific need. Denote $\mathrm{Diag}_\varepsilon = \mathbb{I}\{(\lambda, \tau) : |\lambda - \tau| < \varepsilon\}$.

Claim 7. *For every $\delta > 0$ there exist a set $F = F_\delta = \mathbb{R} \setminus (I_1 \cup I_2)$ such that I_j is an interval containing z_j and of length at most δ ($j = 1, 2$), $\rho(F) > 0$, and there exists $c_\delta > 0$ such that for all small enough ε,*

$$h(\lambda + \tau) \geq c_\delta (1 + (\lambda + \tau)^2) \max\left(e^{2a\cdot 2\pi(\lambda+\tau)}, e^{2b\cdot 2\pi(\lambda+\tau)}\right),$$

for all $\lambda, \tau \in (F \times F) \cap \mathrm{Diag}_\varepsilon$.

Proof. Choose $F = \mathbb{R} \setminus \left(B\left(\frac{z_1}{2}, \delta_0\right) \cup B\left(\frac{z_2}{2}, \delta_0\right)\right)$, where $\delta_0 \leq \delta$ is small enough so that $\rho(F) > 0$. Then, for $\varepsilon \leq \delta_0$ and $(\lambda, \tau) \in (F \times F) \cap \mathrm{Diag}_\varepsilon$, we have

$$|\lambda + \tau - z_j| \geq |2\tau - z_j| - |\lambda - \tau| \geq 2\delta_0 - \varepsilon \geq \delta_0.$$

Choosing the constant $c_\delta > 0$ which is the consequence of applying Claim 6 will end our proof. □

4.1.3. *Convergence properties of the functions S_k.* Recall the definition of $S_k^{a,b}$ in (14). We stress, once again, that $S_k^{a,a}(x)$ denotes the evaluation of the same mixed partial derivative at the point (x, a, a). Our goal in this subsection is to prove the following Lemma.

Lemma 4.2. *If condition (2) is satisfied, then for every $k \in \mathbb{N}$ the functions $S_k^{a,a}(x)$, $S_k^{a,b}(x)$ and $S_k^{b,b}(x)$ are in $L^1(\mathbb{R})$ with respect to the variable x. Moreover,*

$$\sum_{k\geq 1} \frac{1}{k^2} \int_{\mathbb{R}} S_k(x) dx \ \textit{converges},$$

with any of the three possible superscripts on the letter S.

We will need some convergence properties of the function q and its partial derivatives. The first follows from part 3 of Lemma 3.3.

Observation 4.3. *Let g be one of the functions q, q_a, q_b or q_{ab}. Then $g(x, a, a)$, $g(x, a, b)$ and $g(x, b, b)$ are all in $\left(L^1 \cap L^\infty\right)(\mathbb{R})$ with respect to the variable x.*

The next two claims require some more effort.

Claim 8. *The sum* $\sum_{m\ge 1}\int_{\mathbb{R}} q^m q_a q_b \; dx$ *converges.*

Proof. We will show, in fact, that the positive series $\sum_{m\ge 1}\int_{\mathbb{R}} q^m |q_a q_b| \; dx$ converges.

First, in case we are evaluating at (x,a,b) $(a<b)$, our series converges due to (21) and the bound in part 2 of Lemma 3.3. Now assume we are evaluating at (x,t,t) (where $t\in\{a,b\}$). As we deal with a positive series, it is enough to show that both

(I) $\sum_{m\ge 1}\int_{-1}^{1} q^m |q_a q_b| \; dx<\infty$, and
(II) $\sum_{m\ge 1}\int_{|x|\ge 1} q^m |q_a q_b| \; dx<\infty$.

Denote by $C=\sup_{x\in\mathbb{R}} |q_a q_b(x,t,t)|\in(0,\infty)$. The sum in (II) is bounded by

$$C\sum_{m\ge 1}\int_{|x|\ge 1} q^m(x,t,t)\; dx = C\int_{|x|\ge 1}\frac{q}{1-q}(x,t,t)\; dx \le C'\int_{\mathbb{R}} q(x,t,t)\; dx,$$

where $C'\in(0,\infty)$ is another constant. C, C' and $\int_{\mathbb{R}} q(x,t,t)dx$ are all finite by part 3 of Lemma 3.3.

We turn to show (I). By parts 1 and 4 of Lemma 3.3, the sum

$$\sum_{m\ge 1} q^m |q_a q_b| \; dx = \frac{|q_a q_b|}{1-q}$$

is well-defined for all x (including $x=0$). By the monotone convergence theorem, item (I) is then equivalent to

$$\int_{-1}^{1}\frac{|q_a q_b|}{1-q}(x,t,t)\; dx<\infty,$$

which is indeed finite as an integral of a continuous function on $[-1,1]$. □

Claim 9. *The sum* $\sum_{m\ge 1}\frac{1}{m+2}\int_{\mathbb{R}} q^m \; dx$ *converges.*

Proof. We use a fact which appears in standard proofs of the Central Limit Theorem (CLT). For completeness, we include a proof in appendix C.

Lemma 4.3. *Let* $g\in L^1(\mathbb{R})$ *be a probability density, i.e.,* $g\ge 0$ *and* $\int_{\mathbb{R}} g=1$. *Suppose further that*

(a) $\int_{\mathbb{R}} |\lambda|^k g(\lambda) d\lambda<\infty$ *for* $k=1,2,3$ *and*
(b) $\int_{\mathbb{R}} |\mathcal{F}[g](x)|^\nu \; dx<\infty$ *for some* $\nu\ge 1$.

Then there exists $C>0$ *such that for all* $m\ge\nu$,

$$\int_{\mathbb{R}} |\mathcal{F}[g](x)|^m dx<\frac{C}{\sqrt{m}}.$$

We would like to apply Lemma 4.3 to

$$g^{a,b}(\lambda)=\frac{e^{2\pi(a+b)\lambda}p(\lambda)}{r(ia+ib)}.$$

Notice that this is the density of a probability measure. This choice also obeys the extra integrability conditions in the lemma (Condition (a) follows from the exponential moment assumption (1), and condition (b) with $\nu = 2$ follows from the L^2 assumption (2)). We see now that

$$q(x,a,b) = \frac{r(ia+ib)^2}{r(2ia)r(2ib)} \cdot |\mathcal{F}[g^{a,b}](x)|^2 \le |\mathcal{F}[g^{a,b}](x)|^2,$$

the last inequality following from the log-convexity of $y \mapsto r(iy)$. Similarly we define $g^{a,a}$ and have $q(x,a,a) = |\mathcal{F}[g^{a,a}](x)|^2$. Thus in all three cases of evaluation, using the lemma with the appropriate function g yields:

$$\sum_{m\ge 1} \frac{1}{m+2} \int_{\mathbb{R}} q^m \ dx \le \sum_{m\ge 1} \frac{1}{m+2} \int_{\mathbb{R}} |\mathcal{F}[g](x)|^{2m} dx$$
$$< C \sum_{m\ge 1} \frac{1}{(m+2)\sqrt{2m}} < \infty,$$

as required. □

Proof of Lemma 4.2. Taking derivative by the chain rule, we see that:

$$S_k = \frac{\partial^2}{\partial a\ \partial b}\left\{q^k\right\} = \begin{cases} q_{ab}, & k=1 \\ k(k-1)q^{k-2}q_a q_b, + kq^{k-1}q_{ab} & k>1. \end{cases} \tag{19}$$

Now, the fact that $S_k^{a,a}$, $S_k^{a,b}$ and $S_k^{b,b}$ are in $L^1(\mathbb{R})$ with respect to x follows from Observation 4.3.

We turn now to prove the "moreover" part of the claim. We use (19) in order to rewrite the desired series:

$$\begin{aligned} &\sum_{k\ge 1} \frac{1}{k^2} \int_{\mathbb{R}} S_k(x) dx \\ &= \int_{\mathbb{R}} q_{ab}\ dx + \sum_{k\ge 2} \int_{\mathbb{R}} q^{k-2} q_a q_b\ dx + \sum_{k\ge 2} \frac{1}{k} \int_{\mathbb{R}} q^{k-2}\left(q q_{ab} - q_a q_b\right)\ dx. \end{aligned} \tag{20}$$

Once again, all functions are evaluated at (x,a,a), (x,a,b) or (x,b,b) and what follows holds for each of the three options. By Observation 4.3,

$$\int_{\mathbb{R}} |q|\ dx < \infty, \quad \int_{\mathbb{R}} |q_{ab}|\ dx < \infty, \quad \int_{\mathbb{R}} |q_a q_b|\ dx < \infty, \tag{21}$$

and in particular the left-most term in (20) is finite. For the middle sum in (20), convergence follows from Claim 8 and (21). Convergence of the right-most sum in (20) follows from Claim 9 and (21). This ends the proof of Lemma 4.2. □

4.2. **Existence and Positivity.** In this section we prove that L_1 exists and belongs to $(0,\infty]$. If ρ has at least one atom, Theorem 1 implies that $\lim_{T\to\infty} \frac{V_f^{a,b}(T)}{T^2} > 0$, and therefore $L_1 = \lim_{T\to\infty} \frac{V_f^{a,b}(T)}{T} = \infty$. We thus assume that ρ has no atoms.

Using the formula for the variance obtained in Proposition 2, and recalling the functions $h_k^{a,b}$ are non-negative, we see that the limit L_1 exists and is in $[0,\infty]$. More effort is needed in order to establish that $L_1 > 0$. We begin with a simple bound arising from Proposition 2:

$$\begin{aligned}
\liminf_{T\to\infty} \frac{V_f^{a,b}(T)}{2T} &= \frac{1}{4\pi^2}\liminf_{T\to\infty}\sum_{k\geq 1}\frac{1}{k^2}\int_{\mathbb{R}}\int_{\mathbb{R}} T\operatorname{sinc}^2(2\pi T(\lambda-\tau))h_k^{a,b}(\lambda+\tau)d\rho^{*k}(\lambda)d\rho^{*k}(\tau)\\
&\geq \frac{1}{4\pi^2}\liminf_{T\to\infty}\int_{\mathbb{R}}\int_{\mathbb{R}} T\ \operatorname{sinc}^2(2\pi T(\lambda-\tau))\ h_1^{a,b}(\lambda+\tau)\ d\rho(\lambda)d\rho(\tau)\\
&\geq C_0\ \liminf_{\varepsilon\to 0+}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{1}{2\varepsilon}\mathbb{1}\{(\lambda,\tau):\ |\lambda-\tau|<\varepsilon\}h_1^{a,b}(\lambda+\tau)\ d\rho(\lambda)d\rho(\tau), \qquad (22)
\end{aligned}$$

where $C_0 > 0$ is an absolute constant. the first inequality follows from considering only the first term in the original non-negative series. The second inequality follows by noticing that $T\operatorname{sinc}^2(2\pi T\,\lambda) \geq \frac{4T}{\pi^2}\mathbb{1}\{\lambda : |\lambda| < \frac{1}{4T}\}$.

Fix a parameter $\delta > 0$, and fix $F = F_\delta$ to be the set provided by Claim 7. Continuing (22), we have

$$\begin{aligned}
\liminf_{T\to\infty}\frac{V_f^{a,b}(T)}{2T} &\geq c_\delta \liminf_{\varepsilon\to 0}\iint_{F\times F}\frac{1}{2\varepsilon}\mathbb{1}_{\mathrm{Diag}_\varepsilon}(\lambda,\tau)\ e^{2\pi\cdot 2a(\lambda+\tau)}d\rho(\lambda)d\rho(\tau)\\
&= c_\delta \liminf_{\varepsilon\to 0}\int_F \frac{1}{2\varepsilon}\rho_{2a}\left((\tau-\varepsilon,\tau+\varepsilon)\cap F\right)d\rho_{2a}(\tau)\\
&= c_\delta \liminf_{\varepsilon\to 0}\int_{\mathbb{R}}\frac{1}{2\varepsilon}\mu\left(\tau-\varepsilon,\tau+\varepsilon\right)d\mu(\tau), \qquad (23)
\end{aligned}$$

where μ is the restriction of ρ_{2a} to F, i.e. $\mu(\varphi) = \rho_{2a}(\mathbb{1}_F\cdot\varphi)$ for any test-function φ. Notice that by the choice of F, $\mu(\mathbb{R}) = \rho_{2a}(F) > 0$. By Lemma 4.1, the RHS of (23) is strictly positive, and we conclude that $L_1 > 0$.

4.3. **Linear Variance.** Consider again the first line of (17). Recall that, as we saw in Section 3.6, each term of the series in the RHS of (17) is non-negative. Therefore, by the monotone convergence theorem:

$$\lim_{T\to\infty}\frac{V_f^{a,b}(T)}{2T} = \frac{1}{8\pi^2}\sum_{k\geq 1}\frac{1}{k^2}\lim_{T\to\infty}\int_{-2T}^{2T}\left(1-\frac{|x|}{2T}\right)\ \left(S_k^{a,a}(x) - 2S_k^{a,b}(x) + S_k^{b,b}(x)\right)\,dx,$$

provided that the limit of each term on the RHS exists. These limits can be computed using Observation 4.1:

$$\lim_{T\to\infty}\frac{V_f^{a,b}(T)}{2T} = \frac{1}{8\pi^2}\sum_{k\geq 1}\frac{1}{k^2}\int_{\mathbb{R}}\left(S_k^{a,a}(x) - 2S_k^{a,b}(x) + S_k^{b,b}(x)\right)\,dx,$$

which is finite by Lemma 4.2.

Lastly, we explain how to obtain the form of L_1 appearing in Remark 1.3. By monotone convergence theorem, we may take term-by-term limit as $T\to\infty$ in Proposition 2, and get:

$$\lim_{T\to\infty}\frac{V_f^{a,b}(T)}{2T} = \frac{1}{8\pi^3}\sum_{k\geq 1}\frac{1}{k^2}\int_{\mathbb{R}}\left(p^{*k}(\lambda)\right)^2 h_k^{a,b}(2\lambda)d\lambda \in (0,\infty).$$

5. Theorem 3: Super-linear variance

In this section we prove the two items of Theorem 3, in reverse order.

5.1. **Item** (ii)**: Super-linear variance for particular** a, b**.**

Assume condition (3) holds for the particular a and b at hand. Fix a parameter $\delta > 0$, and let $F = F_\delta$ be the set provided by Claim 7. The premise ensures that, if δ is small enough, at least one of the measures $(1+\lambda)\rho_{2a}|_{F_\delta}$ and $(1+\lambda)\rho_{2b}|_{F_\delta}$ does not have L^2-density. WLOG assume it is the former. At first, assume also $\rho_{2a}|_F$ is not in L^2. Repeating the arguments of the Subsection 4.2 we get the lower bound

$$\liminf_{T\to\infty} \frac{V_f^{a,b}(T)}{2T} \geq c_\delta \int_{\mathbb{R}} |\mathcal{F}[\mu]|^2(x)dx,$$

where $\mu = \rho_{2a}|_F$ and $c_\delta > 0$. The LHS is therefore infinite, and so $L_1 = \infty$.

We are left with the case that $\lambda\rho_{2a}|_{F_\delta}$ does not have L^2-density, but $\rho_{2a}|_{F_\delta}$ does (denote it by p_{2a}). The argument is similar. Continuing from (22) and employing Claim 7, we get

$$\begin{aligned}\liminf_{T\to\infty} \frac{V_f^{a,b}(T)}{2T} &\geq c_\delta \liminf_{\varepsilon\to 0} \int_F \int_F \frac{1}{2\varepsilon} 1\!\!1_{(\tau-\varepsilon,\tau+\varepsilon)}(\lambda)(\lambda+\tau)^2 p_{2a}(\lambda)\ p_{2a}(\tau) d\lambda d\tau \\ &\geq c_\delta \cdot 4 \int_K \lambda^2 p_{2a}(\lambda)^2 d\lambda,\end{aligned}$$

where $K \subset F$ is compact. But, by our assumption, by choosing K properly the last bound can be made arbitrarily large, so that $\lim_{T\to\infty} \frac{V_f^{a,b}(T)}{2T} = \infty$.

5.2. **Item** (i)**: Super-linear variance for almost all** a, b**.**

Let ρ be such that the condition in item (i) holds. If ρ has a singular component, then the condition in item (ii) holds for all possible a, b and so $L_1(a, b) = \infty$ with no exceptions. Otherwise, ρ has density $p(\lambda)$. Define the set

$$E = \{(a, b) :\ a, b \in J,\ a < b,\ \text{the condition in item (ii) fails for } a, b\}.$$

If $E = \emptyset$, once again $L_1(a, b) = \infty$ for all $a, b \in J$ with no exceptions.

Assume then there is some $(a_0, b_0) \in E$. This means there exists λ_1, λ_2 such that for any pair of intervals I_1, I_2 such that $\lambda_j \in I_j$ $(j = 1, 2)$, both the functions $(1+\lambda^2)e^{2\pi\cdot 2a_0\lambda}p(\lambda)$ and $(1+\lambda^2)e^{2\pi\cdot 2b_0\lambda}p(\lambda)$ are in $L^2(\mathbb{R} \setminus (I_1 \cup I_2))$, but at least one of them (WLOG, the former) is not in $L^2(\mathbb{R})$. Observe that the existence of such λ_1, λ_2 depends solely on $p(\lambda)$, and may therefore be regarded as independent of the point $(a_0, b_0) \in E$. Moreover, at least one among λ_1 and λ_2 (say, λ_1) is such that for any neighborhood I containing it, $p \notin L^2(I)$.

Suppose now $a, b \in E$ are such that

$$h_1^{a,b}(\lambda_1) > 0, \tag{24}$$

where $h_1^{a,b}(\lambda) = \left(l_1^a(\lambda)e^{2\pi a\lambda} - l_1^b(\lambda)e^{2\pi b\lambda}\right)^2$ is the function appearing in the the first term of our asymptotic formula, and in the lower bound in inequality (22). Recall $h_1^{a,b}$ is non-negative and has only two zeroes by Claim 5.

We may choose $\delta > 0$ smaller than the minimal distance between λ_1 and a zero of $h_1^{a,b}$, and then construct $F = F_\delta$ as in Claim 7. Certainly $\lambda_1 \in F_\delta$, and so the measure $\mu = \rho_{2a}|_{F_\delta}$ is not in $L^2(\mathbb{R})$ (it is even not in $L^2(I)$ for any neighborhood I of λ_1). Just as in subsection 4.2 we shall get

$$\liminf_{T\to\infty} \frac{V_f^{a,b}(T)}{2T} \geq c_\delta \int_{\mathbb{R}} |\mathcal{F}[\mu]|^2(x)dx = \infty.$$

We end by showing that for a given point $\lambda_1 \in \mathbb{R}$ and a given $a \in J$, the set of $b \in J$ which do not obey (24) is finite. Indeed, this is the set

$$\{b \in J : \ h^{a,b}(\lambda_1) = 0\} = \{b \in J : \ \varphi(a) = \varphi(b)\}$$

where

$$\varphi(y) = e^{2\pi y\lambda_1} l_1^y(\lambda_1) = \frac{\partial}{\partial y}\left(\frac{e^{2\pi\lambda_1 y}}{r(2iy)}\right).$$

Suppose the desired set is not finite. Since φ is real-analytic, it must be constant on J. But then $r(2iy) = \frac{e^{2\pi\lambda_1 y}}{cy+d}$ for some $c, d \in \mathbb{R}$, and the corresponding spectral density would satisfy condition (2) for all relevant a, b. This contradiction ends the proof.

Appendix A. Uniform exponential decay of tails

In this appendix we prove Proposition 1. We follow closely [10, Proposition 5.1] and [16, Ch. 7] which prove similar concentration bounds, known as "Offord-type estimates". We rely on the following lemma, which follows either from [10, Lemma 6.1] or [16, Lemma 7.1.2].

Lemma A.1. *If $\eta \sim \mathcal{N}_{\mathbb{C}}(0, \sigma^2)$, and E is an event in the probability space with $\mathbb{P}(E) = p$, then:*

$$|\mathbb{E}(\chi_E \log|\eta|)| \leq p\left[\frac{p}{2} - 2\log p + \log\sigma\right].$$

Proof of Proposition 1. Take $\phi(z) = \phi_T(z)$ to be a real C^2 function, whose support is $[-\frac{1}{2} - T, T + \frac{1}{2}] \times [a', b']$ with $-\Delta < a' < a < b < b' < \Delta$, and which takes the value 1 on $R_T = [-T, T] \times [a, b)$. We may build such $\phi_T(z)$ that will obey also the bound $\|\Delta\phi\|_{L^1} < c_0(T + b - a)$, where $c_0 > 0$ is a constant (depending on $|\Delta - b|$ and $|a + \Delta|$). Fix $s > 0$. We are interested in dominating the probability of the event $A_T = \{n_f(R_T) > sT\}$. Write $p = p_T = \mathbb{P}(A_T)$.

We have

$$n_f(R_T) < \frac{1}{2\pi}\int \Delta\phi_T(z)\log|f(z)|dm(z),$$

and therefore,

$$\begin{aligned} sT \cdot p &\leq \mathbb{E}(\chi_{A_T} n_f(R_T)) \leq \mathbb{E}\left(\chi_{A_T}\frac{1}{2\pi}\int \Delta\phi(z)\log|f(z)|dm(z)\right) \\ &= \frac{1}{2\pi}\int \Delta\phi\,\mathbb{E}\left(\chi_{A_T}\log|f(z)|\right)dm(z) \\ &\leq \frac{1}{2\pi}\|\Delta\phi\|_{L^1}\sup_{z\in\mathbb{R}\times[a',b']}\mathbb{E}\left(\chi_{A_T}\log|f(z)|\right) \end{aligned}$$

The exchange of expectation and integral is justified by Fubini's theorem, as follows:

$$\int_D \mathbb{E}\big|\Delta\phi(z)\cdot\log|f(z)|\,\big|dm(z) = \int_D |\Delta\phi(z)|\,\mathbb{E}\big|\log|f(z)|\,\big| < \sup\mathbb{E}\big|\log|f(z)|\big|\cdot\|\Delta\phi\|_{L^1} < \infty.$$

Applying lemma A.1 with $\eta = f(z)$, we get:

$$\sup_{z\in\mathbb{R}\times[a',b']} \mathbb{E}(\chi_{A_T}\log|f(z)|) < p(c_1 - 2\log p).$$

for some constant $c_1 > 0$ (depending on $\sup_{y\in[a',b']}\mathbb{E}|f(iy)|^2$). We conclude that:

$$sT\cdot p \le \frac{c_0}{2\pi}(T+b-a)p(c_1 - 2\log p),$$

which leads to the exponential bound we strived for:

$$\exists c, C > 0 \text{ such that } p_T = \mathbb{P}(n_f(R_T) > T\,s) \le Ce^{-cs},\ \forall T \ge 1\,.$$

□

Appendix B. Justification for changing operations

In this section we prove Lemmas 3.4 and 3.5.

B.1. **Proof of Lemma 3.4.** We begin by showing (A-I). Let $1 < p < 2$ be arbitrary, and let $q > 2$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Hölder's inequality implies:

$$\begin{aligned}\int_{-T}^{T} \mathbb{E}\left|\frac{f'(t+ia)}{f(t+ia)}\right| &\le \int_{-T}^{T} \mathbb{E}[|f'(t+ia)|^q]^{1/q}\mathbb{E}[|f(t+ia)|^{-p}]^{1/p}dt \\ &\le \mathbb{E}[|f'(ia)|^q]^{1/q}\mathbb{E}[|f(ia)|^{-p}]^{1/p}\cdot T < \infty,\end{aligned}$$

where finiteness follows from $f'(ia)$ and $f(ia)$ being complex Gaussian random variables, thus having finite moments of any order.

We now turn to prove (A-II). We use the notation $f \lesssim g$ to stand for the inequality $f \le C\cdot g$, where $C > 0$ is a constant (which may vary from line to line). Similarly, $f \eqsim g$ stands for $f = C\cdot g$ with some $C > 0$.

As before, let $1 < p < 2$ and take $q > 2$ to obey $\frac{1}{p} + \frac{1}{q} = 1$. By Hölder inequality we have

$$\begin{aligned}&\int_{-T}^{T}\int_{-T}^{T} \mathbb{E}\left|\frac{f'(t+ia)\ f'(s+ib)}{f(t+ia)\ f(s+ib)}\right| dtds \\ &\le \int_{-T}^{T}\int_{-T}^{T} \left[\mathbb{E}\Big|f'(t+ia)f'(s+ib)\Big|^q\right]^{1/q} \left[\mathbb{E}\Big|f(t+ia)f(s+ib)\Big|^{-p}\right]^{1/p} dt\ ds \\ &\lesssim \int_{-T}^{T}\int_{-T}^{T} \mathbb{E}\left[\Big|f(t+ia)f(s+ib)\Big|^{-p}\right]^{1/p} dt\ ds. \qquad (25)\end{aligned}$$

The last inequality is an application of Cauchy-Schwarz inequality and stationarity, as follows:

$$\begin{aligned}&\left[\mathbb{E}\left|f'(t+ia)f'(s+ib)\right|^q\right]^{\frac{1}{q}}\\&\le\left(\sqrt{\mathbb{E}|f'(t+ia)|^{2q}\ \mathbb{E}|f'(s+ib)|^{2q}}\right)^{\frac{1}{q}}=\left(\mathbb{E}[|f'(ia)|^{2q}]\mathbb{E}[|f'(ib)|^{2q}]\right)^{\frac{1}{2q}}<\infty,\end{aligned}$$

where again finiteness follows from $f'(ia)$ and $f'(ib)$ being Gaussian.

Let

$$A=\left\{(t,s)\in[-T,T]^2:\ |r(t-s+ia+ib)|^2\le\frac{2}{3}r(2ia)r(2ib).\right\}.$$

We split the last integral in (25) into two parts: on A and on $A^c=[-T,T]^2\setminus A$. For the integral on A we use the following lemma:

Lemma B.1. *Suppose ξ_1, ξ_2 are independent $\mathcal{N}_{\mathbb{C}}(0,1)$ random variables, and let $Z_1=\alpha\xi_1$ and $Z_2=\beta\xi_1+\gamma\xi_2$ where $\alpha,\beta,\gamma\in\mathbb{C}\setminus\{0\}$. Let $1<p<2$. Then*

$$\mathbb{E}\left[\ |Z_1Z_2|^{-p}\ \right]\le|\alpha\gamma|^{-p}\ \Gamma\left(1-\frac{p}{2}\right)^2$$

We include the proof in Section B.3. We apply Lemma B.1 with $Z_1=f(t+ia)$ and $Z_2=f(s+ib)$, which yields the choice of parameters α,β,γ so that

$$\alpha=\sqrt{r(2ia)},\quad\alpha\overline{\beta}=\overline{r(t-s+ia+ib)},\quad|\beta|^2+|\gamma|^2=r(2ib).\tag{26}$$

In particular,

$$|\alpha\gamma|=\sqrt{r(2ia)r(2ib)-|r(t-s+ia+ib)|^2}.$$

Using this in Lemma B.1, we have:

$$\begin{aligned}&\iint_A\left(\mathbb{E}\left|f(t+ia)f(s+ib)\right|^{-p}\right)^{1/p}dt\ ds\\&\lesssim\iint_A\left(r(2ia)r(2ib)-|r(t-s+ia+ib)|^2\right)^{-p/2}dt\ ds,\end{aligned}$$

which is bounded by the definition of A. In order to bound the integration on A^c, we use another lemma (which is also proved in Section B.3).

Lemma B.2. *Suppose ξ_1, ξ_2 are independent $\mathcal{N}_{\mathbb{C}}(0,1)$ random variables, and let $Z_1=\alpha\xi_1$ and $Z_2=\beta\xi_1+\gamma\xi_2$ where $\alpha,\beta,\gamma\in\mathbb{C}\setminus\{0\}$. Suppose $M>0$ is such that $|\frac{\gamma}{\beta}|<M$, and let $1<p<2$. Then there exists a constant $c>0$, depending only on M and p, such that*

$$\mathbb{E}\left[\frac{1}{|Z_1Z_2|^p}\right]\le\frac{c}{|\alpha\beta|^p}\left|\frac{\gamma}{\beta}\right|^{2-2p}.$$

We apply this lemma again to $Z_1=f(t+ia)$ and $Z_2=f(s+ib)$, so the choice of parameters in (26) remains valid. Thus,

$$\left|\frac{\gamma}{\beta}\right|^2=\frac{r(2ia)r(2ib)}{|r(t-s+ia+ib)|^2}-1$$

is uniformly bounded for $(t,s)\in A^c$. Applying Lemma B.2 we get that for some $c>0$,

$$
\begin{aligned}
&\iint_{A^c}\left(\mathbb{E}\left|f(t+ia)f(s+ib)\right|^{-p}\right)^{1/p} dt\ ds\\
&\quad\lesssim \iint_{A^c}\left(\frac{c}{|r(t-s+ia+ib)|^p}\ \frac{(r(2ia)r(2ib)-|r(t-s+ia+ib)|^2)^{1-p}}{|r(t-s+ia+ib)|^{2-2p}}\right)^{\frac{1}{p}} dt\ ds\\
&\quad\lesssim \iint_{A^c}\left(r(2ia)r(2ib)-|r(t-s+ia+ib)|^2\right)^{-\frac{p-1}{p}} dt\ ds \qquad (27)\\
&\quad\lesssim \int_{\tilde{A}^c}\left(r(2ia)r(2ib)-|r(x+ia+ib)|^2\right)^{-\frac{p-1}{p}} dx,
\end{aligned}
$$

where $\tilde{A}^c$ is the one-dimensional set

$$\tilde{A}^c=\{x\in[-T,T]:\ |r(x+ia+ib)|^2>\frac{2}{3}r(2ia)r(2ib)\}.$$

The last inequality in (27) is obtained by a simple change of variables. One step before that in (27) we bounded $|r(t-s+ia+ib)|^{-1}$ from above by a constant, using the definition of A^c.

Before continuing, we notice that

$$
\begin{aligned}
|r(x+ia+ib)|^2 &= \left|\int_{\mathbb{R}} e^{-2\pi ix\lambda}e^{2\pi\cdot(a+b)\lambda}d\rho(\lambda)\right|^2\\
&\le \left(\int_{\mathbb{R}} e^{2\pi\cdot(a+b)\lambda}d\rho(\lambda)\right)^2 \quad (=r(ia+ib)^2)\\
&\le \int_{\mathbb{R}} e^{2\pi\cdot 2a\lambda}d\rho(\lambda)\cdot\int_{\mathbb{R}} e^{2\pi\cdot 2a\lambda}d\rho(\lambda)\\
&= r(2ia)r(2ib), \qquad (28)
\end{aligned}
$$

and the inequality is sharp when $a\neq b$ (see also part 2 of Lemma 3.3 below). Therefore, if $a\neq b$, the last integral in (27) is finite.

In case $a=b$, there may be only a finite number of isolated points x_0 for which $|r(x_0+2ia)|^2=r(2ia)^2$. Taylor expansion near any of those points gives $|r(x+2ia)|^2=r(2ia)^2-C(x-x_0)^2+o((x-x_0)^2)$ as x tends to x_0 (here $C\ge 0$ since $|r(x+2ia)|^2\le r(2ia)^2$ by taking $a=b$ in (28)). So, in this case the finiteness of the integral (27) is equivalent to that of

$$\int_{|x-x_0|<\delta}(x-x_0)^{-2(p-1)/p}dx$$

(with some $\delta>0$), which is indeed finite for $1<p<2$.

B.2. **Proof of Lemma 3.5.** We will justify in detail the second item, as the proof of the first is similar and simpler.

Fix $(t,s)\in[-T,T]^2$. We may assume that $|r(t-s+ia+ib)|^2<r(2ia)r(2ib)$ as the set of $(t,s)\in[-T,T]^2$ where this inequality does not hold is of measure zero (see discussion

following (28)). Consider the random variables

$$U(h_1,h_2)=\frac{\log|f(t+ia+ih_1)|-\log|f(t+ia)|}{h_1}\cdot\frac{\log|f(s+ib+ih_2)|-\log|f(s+ib)|}{h_2}.$$

These are well-defined for $0<h_1<\delta_1$ and $0<h_2<\delta_2$, where δ_1,δ_2 are properly chosen numbers (we use here that almost surely, there are no zeroes on the vertical lines $\{t\}\times[a,a+\delta_1]$ and $\{s\}\times[b,b+\delta_2]$, as was explained in Section 3.2). Notice that almost surely,

$$\lim_{h_1\to 0+}\lim_{h_2\to 0+}U(h_1,h_2)=\frac{\partial^2}{\partial a\,\partial b}[\,\log|f(t+ia)|\log|f(s+ib)|\,].$$

Our goal (B-II) can be understood as convergence in $L^1(\mathbb{P})$ of the above limit. This will follow if the family $\{U(h_1,h_2)\}_{h_1,h_2}$ is uniformly integrable, i.e., if for every $\varepsilon>0$ there exists $k>0$ such that for all relevant h_1,h_2 it holds that

$$\mathbb{E}\left(|U(h_1,h_2)|\mathbb{1}_{\{|U(h_1,h_2)|\geq k\}}\right)<\varepsilon.$$

Uniform integrability, in turn, would follow from the following statement: [3]

$$\exists p>1:\ \sup_{h_1,h_2}\mathbb{E}|U(h_1,h_2)|^p<\infty. \tag{29}$$

Applying the Newton-Leibniz formula, the bound (8) and Jensen's inequality we get:

$$\begin{aligned}|U(h_1,h_2)|&=\left|\frac{1}{h_1h_2}\int_0^{h_2}\int_0^{h_1}\frac{\partial}{\partial y_1}\log|f(t+ia+iy_1)|\frac{\partial}{\partial y_2}\log|f(s+ib+iy_2)|dy_1\,dy_2\right|\\&\leq\int_0^{h_2}\int_0^{h_1}\left|\frac{f'}{f}(t+ia+iy_1)\right|\left|\frac{f'}{f}(s+ib+iy_1)\right|\frac{dy_1}{h_1}\frac{dy_2}{h_2}\\&\leq\left(\int_0^{h_2}\int_0^{h_1}\left|\frac{f'}{f}(t+ia+iy_1)\right|^p\left|\frac{f'}{f}(s+ib+iy_1)\right|^p\frac{dy_1}{h_1}\frac{dy_2}{h_2}\right)^{1/p}.\end{aligned}$$

Taking $1<p'<\frac{2}{p}$ and q' such that $\frac{1}{p'}+\frac{1}{q'}=1$, we apply Hölder's inqeuality to bound the last expression by

$$\begin{aligned}&\left(\int_0^{h_2}\int_0^{h_1}\mathbb{E}|f'(t+ia+iy_1)\ f'(s+ib+iy_2)|^{pq'}\frac{dy_1}{h_1}\frac{dy_2}{h_2}\right)^{\frac{1}{pq'}}\\&\times\left(\int_0^{h_2}\int_0^{h_1}\mathbb{E}|f(t+ia+iy_1)\ f(s+ib+iy_2)|^{-pp'}\frac{dy_1}{h_1}\frac{dy_2}{h_2}\right)^{\frac{1}{pp'}}\end{aligned}$$

Using Cauchy-Schwarz, the first integral is bounded by

$$\left(\max_{y_1\in[0,\delta_1]}\mathbb{E}|f'(t+ia+iy_1)|^{2pq'}\max_{y_2\in[0,\delta_2]}\mathbb{E}|f'(s+ib+iy_2)|^{2pq'}\right)^{\frac{1}{2pq'}},$$

[3] Indeed, suppose (29) holds. Denoting by q the number such that $\frac{1}{p}+\frac{1}{q}=1$, we apply Hölder's inequality to get:

$$\mathbb{E}\big(|U(h_1,h_2)|\mathbb{1}\{|U(h_1,h_2)|\geq k\}\big)\leq(\mathbb{E}|U(h_1,h_2)|^p)^{1/p}\,\mathbb{P}\,(|U(h_1,h_2)|\geq k)^{1/q}\lesssim\frac{1}{k^{p/q}},$$

so the definition of uniform integrability is satisfied.

which is finite and independent of h_1 and h_2. Applying Lemma B.1 with the same choice of parameters as before, we may bound the second integral (up to a constant factor depending on p and p') by:

$$\left(\int_0^{h_1}\int_0^{h_2}\left\{r(2ia+2iy_1)r(2ib+2iy_2)-|r(t-s+ia+ib+iy_2+iy_2)|^2\right\}^{-\frac{pp'}{2}}\frac{dy_1}{h_1}\frac{dy_2}{h_2}\right)^{\frac{1}{pp'}}$$
$$\lesssim \max_{y_1\in[0,\delta_1],y_2\in[0,\delta_2]}\left\{r(2ia+2iy_1)r(2ib+2iy_2)-|r(t-s+ia+ib+iy_2+iy_2)|^2\right\}^{-\frac{1}{2}},$$

which is again finite and independent of h_1 and h_2. Our proof is complete.

B.3. **Proofs of auxiliary Lemmas.** This part is dedicated to prove Lemmas B.1 and B.2 that were used earlier in this section.

Proof of Lemma B.1. Using the notations in the statement of the Lemma, we have:

$$\mathbb{E}\left[|Z_1Z_2|^{-p}\right]=\frac{1}{\pi^2}\iint_{\mathbb{C}^2}|\alpha\xi_1(\beta\xi_1+\gamma\xi_2)|^{-p}e^{-|\xi_1|^2-|\xi_2|^2}dm(\xi_1)dm(\xi_2)$$

Now, by the Hardy-Littlewood re-arrangement inequality, we have:

$$\begin{aligned}\frac{1}{\pi}\int_{\mathbb{C}}|\beta\xi_1+\gamma\xi_2|^{-p}e^{-|\xi_2|^2}dm(\xi_2)&\le|\gamma|^{-p}\cdot\frac{1}{\pi}\int_{\mathbb{C}}|\xi_2|^{-p}e^{-|\xi_2|^2}dm(\xi_2)\\&=|\gamma|^{-p}\,\Gamma\left(1-\frac{p}{2}\right).\end{aligned}$$

So,

$$\begin{aligned}\mathbb{E}\left[|Z_1Z_2|^{-p}\right]&\le|\alpha\gamma|^{-p}\,\Gamma\left(1-\frac{p}{2}\right)\cdot\frac{1}{\pi}\int_{\mathbb{C}}|\xi_1|^{-p}e^{-|\xi_1|^2}dm(\xi_1)\\&=|\alpha\gamma|^{-p}\,\Gamma\left(1-\frac{p}{2}\right)^2.\end{aligned}$$

□

Proof of Lemma B.2. In this proof, the constant hidden by the "$\lesssim$" and "$\simeq$" notation depends only on M and p. We begin by writing-out the desired expectation explicitly.

$$\begin{aligned}\mathbb{E}\left[\ |Z_1Z_2|^{-p}\ \right]&=|\alpha\beta|^{-p}\,\mathbb{E}\left[\ \left|\xi_1^2+\frac{\gamma}{\beta}\xi_1\xi_2\right|^{-p}\ \right]\\&=|\alpha\beta|^{-p}\cdot\frac{1}{\pi^2}\iint_{\mathbb{C}^2}\left|z^2+\frac{\gamma}{\beta}zw\right|^{-p}e^{-|z|^2-|w|^2}\ dm(z)\ dm(w)\\&=|\alpha\beta|^{-p}\pi^{-2}\int_{\mathbb{C}}|z|^{-p}\left(\int_{\mathbb{C}}\left|z+\frac{\gamma}{\beta}w\right|^{-p}e^{-|w|^2}dm(w)\right)e^{-|z|^2}dm(z).\end{aligned}\tag{30}$$

We bound the inner integral as follows:

$$\int_{\mathbb{C}} \left|z+\frac{\gamma}{\beta}w\right|^{-p} e^{-|w|^2} dm(w)$$
$$\lesssim \int_{|w|\le \frac{1}{2}\left|\frac{\beta}{\gamma}z\right|} |z|^{-p} e^{-|w|^2} dm(w) + |z|^{-p} e^{-\frac{1}{4}\left|\frac{\beta}{\gamma}z\right|^2} \left|\frac{\beta}{\gamma}z\right|^2 + \int_{|w|>2\left|\frac{\beta}{\gamma}z\right|} \left|\frac{\gamma}{\beta}w\right|^{-p} e^{-|w|^2} dm(w)$$
$$\eqsim |z|^{-p}\left(1-e^{-\frac{1}{4}\left|\frac{\beta}{\gamma}z\right|^2}\right) + \left|\frac{\beta}{\gamma}\right|^2 |z|^{2-p} e^{-\frac{1}{4}\left|\frac{\beta}{\gamma}z\right|^2} + \left|\frac{\beta}{\gamma}\right|^p I\left(\left|\frac{\beta}{\gamma}z\right|\right),$$

where

$$I(s) = \int_{|w|>2s} |w|^{-p} e^{-|w|^2} dm(w) \lesssim \begin{cases} 1, & 0<s\le 1, \\ s^{-p}e^{-4s^2}, & s>1. \end{cases}$$

The last bound is achieved by changing to polar coordinates, as follows:

$$I(s) \eqsim \int_{2s}^{\infty} r^{-p+1} e^{-r^2} dr \lesssim s^{-p+1} \int_{2s}^{\infty} e^{-r^2} dr \le s^{-p+1} \frac{1}{2s} e^{-4s^2}.$$

Returning to the double integral in (30), we have:

$$\mathbb{E}\left[\left|\xi_1^2 + \frac{\gamma}{\beta}\xi_1\xi_2\right|^{-p}\right]$$
$$\lesssim \int_{\mathbb{C}} \left\{ |z|^{-2p}\left(1-e^{-\frac{1}{4}\left|\frac{\beta}{\gamma}z\right|^2}\right) + \left|\frac{\beta}{\gamma}\right|^2 |z|^{2-2p} e^{-\frac{1}{4}\left|\frac{\beta}{\gamma}z\right|^2} + |z|^{-p}\left|\frac{\beta}{\gamma}\right|^p I\left(\left|\frac{\beta}{\gamma}z\right|\right) \right\} e^{-|z|^2} dm(z)$$

This is the sum of three integrals, which we bound separately. For the first, we have:

$$\int_{\mathbb{C}} |z|^{-2p}\left(1-e^{-\frac{1}{4}\left|\frac{\beta}{\gamma}z\right|^2}\right) e^{-|z|^2} dm(z)$$
$$\lesssim \int_{|z|\le\left|\frac{\gamma}{\beta}\right|} \left|\frac{\beta}{\gamma}\right|^2 |z|^{2-2p} dm(z) + \int_{|z|>\left|\frac{\gamma}{\beta}\right|} |z|^{-2p} e^{-|z|^2} dm(z)$$
$$\eqsim \left|\frac{\beta}{\gamma}\right|^2 \left|\frac{\gamma}{\beta}\right|^{4-2p} + O(1) \eqsim \left|\frac{\gamma}{\beta}\right|^{2-2p}$$

Denote $A = 1 + \frac{1}{4}\left|\frac{\beta}{\gamma}\right|^2$. Before estimating the second integral, we compute

$$\int_{\mathbb{C}} |z|^{2-2p} e^{-A|z|^2} dm(z)$$
$$\eqsim \int_0^\infty r^{2-2p} e^{-Ar^2} r\,dr \qquad [r = |z|\ ]$$
$$= \frac{1}{2A}\int_0^\infty \left(\frac{s}{A}\right)^{1-p} e^{-s} ds \qquad [s = Ar^2]$$
$$\eqsim A^{-(2-p)}.$$

Thus, the second integral is

$$\left|\frac{\beta}{\gamma}\right|^2 \int_{\mathbb{C}} |z|^{2-2p} e^{-\left(1+\frac{1}{4}\left|\frac{\beta}{\gamma}\right|^2\right)|z|^2} dm(z) \eqsim \left|\frac{\beta}{\gamma}\right|^2 \left(1+\frac{1}{4}\left|\frac{\beta}{\gamma}\right|^2\right)^{-(2-p)} \eqsim \left|\frac{\gamma}{\beta}\right|^{2-2p}.$$

For the third integral we first compute

$$\begin{aligned}
&\int_{|z|>|\frac{\gamma}{\beta}|} |z|^{-2p} e^{-A|z|^2} dm(z) \\
&\qquad \asymp \int_{|\frac{\gamma}{\beta}|}^{\infty} r^{-2p} e^{-Ar^2} r dr && [r = |z|\ ] \\
&\qquad = \frac{1}{2A} \int_{A|\frac{\gamma}{\beta}|}^{\infty} \left(\frac{s}{A}\right)^{-p} e^{-s} ds && [s = Ar^2] \\
&\qquad \lesssim A^{p-1} \int_{1/4}^{\infty} s^{-p} e^{-s} ds \asymp A^{p-1}.
\end{aligned}$$

Finally, the third integral is

$$\begin{aligned}
&\left|\frac{\beta}{\gamma}\right|^p \int_{\mathbb{C}} |z|^{-p}\, I\left(\left|\frac{\beta}{\gamma} z\right|\right) e^{-|z|^2} dm(z) \\
&\asymp \left|\frac{\beta}{\gamma}\right|^p \left\{ \int_{|z|<|\frac{\gamma}{\beta}|} |z|^{-p} e^{-|z|^2} dm(z) + \left|\frac{\beta}{\gamma}\right|^{-p} \int_{|z|>|\frac{\gamma}{\beta}|} |z|^{-2p} e^{-(1+4\left|\frac{\beta}{\gamma}\right|^2)|z|^2} dm(z) \right\} \\
&\asymp \left|\frac{\beta}{\gamma}\right|^p \left|\frac{\gamma}{\beta}\right|^{2-p} + \left(1 + 4\left|\frac{\beta}{\gamma}\right|^2\right)^{p-1} \asymp \left|\frac{\gamma}{\beta}\right|^{2-2p}.
\end{aligned}$$

The proof is complete.

□

Appendix C. Moments of the characteristic function

Here we prove Lemma 4.3, which estimates moments of the characteristic function (or Fourier transform) of a probability distribution. We adapt the proof of the Central Limit Theorem appearing in [11, Ch. XV.5].

Proof of Lemma 4.3. Write $G(x) = \mathcal{F}[g](x)$. We may assume that $\int_{\mathbb{R}} \lambda g(\lambda) = 0$ (otherwise we shall consider, instead of g, the function $g_\mu(\lambda) = g(\lambda + \mu)$ where $\mu := \int_{\mathbb{R}} \lambda g(\lambda) d\lambda$. There is no penalty since $|\mathcal{F}[g_\mu](x)| = |\mathcal{F}[g](x)|$ for all $x \in \mathbb{R}$). By assumption (a), $G(x)$ is thrice differentiable, and by the above assumptions $G(0) = 1$ and $G'(0) = 0$.

To prove the lemma, it is enough to show that

$$\lim_{m\to\infty} \sqrt{m} \int_{\mathbb{R}} |G(x)|^m dx \text{ exists and is finite.}$$

Notice that $\sqrt{m} \int_{\mathbb{R}} |G(x)|^m dx = \int_{\mathbb{R}} |G(x/\sqrt{m})|^m dx$, and so it is enough to show that

$$\lim_{m\to\infty} \int_{\mathbb{R}} \left| \left| G\left(\frac{x}{\sqrt{m}}\right)\right|^m - e^{-\frac{\alpha x^2}{2}} \right| dx = 0, \tag{31}$$

for some value of $\alpha > 0$, which in fact is $\alpha := G''(0)$.

We shall achieve (31) by splitting the integral into three parts, and showing each could be made less than a given $\varepsilon > 0$ if $m \geq \nu$ is chosen large enough.

Fix $R>0$ (to be determined later). By Taylor expansion,

$$G(x)=G(0)+xG'(0)+\frac{x^2}{2}G''(0)+o(x^2)=1+\frac{\alpha x^2}{2}+o(x^2),\ x\to 0 \tag{32}$$

and so $|G(x/\sqrt{m})|^m \to e^{-\alpha x^2/2}$ as $m\to\infty$, uniformly in $x\in[-R,R]$. Thus the integral in (31) computed on $[-R,R]$ converges to zero as $m\to\infty$.

From the expansion (32) we get

$$\exists\delta>0\ \forall|x|<\delta:\ |G(x)|\le e^{-\frac{\alpha x^2}{4}}.$$

Consider the integration in (31) for $R\le|x|\le\delta\sqrt{m}$. For such x we have $|G(x/\sqrt{m})|^m\le e^{-\frac{\alpha x^2}{4}}$, and so the integrand is less than $2e^{-\frac{\alpha x^2}{4}}$. Choosing R so that $4\int_R^\infty e^{-\frac{\alpha x^2}{4}}<\varepsilon$ will satisfy our needs.

Lastly, consider the integration on $\delta\sqrt{m}\le|x|<\infty$. By properties of Fourier transform, $\eta:=\sup_{|x|\ge\delta}|G(x)|\in(0,1)$. Thus

$$\int_{|x|\ge\delta\sqrt{m}}\left|\left|G\left(\frac{x}{\sqrt{m}}\right)\right|^m-e^{-\frac{\alpha x^2}{2}}\right|\,dx\le\eta^{m-\nu}\sqrt{m}\int_{\mathbb{R}}|G|^\nu+\int_{|x|\ge\delta\sqrt{m}}e^{-\frac{\alpha x^2}{2}}dx<\varepsilon,$$

for m large enough. Here we have used condition (b).

□

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