The small ball inequality and binary nets

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Outline

- What is the "Small Ball Inequality" (SBI)
- Motivation
- Results: new connection with nets
- Proofs
- Ideas for higher dimensions
- Related methods in analysis

Definitions

- $\mathcal{D} = \{[\frac{m}{2^k}, \frac{m+1}{2^k}) : \quad k \in \mathbb{N}_0, \quad m = 0, 1, \ldots, 2^k - 1\}$
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-1	1
1	-1

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- The constant in $\gtrsim$ depends on d, not on n
- "reverse triangle inequality"

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SBI $\Rightarrow$ SSBI: Notice:

$$\sum |\varepsilon_R| = \#\{R \in \mathcal{D}^d : |R| = 2^{-n}\} \asymp n^{d-1} \cdot 2^n$$

(=shape · placement).

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Notice that $\|h_R\|_2^2 = |R|$, and $\langle h_{R_1}, h_{R_2} \rangle = 0$ for $R_1 \neq R_2$.

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- best power known: $\frac{d-1}{2} + \eta(d)$ for $d \geq 3$ (Bilyk-Lacey-Vagharshakyan 2008)

Motivation 1: Probability

Let $X_t : T \to \mathbb{R}$ be a random process (usually Gaussian), estimate the *small ball probability*

$$\mathbb{P}\left(\sup_{t\in T} |X_t| < \varepsilon\right) \approx ?, \quad \varepsilon \to 0.$$

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- SBP $\leftrightarrow$ metric entropy (Kuelbs-Li '93)

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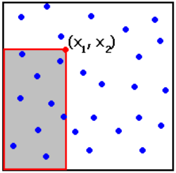

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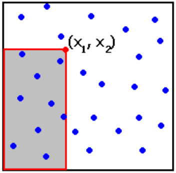

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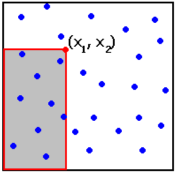

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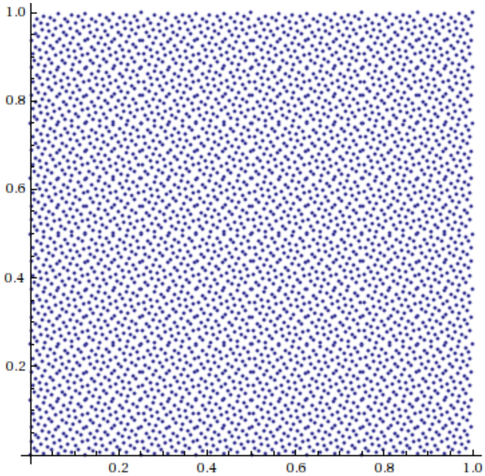
- construct a set with "low" discrepancy
- universal lower bounds on discrepancy

Low discrepancy sets



The van der Corput set with $N = 2^{12}$ points

$\left(0.x_1 x_2 \ldots x_n,\ 0.x_n x_{n-1} \ldots x_2 x_1\right), \quad x_k = 0 \text{ or } 1.$

Discrepancy $\approx \log N$

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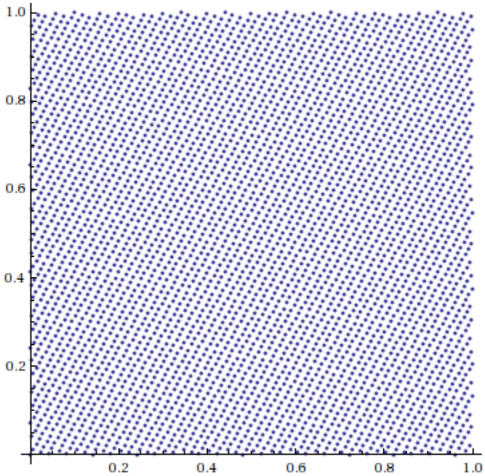

The irrational ($\alpha = \sqrt{2}$) lattice with $N = 2^{12}$ points

$$\left(n/N, \{n\alpha\}\right), \quad n = 0, 1, ..., N-1.$$

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Discrepancy estimates

L^p norm ($1 < p < \infty$):

$\|D_N\|_p \gtrsim (\log N)^{\frac{d-1}{2}}$ [Roth '54, Schmidt '77]

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Main idea: $D_N \approx \sum_{R:\,|R|\approx\frac{1}{N}} \frac{\langle D_N, h_R\rangle}{|R|} h_R$

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Previously, no formal connection between SBI and discrepancy.

Discrepancy estimates	**Small Ball inequality (signed)**
Dimension $d = 2$	
$\|D_N\|_\infty \gtrsim \log N$	$\Big\| \sum_{\lvert R\rvert = 2^{-n}} \varepsilon_R h_R \Big\|_\infty \gtrsim n$
Higher dimensions, L^2 bounds	
$\|D_N\|_2 \gtrsim (\log N)^{(d-1)/2}$	$\Big\| \sum_{\lvert R\rvert = 2^{-n}} \varepsilon_R h_R \Big\|_2 \gtrsim n^{(d-1)/2}$
Higher dimensions, conjecture	
$\|D_N\|_\infty \gtrsim (\log N)^{d/2}$	$\Big\| \sum_{\lvert R\rvert = 2^{-n}} \varepsilon_R h_R \Big\|_\infty \gtrsim n^{d/2}$
Higher dimensions, known results	
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- Best known: $C \approx \frac{\varepsilon}{\log(1/\varepsilon)}$
- Conjecture: $C \approx \varepsilon$.
- Construct "extremal" sequences n_k (best construction - $C \approx \sqrt{\varepsilon}$).

Discrepancy function $D_N(x) = \#\{\mathcal{P}_N \cap [0,x)\} - Nx_1x_2$	**Lacunary Fourier series** $f(x) \sim \sum_{k=1}^{\infty} c_k \sin n_k x$, $\frac{n_{k+1}}{n_k} > \lambda > 1$
$\|D_N\|_2 \gtrsim \sqrt{\log N}$ (Roth, '54)	$\|f\|_2 \equiv \sqrt{\sum \|c_k\|^2}$
$\|D_N\|_\infty \gtrsim \log N$ (Schmidt, '72; Halász, '81)	$\|f\|_\infty \gtrsim \sum \|c_k\|$ (Sidon, '27)
$\|D_N\|_1 \gtrsim \sqrt{\log N}$ (Halász, '81)	$\|f\|_1 \gtrsim \|f\|_2$ (Sidon, '30)

Nets

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- there are no perfect $(t = 0)$ b-adic nets in $d > b + 1$
- for $d \geq 2$ there is $t = t(d)$ so that (t,m,d)-nets exist in any base b

Main result

Theorem (Bilyk, F.)

- *the SBI holds in* $d = 2$ *(new, elementary proof).*
- $(0, n+1, 2)$*-net* $\iff$ *extremal set for SSBI*
  (i.e., $\arg\max \sum_{|R|=2^{-n}} \varepsilon_R h_R$ *with some* $\varepsilon_R = \pm 1$*)*
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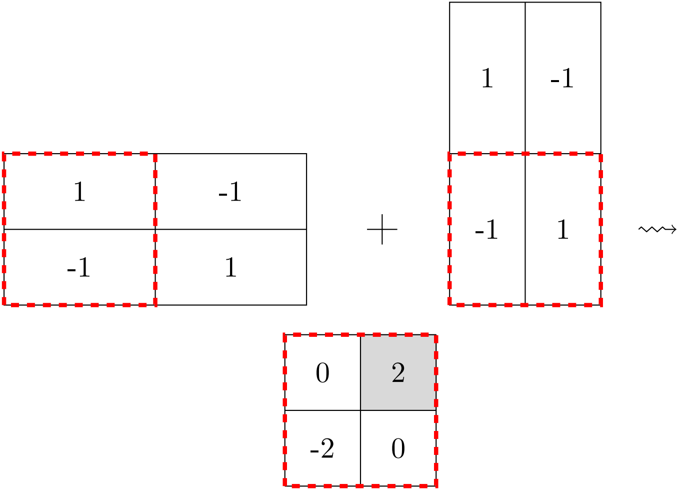
Reminder: SSBI in $d = 2$

For any $n \in \mathbb{N}$ and $\varepsilon_R \in \{\pm 1\}$:

$$\Big\| \sum_{|R|=2^{-n}} \varepsilon_R h_R \Big\|_\infty = n + 1$$

A new proof in $d = 2$: signed case



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- In the end we have 2^{n+1} cubes Q_j of size $2^{-(n+1)} \times 2^{-(n+1)}$, on which all $F_k = +2$. Then on each Q_j

$$\sum_{|R|=2^{-n}} \varepsilon_R h_R(x) = \sum_{k=\frac{n+1}{2}}^{n} F_k(x) = \frac{n+1}{2} \cdot 2 = n+1.$$

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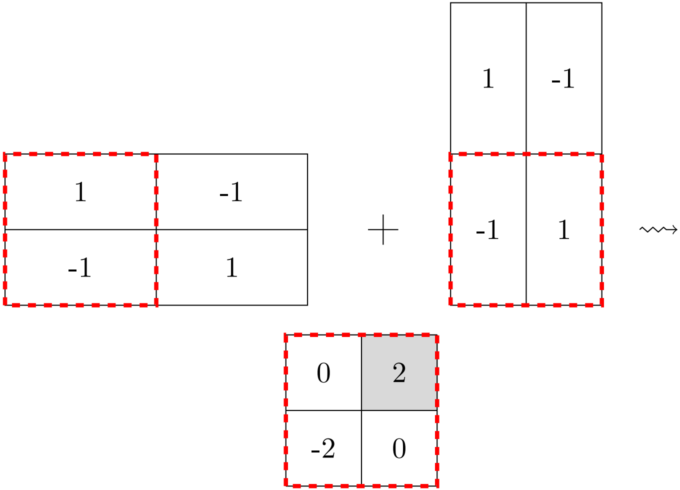

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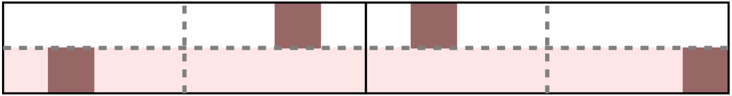

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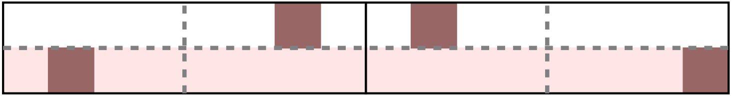
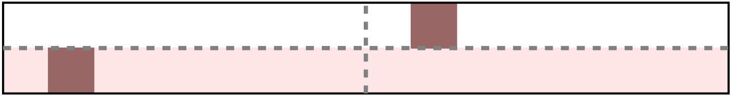


- We further choose a sub square in each of those and they have to lie in the opposite quarters of R.

Connection to binary nets

- Since every dyadic R with $|R| = 2^{-n}$ contains exactly two of the 2^{n+1} chosen squares, the extremal set is a $(1, n+1, 2)$-net in base $b = 2$.



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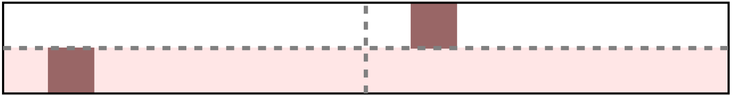

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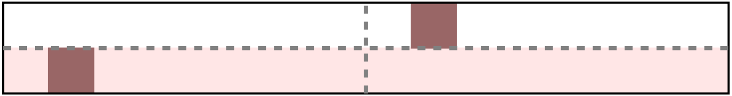

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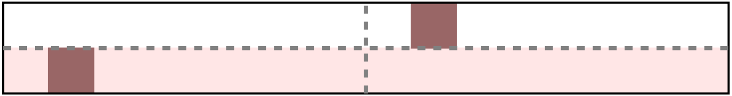

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- Each dyadic $(0, n+1, 2)$-net $\mathcal{P}$ may be obtained this way (may choose ε_R so that all terms are $+1$ on the net!)
- The total number of different binary $(0, m, 2)$-nets is

$$2^{\#\{R:|R|=2^{-n}\}} = 2^{(n+1)2^n}$$

Examples of two-dimensional nets

- $\varepsilon_R \equiv +1$: Van der Corput set.
  $\big(0.x_1 x_2 ... x_n,\ 0.x_n x_{n-1} ... x_2 x_1\big), \quad x_k = 0 \text{ or } 1$

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- If the coefficients have product structure, i.e.
  $\varepsilon_{R_1 \times R_2} = \varepsilon_{R_1} \cdot \varepsilon_{R_2}$: Owen's scrambling of VdC.
  SSBI proved in all dimensions [Karslidis 2015].

A new proof in $d = 2$: general case

At each step choose the subcube Q_j where

$$F_k(x) = |\alpha_{R'}| + |\alpha_{R''}|.$$

Then

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Let $d \geq 2$. Assume that in dimension $d' = d - 1$ for all $\varepsilon_R = \pm 1$ we have:

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- In dimension $d = 2$ equivalent.
- $\big\| \sum_{|R| \geq 2^{-n}} \varepsilon_R h_R \big\|_2 \gtrsim n^{d'/2}$
- $d = 2 \Rightarrow d' = 1$: the bound $\big\| \sum_{|I| \geq 2^{-n}} \varepsilon_I h_I \big\|_\infty \geq n$ is trivial.
- $d = 3$: $\sum_k \sum_{|R| = 2^{-k}} g_k$, where $g_k \sim \text{Bin}(k, 1/2)$ - perhaps $\cap \{g_k > \sqrt{k}\} \neq \phi$?

Dimension reduction: general case

In dimension $d' = 1$ a proper analog would be:

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- Unfortunately this inequality is NOT true in general! (counter-example by Ohad Feldheim, with $\alpha_I \in \{0, 1\}$)

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- $\Psi \stackrel{\text{def}}{=} \prod_{k=0}^{n} (1 + f_k) = \begin{cases} 2^{n+1} & \text{if } f_k = +1 \text{ for all } k, \\ 0 & \text{otherwise.} \end{cases}$

Proofs of Sidon's theorem:

1. Riesz product: $\prod_{k=1}^{K}(1+\varepsilon_k \cos n_k x)$
2. "zooming in": suppose $n_{k+1}/n_k \geq 9$. At step k look at

$$B_k = \{x \in [0,1] : \quad \alpha_k \sin(2\pi n_k x) \geq \frac{1}{2}|\alpha_k|\}$$

Each interval of B_k contains at least 3 periods of $\sin(2\pi n_{k+1}x)$, in particular contains an interval of B_{k+1}. On $\cap B_k$ we have $\sum_k \alpha_k \sin(2\pi n_k x) \geq \frac{1}{2}\sum_k |\alpha_k|$.

Thank you.

b-adic nets

Theorem

Fix $m \in \mathbb{N}$ and $b \geq 2$. For each $R \in \mathcal{D}_b^2$ with $|R| = b^{-(m-1)}$, choose a function $\phi_R \in \mathcal{H}_R$.

(i) *b-adic SSBI holds:* $\max_{x \in [0,1)^2} \sum_{|R| = b^{-(m-1)}} \phi_R(x) \ = m$.

(ii) *The set on which the maximum is achieved is a $(0, m, 2)$-net in base b.*

(iii) *Each $(0, m, 2)$-net in base b may be obtained this way*

(iv) *The number of different $(0, m, 2)$-nets in base b is $(b!)^{mb^{m-1}}$.*

$\phi_R \in \mathcal{H}_R$

-1	1	-1
-1	-1	1
1	-1	-1