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Typical height of the (2+1)-D Solid-on-Solid surface with pinning above a wall in the delocalized phase

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Abstract

We study the typical height of the (2+1)-dimensional solid-on-solid surface with pinning interacting with an impenetrable wall in the delocalization phase. More precisely, let Λ_N be a $N \times N$ box of $\mathbb{Z}^2$, and we consider a nonnegative integer-valued field $(\phi(x))_{x\in\Lambda_N}$ with zero boundary conditions (*i.e.* $\phi|_{\Lambda_N^{\complement}} = 0$) associated with the energy functional

$$\mathcal{V}(\phi) = \beta \sum_{x\sim y} |\phi(x) - \phi(y)| - \sum_{x} h\mathbf{1}_{\{\phi(x)=0\}},$$

where $\beta > 0$ is the inverse temperature and $h \geq 0$ is the pinning parameter. Lacoin has shown that for sufficiently large β, there is a phase transition between delocalization and localization at the critical point

$$h_w(\beta) = \log\left(\frac{e^{4\beta}}{e^{4\beta} - 1}\right).$$

In this paper we show that for $\beta \geq 1$ and $h \in (0, h_w)$, the values of ϕ concentrate at the height $H = \lfloor (4\beta)^{-1} \log N \rfloor$ with constant order fluctuations. Moreover, at criticality $h = h_w$, we provide evidence for the conjectured typical height $H_w = \lfloor (6\beta)^{-1} \log N \rfloor$.

Keywords: Random surface; Solid-On-Solid; Wetting; Typical height; Delocalization behavior

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1. Introduction

1.1. Background

The solid-on-solid (SOS) model, introduced in [5,21], is a crystal surface model, which acts as a qualitative approximation of the Ising model in low temperature (see [6] for more details).

Now we formally describe the $(d+1)$-dimensional solid-on-solid model on the lattice $\mathbb{Z}^d$. Let $\Lambda_N := [\![1, N]\!]^d$ denote a box of size N in the lattice $\mathbb{Z}^d$ and we define its external boundary to be

$$\partial\Lambda_N := \left\{x \in \mathbb{Z}^d \setminus \Lambda_N : \ \exists y \in \Lambda_N, x \sim y\right\}$$

where $x \sim y$ denotes that x and y are nearest neighbors in the lattice $\mathbb{Z}^d$. Given $\phi \in \widetilde{\Omega}_{\Lambda_N} := \mathbb{Z}^{\Lambda_N}$, we define the Hamiltonian for the solid-on-solid model with zero boundary condition as

$$\mathcal{H}_N(\phi) := \sum_{\substack{\{x,y\}\subset\Lambda_N \\ x\sim y}} |\phi(x)-\phi(y)| + \sum_{\substack{x\in\Lambda_N,\ y\in\partial\Lambda_N \\ x\sim y}} |\phi(x)|. \tag{1}$$

Then for $\beta > 0$ (inverse temperature), we define a probability measure on $\widetilde{\Omega}_N = \mathbb{Z}^{\Lambda_N}$ as follows

$$\forall\phi \in \widetilde{\Omega}_N, \qquad \mathbf{P}_N^\beta(\phi) := \frac{1}{\widetilde{\mathcal{Z}}_N^\beta} e^{-\beta\mathcal{H}_N(\phi)} \tag{2}$$

where

$$\widetilde{\mathcal{Z}}_N^\beta := \sum_{\psi\in\widetilde{\Omega}_N} e^{-\beta\mathcal{H}_N(\psi)} \leq \left(\frac{1+e^{-d\beta}}{1-e^{-d\beta}}\right)^{|\Lambda_N|}$$

and we refer to [18, Equations (3.8)-(3.10)] for a proof of the last inequality. It is known (see [9,21,22]) that for any $\beta > 0$, the $(1+1)$-dimensional SOS surface is rough (delocalized), which means the expectation of the absolute value of the height at the center diverges in the thermodynamic limit. However, for $d \geq 3$, it is shown in [4] by Peierls argument that for any $\beta > 0$, the $(d+1)$-dimensional SOS surface is rigid (localized), that is, the expectation of the absolute value of the height at the center is uniformly bounded. The interesting case is $d = 2$ which exhibits a phase transition between rough (for small β, cf. [10–12]) and rigid (for large β, cf. [2,13]). Moreover, numerical simulations suggest that $\beta_c \approx 0.806$ is where the delocalization/localization transition occurs [6].

1.2. The $(2+1)$-dimensional SOS surface above a wall

The probability distribution of the $(2+1)$-dimensional SOS interface above an impenetrable wall (taking non-negative integer values) is the conditional distribution

$$\forall\phi \in \Omega_N := \left\{\phi \in \widetilde{\Omega}_N : \phi \geq 0\right\}, \qquad \mathbb{P}_N^\beta(\phi) := \mathbf{P}_N^\beta(\phi)/\mathbf{P}_N^\beta(\Omega_N). \tag{3}$$

In [3, Theorem 4.1], Bricmont, Mellouki, and Fröhlich showed that for large β, the average height of the surface satisfies

$$\frac{1}{C\beta}\log N \leq \frac{1}{N^2}\mathbb{E}_N^\beta\left[\sum_{x\in\Lambda_N}\phi(x)\right] \leq \frac{C}{\beta}\log N$$

where $\mathbb{E}_N^\beta$ is the expectation corresponding to the law $\mathbb{P}_N^\beta$. Later in [6], Caputo, Lubetzky, Martinelli, Sly and Toninelli showed that for $\beta \geq 1$, the typical height of the surface concentrates at

$$H = \left\lfloor \frac{1}{4\beta} \log N \right\rfloor$$

with fluctuations of order $O(1)$, where $\lfloor x \rfloor := \sup\{n \in \mathbb{Z} : \ n \leq x\}$, as follows.

Theorem A (*[6, Theorem 3.1]*). *There exist two universal constants $C, K > 0$ such that for all $\beta \geq 1$ and all integer $k \geq K$, we have for all N,*

$$\mathbb{P}_N^\beta \left(|\{x \in \Lambda_N : \ \phi(x) \geq H + k\}| > e^{-2\beta k} N^2 \right) \leq e^{-Ce^{-2\beta k} N \left(1 \wedge e^{-2\beta k} N \log^{-8} N\right)},$$

and

$$\mathbb{P}_N^\beta \left(|\{x \in \Lambda_N : \ \phi(x) \leq H - k\}| > e^{-2\beta k} N^2 \right) \leq e^{-e^{\beta k} N}.$$

This result describes the effect of the impenetrable wall in the large β regime, as the surface is pushed up to the height of order $\frac{1}{4\beta} \log N$, instead of remaining uniformly bounded when no wall is present. This effect is often called *entropic repulsion*. Furthermore, in [7] these authors provided a full description of the macroscopic shape of the SOS surface, including the scaling limit and fluctuations of the rescaled macroscopic level lines. In particular, they show in [7, Theorem 1] that the surface concentrates on two values: H and $H - 1$. Moreover, concerning 3D Ising interfaces conditioned to stay above a floor at a negative level, Gheissari and Lubetzky [14] proved a phase transition in the occurrence of entropic repulsion when variating the level of the floor. Thus, similar results are believed to hold for $(2 + 1)$-D SOS surface above a hard wall.

1.3. The $(2 + 1)$-dimensional SOS surface with pinning above a wall

In this paper, we are interested in the case where the $(2+1)$-dimensional SOS surface above a wall interacts with a pinning (or wetting) attraction to the wall.

More precisely, we model this surface in the box $\Lambda_N \subset \mathbb{Z}^2$ by an element of $\Omega_N = \mathbb{Z}_+^{\Lambda_N}$, where $\mathbb{Z}_+ := \mathbb{Z} \cap [0, \infty)$. Given $\beta > 0$ and $h \geq 0$, we define the probability measure for the $(2 + 1)$-dimensional SOS surface above a wall with zero boundary conditions and pinning reward h, namely $\mathbb{P}_N^{\beta,h}$ on Ω_N, by

$$\mathbb{P}_N^{\beta,h}(\phi) := \frac{1}{\mathcal{Z}_N^{\beta,h}} e^{-\beta \mathcal{H}_N(\phi) + h|\{x \in \Lambda_N : \ \phi(x)=0\}|}, \tag{4}$$

where

$$\mathcal{Z}_N^{\beta,h} := \sum_{\phi \in \Omega_N} e^{-\beta \mathcal{H}_N(\phi) + h|\{x \in \Lambda : \ \phi(x)=0\}|} \leq e^{h|\Lambda_N|} \left(\frac{1 + e^{-2\beta}}{1 - e^{-2\beta}} \right)^{|\Lambda_N|}. \tag{5}$$

By [18, Equation (2.9)], we know the existence of the following limit

$$\mathrm{F}(\beta, h) := \lim_{N \to \infty} \frac{1}{N^2} \log \mathcal{Z}_N^{\beta,h}$$

which is called the free energy. By Hölder's inequality, for $\theta \in [0, 1]$ we have

$$\mathcal{Z}_N^{\beta,\theta h_1 + (1-\theta) h_2} \leq \left(\mathcal{Z}_N^{\beta,h_1} \right)^\theta \cdot \left(\mathcal{Z}_N^{\beta,h_2} \right)^{1-\theta},$$

and then $\mathrm{F}(\beta, h)$ is increasing and convex in h since $\mathrm{F}(\beta, h)$ is the limit of a sequence of increasing and convex functions in h. Therefore, at points where $\mathrm{F}(\beta, h)$ is differentiable in h, the convexity (cf. [15, Appendix A.1.1]) allows us to exchange the order of limit and derivative to obtain the asymptotic contact fraction

$$\partial_h \mathrm{F}(\beta, h) = \lim_{N\to\infty} \frac{1}{N^2} \mathbb{E}_N^{\beta,h}\left[|\phi^{-1}(0)|\right],$$

where we have used the notation $\phi^{-1}(A) := \{x \in \Lambda_N : \phi(x) \in A\}$ for $A \subset \mathbb{Z}$, and $\phi^{-1}(k) := \phi^{-1}(\{k\})$ for $k \in \mathbb{Z}$. In [8], Chalker showed that there exists a critical value

$$h_w(\beta) := \sup\{h \in \mathbb{R}_+ : \mathrm{F}(\beta, h) = \mathrm{F}(\beta, 0)\} \tag{6}$$

which is positive for all $\beta > 0$, thus separating the delocalized phase ($\partial_h \mathrm{F}(\beta, h) = 0$) from the localized phase ($\partial_h \mathrm{F}(\beta, h) > 0$). We refer to the surveys [17,23] for a comprehensive bibliography on the subject of localization/delocalization of surface models. Chalker further showed that for all $\beta > 0$,

$$\log\left(\frac{e^{4\beta}}{e^{4\beta}-1}\right) \le h_w(\beta) \le \log\left(\frac{16(e^{4\beta}+1)}{e^{4\beta}-1}\right). \tag{7}$$

Later, Alexander, Dunlop and Miracle-Solé [1] showed that the lower bound in (7) is asymptotically sharp, and when h decreases to h_w the system undergoes a sequence of layering transitions (*i.e.* the typical height of the surface varies as h decreases to h_w). More recently, Lacoin proved in [18, Proposition 5.1] that for $\beta > \beta_1$ (where $\beta_1 \in (\log 2, \log 3)$ is given by [18, (2.20)]), we have

$$h_w(\beta) = \log\left(\frac{e^{4\beta}}{e^{4\beta}-1}\right), \tag{8}$$

and there exists a constant C_β such that

$$\forall u \in (0, 1], \quad C_\beta^{-1} u^3 \le \mathrm{F}(\beta, u + h_w(\beta)) - \mathrm{F}(\beta, h_w(\beta)) \le C_\beta u^3.$$

In fact, this constant C_β can be determined more precisely under additional conditions on β, for which we refer to [18, Theorem 2.1]. Furthermore, when $h > h_w$, a complete picture of the typical height, the Gibbs states and regularity of the free energy is provided in [19].

1.4. Subcritical regime

In this paper, our goal is to describe the typical height of the $(2+1)$-dimensional SOS surface above a wall with pinning parameter $h \in (0, h_w)$. Our main result is a generalization of Theorem A to the subcritical pinning regime. We note that for $h \in (0, h_w)$ we have $e^{-h} + e^{-4\beta} > 1$, and then define for $\delta > 0$,

$$\kappa(\beta, h, \delta) := \frac{4\beta + \delta}{\log\left(e^{-h} + e^{-4\beta}\right)}. \tag{9}$$

Theorem 1.1. *Fix $\beta \ge 1$, $h \in (0, h_w)$ and $N \ge 1$. Let $H = \left\lfloor \frac{1}{4\beta} \log N \right\rfloor$.*

(i) There exist two universal constants $C, K > 0$ such that for all integer $m \ge K$,

$$\mathbb{P}_N^{\beta,h}\left(\left|\phi^{-1}([H+m, \infty))\right| > e^{-2\beta m} N^2\right) \le e^{-C e^{-2\beta m} N \left(1 \wedge e^{-2\beta m} N \log^{-8} N\right)}.$$

(ii) For $\delta > 0$ and $m \in \mathbb{N}$ we have

$$\mathbb{P}_N^{\beta,h}\left(\left|\phi^{-1}([0, H-m])\right| > 2e^{-2\beta m}N^2\right) \le 3e^{-\min\left(\frac{1}{2}e^{2\beta m}-4\beta(1+\kappa),\ \delta\right)N},$$

where κ is defined in (9).

We expect that similarly, for $h \in (0, h_w)$, the surface concentrates on two values $H - c(h)$ and $H - c(h) - 1$ where $c(h) \in \mathbb{Z}_+$ tends to infinity when h tends to h_w.

1.5. Behavior at criticality

Next we consider the behavior at critical $h = h_w$, as defined in (6). Our main result is that the amount of non-isolated zeros is at most of order N with high probability. For $\phi \in \Omega_N$, we define its isolated and non-isolated zeros to be respectively

$$\begin{aligned} q_1(\phi) &:= \{x \in \Lambda_N : \ \phi(x) = 0, \forall y \in \Lambda_N, y \sim x, \phi(y) \ge 1\}, \\ q_{2+}(\phi) &:= \{x \in \Lambda_N : \ \phi(x) = 0, \exists y \in \Lambda_N, y \sim x, \phi(y) = 0\}. \end{aligned} \tag{10}$$

We prove the following theorem.

Theorem 1.2. *For $\beta \ge 1$ and $h = h_w$, we have for all $N \in \mathbb{N}$ and $C > 0$,*

$$\mathbb{P}_N^{\beta,h_w}\left(\phi \in \Omega_N : \ |q_{2+}(\phi)| \ge CN\right) \le e^{-N\left(\frac{C}{20}e^{-6\beta}-4\beta\right)}.$$

When $h = h_w$, it is conjectured that the surface height concentrates around the value

$$H_w = \left\lfloor \frac{1}{6\beta}\log N \right\rfloor, \tag{11}$$

with fluctuations similar to Theorem 1.1 [20]. The intuition for this different typical height is a balance at criticality between entropic cost of lifting the surface up and the reward for isolated zeros. Theorem 1.2 indicates that non-isolated zeros should not contribute to this balance.

Our last result gives further evidence for the conjecture. We show that the probability of downwards fluctuations from the conjectured typical height H_w is very small, if the amount of zeros is at most of order $N^{4/3}$.

Proposition 1.3. *For all $\beta \ge 1$, $C > 0$, $h = h_w$, $N \in \mathbb{N}$ and $m \in \mathbb{N}$, letting $H_w = \lfloor \frac{1}{6\beta}\log N \rfloor$ we have*

$$\begin{aligned} &\mathbb{P}_N^{\beta,h_w}\left(\left\{|\phi^{-1}(0)| \le CN^{\frac{4}{3}}\right\} \bigcap \left\{\left|\phi^{-1}([1, H_w - m])\right| \ge 2e^{-2\beta m}N^2\right\}\right) \\ &\quad \le 2\exp\left(4\beta N + 4\beta CN^{\frac{4}{3}} - \tfrac{1}{2}e^{2\beta m}N^{\frac{4}{3}}\right). \end{aligned}$$

As a consequence of Theorem 1.2 and Proposition 1.3, it is enough to prove that for large enough $C > 0$, we have

$$\mathbb{P}_N^{\beta,h_w}\left(|q_1(\phi)| > CN^{4/3}\right) = o(1), \quad N \to \infty \tag{12}$$

in order to obtain a lower bound on the typical height of the surface at criticality, matching the conjectured height in (11).

1.6. Open problems and heuristic arguments

Subcritical regime. Theorem 1.1 opens the door to more advanced questions about the structure of the surface when $h \in (0, h_w)$. Shape results similar to [7] are expected to hold. In particular, we expect that for $h \in (0, h_w)$, the surface concentrates on two values $H - c(h)$ and $H - c(h) - 1$, where $c(h) \in \mathbb{Z}_+$ tends to infinity when h tends to h_w.

At criticality. The behavior of the surface at $h = h_w$ remains to be analyzed. In particular, proving an analogous result to Theorem 1.1 with typical height as in (11) is of interest, as are more advanced shape results (similar to [7, Theorems 1 and 2]). As mentioned earlier, our results would imply a lower bound matching this typical height, provided that (12) holds. We turn to offer heuristic arguments for the conjectured estimates (11) regarding the typical height and (12) regarding the number of isolated zeros.

For large β (low temperature), neighboring vertices tend to have close values. Thus the surface is expected to be nearly flat — most of the vertices will take a certain value H_w, up to the inner boundary. Nonetheless, some vertices will take the value zero (even without pinning, as proved in [3]). As the zeros are believed to have a spatial mixing property (see the intuitive argument of [18, Equation (3.13)]), their appearance could be approximated by an i.i.d. process, where each vertex is an isolated zero with probability $e^{-4\beta H_w}$ (stemming from having 4 neighbors at height H_w) and two adjacent vertices are a pair of zeros with probability $e^{-6\beta H_w}$ (stemming from having 6 neighbors at height H_w). At criticality, the pinning reward is not influenced at all by isolated zeros (see (24)), and the main contribution is therefore from pairs of neighboring zeros, whose amount is typically $N^2 e^{-6\beta H_w}$. The main penalty to the Hamiltonian (1) is from lifting the inner boundary up to H_w while the outer boundary remains at 0, thus contributing $e^{-4\beta N H_w}$ to the computation of probability in (4). The value of H_w is one which balances these two factors, that is,

$$\exp\left(h_w N^2 e^{-6\beta H_w}\right) e^{-4\beta N H_w} \asymp 1,$$

which implies $H_w = \lfloor \frac{1}{6\beta} \log N \rfloor$ as in (11). Now, by the i.i.d. model for zeros, the mean number of isolated zeros is

$$N^2 e^{-4\beta H_w} = N^{4/3},$$

which implies (12).

1.7. Outline of the paper

The paper is organized as follows. Section 2 is devoted to Theorem 1.1–(i) about upward fluctuations in the subcritical regime. Section 3 is about Theorem 1.1–(ii) concerning downward fluctuations in the subcritical regime. In Section 4, we prove Theorem 1.2 and Proposition 1.3 at criticality.

2. Theorem 1.1-(1): Upward fluctuations for $h \in (0, h_w)$

Intuitively, the height of the (2+1)-dimensional SOS surface above a wall with pinning (*i.e.* $h \geq 0$) is stochastically dominated by that without pinning (*i.e.* $h = 0$). We use this comparison between $\mathbb{P}_N^{\beta,h}$ and $\mathbb{P}_N^{\beta,0}$ to prove part (i), where $\mathbb{P}_N^{\beta,0} = \mathbb{P}_N^{\beta}$ is defined in Section 1.2.

2.1. Partial order and stochastic domination

We define a partial order "$\leq$" on $\Omega_N \times \Omega_N$ as follows

$$\phi \leq \psi \quad \Leftrightarrow \quad \forall x \in \Lambda_N,\ \phi(x) \leq \psi(x).$$

Moreover, a function $f : \Omega_N \mapsto \mathbb{R}$ is increasing if

$$\phi \le \psi \quad \Rightarrow \quad f(\phi) \le f(\psi).$$

Similarly, an event $\mathcal{A} \subset \Omega_N$ is increasing if its indicator function $\mathbf{1}_{\mathcal{A}}$ is increasing. For two probability measures μ_1, μ_2 on Ω_N, we say that μ_2 dominates μ_1, denoted by $\mu_1 \preceq \mu_2$, if for any bounded increasing function $f : \Omega_N \mapsto \mathbb{R}$, we have

$$\mu_1(f) \le \mu_2(f).$$

Lemma 2.1. *For all $\beta > 0$ and $0 \le h_1 \le h_2$, we have*

$$\mathbb{P}_N^{\beta,h_2} \preceq \mathbb{P}_N^{\beta,h_1}. \tag{13}$$

Proof. Since [16, Theorem 6] is applied for finite distributive lattice, we set

$$\mathcal{A}_n := \left\{ \phi \in \Omega_N : \max_{x \in \Lambda_N} \phi(x) \le n \right\}.$$

It is fundamental to verify Holley's condition [16, Equation (7)] to obtain

$$\mathbb{P}_N^{\beta,h_2}\left(\cdot \mid \mathcal{A}_n\right) \preceq \mathbb{P}_N^{\beta,h_1}\left(\cdot \mid \mathcal{A}_n\right),$$

and then for any bounded increasing function $f : \Omega_N \mapsto \mathbb{R}$ we have

$$\frac{\mathbb{E}_N^{\beta,h_2}[f \mathbf{1}_{\mathcal{A}_n}]}{\mathbb{P}_N^{\beta,h_2}(\mathcal{A}_n)} \le \frac{\mathbb{E}_N^{\beta,h_1}[f \mathbf{1}_{\mathcal{A}_n}]}{\mathbb{P}_N^{\beta,h_1}(\mathcal{A}_n)}. \tag{14}$$

Moreover, by the dominate convergence theorem, for all $h \ge 0$ we have

$$\mathbb{E}_N^{\beta,h}\left[f\right] = \lim_{n\to\infty} \frac{\mathbb{E}_N^{\beta,h}[f \mathbf{1}_{\mathcal{A}_n}]}{\mathbb{P}_N^{\beta,h}(\mathcal{A}_n)}. \tag{15}$$

Combining (14) and (15), we conclude the proof. □

2.2. *Proof of Theorem 1.1–(i).*

Note that for any integer m, the event

$$\left\{ \phi \in \Omega_N : \ |\{x \in \Lambda_N : \ \phi(x) \ge H + m\}| > e^{-2\beta m} N^2 \right\}$$

is increasing. We combine Lemma 2.1 and Theorem A to conclude the proof. □

3. Theorem 1.1-(ii): downward fluctuations for $h \in (0, h_w)$

To prove part (ii) of Theorem 1.1, we first show that $|\phi^{-1}(0)|$ is at most of order N, with high probability, adopting the strategy in [6, Theorem 3.1].

Lemma 3.1. *For all $\beta \ge 1$, $h \in [0, h_w)$, $\delta > 0$ and $N \ge 1$, we have*

$$\mathbb{P}_N^{\beta,h}\left(|\phi^{-1}(0)| \ge \kappa N \right) \le e^{-\delta N},$$

where $\kappa = \kappa(\beta, h, \delta)$ is defined in (9).

Proof. For $\phi \in \Omega_N$ and each $A \subseteq \phi^{-1}(0)$, we define $U_A\phi : \Lambda_N \mapsto \mathbb{Z}_+$ as follows

$$(U_A\phi)(x) := \begin{cases} \phi(x)+1, & \text{if } x \notin A, \\ 0, & \text{if } x \in A. \end{cases}$$

Since the action U_A increases the height of each site in $\Lambda_N \setminus A$ by one, we have

$$\begin{aligned} &\mathcal{H}_N(U_A\phi) \le \mathcal{H}_N(\phi) + 4|A| + 4N, \\ &|\phi^{-1}(0)| - |(U_A\phi)^{-1}(0)| = |\phi^{-1}(0) \setminus A|. \end{aligned} \tag{16}$$

Therefore,

$$\mathbb{P}_N^{\beta,h}(U_A\phi) \ge \mathbb{P}_N^{\beta,h}(\phi) \cdot \exp\left(-h|\phi^{-1}(0) \setminus A| - 4\beta|A| - 4\beta N\right),$$

and then

$$\begin{aligned} \sum_{A\subseteq\phi^{-1}(0)} \mathbb{P}_N^{\beta,h}(U_A\phi) &\ge e^{-4\beta N} \cdot \mathbb{P}_N^{\beta,h}(\phi) \sum_{A\subseteq\phi^{-1}(0)} \exp\left(-h|\phi^{-1}(0) \setminus A| - 4\beta|A|\right) \\ &= e^{-4\beta N - h|\phi^{-1}(0)|} \cdot \mathbb{P}_N^{\beta,h}(\phi) \sum_{n=0}^{|\phi^{-1}(0)|} \sum_{\substack{A\subseteq\phi^{-1}(0) \\ |A|=n}} \exp\left(-n(4\beta - h)\right) \\ &= e^{-4\beta N - h|\phi^{-1}(0)|} \left(1 + e^{-(4\beta-h)}\right)^{|\phi^{-1}(0)|} \mathbb{P}_N^{\beta,h}(\phi). \end{aligned} \tag{17}$$

Observe that for $A, A' \subseteq \phi^{-1}(0)$ with $A \neq A'$, we have

$$U_A\phi \neq U_{A'}\phi.$$

Furthermore, for $\phi \neq \psi$, if $A \subseteq \phi^{-1}(0)$ and $B \subseteq \psi^{-1}(0)$, we have

$$U_A\phi \neq U_B\psi,$$

because we can recover A from $U_A\phi$ by zero-value sites and then proceed to recover ϕ. Therefore $\sum_{\phi\in\Omega_N} \sum_{A\subset\phi^{-1}(0)} \mathbb{P}_N^{\beta,h}(U_A\phi) \le 1$. In particular, using (17) we obtain

$$\begin{aligned} 1 &\ge \sum_{\phi:\ |\phi^{-1}(0)|\ge\kappa N} \sum_{A\subset\phi^{-1}(0)} \mathbb{P}_N^{\beta,h}(U_A\phi) \\ &\ge \sum_{\phi:\ |\phi^{-1}(0)|\ge\kappa N} e^{-4\beta N - h|\phi^{-1}(0)|} \left(1 + e^{-(4\beta-h)}\right)^{|\phi^{-1}(0)|} \mathbb{P}_N^{\beta,h}(\phi) \\ &\ge e^{-4\beta N} \left(e^{-h} + e^{-4\beta}\right)^{\kappa N} \mathbb{P}_N^{\beta,h}\left(|\phi^{-1}(0)| \ge \kappa N\right) \end{aligned} \tag{18}$$

where in the last inequality we have used that $e^{-h} + e^{-4\beta} > 1$ for $h \in [0, h_w)$. By the definition of κ in (9) we have

$$\left(e^{-h} + e^{-4\beta}\right)^{\kappa} e^{-4\beta} = e^{\delta}.$$

Plugging this into (18), we conclude the proof of Lemma 3.1. $\square$

Lemma 3.2. *Let $\beta \ge 1$, $h \in [0, h_w)$ and $\kappa > 0$. Then for all $m > \lceil \frac{1}{2\beta} \log(8\beta(1+\kappa)) \rceil$ and $N \ge 1$ we have*

$$\mathbb{P}_N^{\beta,h}\left(\left\{|\phi^{-1}(0)| \le \kappa N\right\} \bigcap \left\{\left|\phi^{-1}([1, H-m])\right| \ge \frac{e^{-2\beta m}}{1 - e^{-2\beta}} N^2\right\}\right)$$
$$\le \frac{1}{1 - e^{-\beta N}} e^{-\left(\frac{1}{2} e^{2\beta m} - 4\beta(1+\kappa)\right)N}.$$

Remark 1. The condition $m > \frac{1}{2\beta}\log(8\beta(1+\kappa))$ is only to ensure that $\frac{1}{2}e^{2\beta m} - 4\beta(1+\kappa) > 0$.

Proof. Fix an integer $\ell \in [1, H-m]$. For any subset $A \subseteq \phi^{-1}(\ell)$, we define $V_A\phi : \Lambda_N \mapsto \mathbb{Z}_+$ as follows

$$(V_A\phi)(x) := \begin{cases} 0, & \text{if } x \in \phi^{-1}(0), \\ 1, & \text{if } x \in A, \\ \phi(x)+1, & \text{if } x \notin A \cup \phi^{-1}(0). \end{cases} \tag{19}$$

Observe that for $x \in A$ and $y \notin A \cup \phi^{-1}(0)$ with $x \sim y$,

$$|(V_A\phi)(x) - (V_A\phi)(y)| = \phi(y) \le |\ell - \phi(y)| + \ell,$$

and then

$$\mathcal{H}_N(V_A\phi) \le \mathcal{H}_N(\phi) + 4N + 4|\phi^{-1}(0)| + 4\ell|A|.$$

Moreover, as $|(V_A\phi)^{-1}(0)| = |\phi^{-1}(0)|$, we obtain

$$\mathbb{P}_N^{\beta,h}(V_A\phi) \ge \mathbb{P}_N^{\beta,h}(\phi)\, e^{-4\beta N - 4\beta|\phi^{-1}(0)| - 4\beta\ell|A|}.$$

Similarly to (17), we have

$$\begin{aligned} \sum_{A\subseteq\phi^{-1}(\ell)} \mathbb{P}_N^{\beta,h}(V_A\phi) &\ge \mathbb{P}_N^{\beta,h}(\phi) \sum_{A\subseteq\phi^{-1}(\ell)} e^{-4\beta N - 4\beta|\phi^{-1}(0)| - 4\beta\ell|A|} \\ &= \mathbb{P}_N^{\beta,h}(\phi)\, e^{-4\beta N - 4\beta|\phi^{-1}(0)|}\left(1 + e^{-4\beta\ell}\right)^{|\phi^{-1}(\ell)|} \\ &\ge \mathbb{P}_N^{\beta,h}(\phi) \exp\left(-4\beta N - 4\beta|\phi^{-1}(0)| + \tfrac{1}{2}e^{-4\beta\ell}|\phi^{-1}(\ell)|\right) \end{aligned} \tag{20}$$

where we have used $(1+x) \ge e^{x/2}$ for $x \in [0,1]$ in the last inequality.

Note that for $A, A' \subseteq \phi^{-1}(\ell)$ with $A \ne A'$, we have

$$V_A\phi \ne V_{A'}\phi.$$

Moreover, for $\phi \ne \psi \in \Omega_N$, $A \subset \phi^{-1}(\ell)$ and $B \subset \psi^{-1}(\ell)$, we have

$$V_A\phi \ne V_B\psi,$$

since we can recover A by $1-$ valued sites of $V_A\phi$ and then proceed to recover ϕ. Therefore, by (20), denoting $j = H - \ell$ we obtain

$$\begin{aligned} 1 &\ge \sum_{\substack{\phi:\, |\phi^{-1}(\ell)| \ge e^{-2\beta j}N^2 \\ |\phi^{-1}(0)| \le \kappa N}} \sum_{A\subset\phi^{-1}(\ell)} \mathbb{P}_N^{\beta,h}(V_A\phi) \\ &\ge \exp\left(-4\beta N - 4\beta\kappa N + \tfrac{1}{2}e^{2\beta j}N\right)\mathbb{P}_N^{\beta,h}\left(\{|\phi^{-1}(\ell)| \ge e^{-2\beta j}N^2\} \cap \{|\phi^{-1}(0)| \le \kappa N\}\right). \end{aligned} \tag{21}$$

Moreover, as

$$\left\{|\phi^{-1}([1, H-m])| \ge \frac{e^{-2\beta m}}{1 - e^{-2\beta}}N^2\right\} \subset \bigcup_{j=m}^{H-1}\left\{|\phi^{-1}(H-j)| \ge e^{-2\beta j}N^2\right\},$$

by union bound and (21) we obtain

$$\begin{aligned}
&\mathbb{P}_N^{\beta,h}\left(\left\{|\phi^{-1}([1,H-m])|\geq\frac{e^{-2\beta m}}{1-e^{-2\beta}}N^2\right\}\bigcap\left\{|\phi^{-1}(0)|\leq\kappa N\right\}\right)\\
&\quad\leq\sum_{j=m}^{H-1}\mathbb{P}_N^{\beta,h}\left(\{|\phi^{-1}(H-j)|\geq e^{-2\beta j}N^2\}\bigcap\left\{|\phi^{-1}(0)|\leq\kappa N\right\}\right)\\
&\quad\leq\sum_{j=m}^{H-1}\exp\left(4\beta N+4\beta\kappa N-\tfrac{1}{2}e^{2\beta j}N\right)\\
&\quad\leq\frac{1}{1-e^{-\beta N}}\exp\left(4\beta N+4\beta\kappa N-\tfrac{1}{2}e^{2\beta m}N\right),
\end{aligned}$$

where in the last inequality we have used that for $j\geq 0$,

$$\frac{\exp\left(-\frac{1}{2}e^{2\beta(j+1)}N\right)}{\exp\left(-\frac{1}{2}e^{2\beta j}N\right)}\leq\exp\left(-\beta e^{2\beta j}N\right)\leq e^{-\beta N}.$$

This concludes the proof. □

3.1. *Proof of* Theorem 1.1–(ii)

For all $N\geq 1$, we have

$$\begin{aligned}
&\mathbb{P}_N^{\beta,h}\left(\left|\phi^{-1}([0,H-m])\right|>2e^{-2\beta m}N^2\right)\\
&\quad\leq\mathbb{P}_N^{\beta,h}\left(\left|\phi^{-1}(0)\right|>\kappa N\right)+\\
&\qquad\mathbb{P}_N^{\beta,h}\left(\left\{|\phi^{-1}(0)|\leq\kappa N\right\}\bigcap\left\{\left|\phi^{-1}([1,H-m])\right|\geq\frac{e^{-2\beta m}}{1-e^{-2\beta}}N^2\right\}\right)\\
&\quad\leq e^{-\delta N}+\frac{1}{1-e^{-\beta N}}\exp\left(-\left(\tfrac{1}{2}e^{2\beta m}-4\beta(1+\kappa)\right)N\right)\\
&\quad\leq 3\exp\left(-\min\left(\tfrac{1}{2}e^{2\beta m}-4\beta(1+\kappa),\delta\right)N\right),
\end{aligned}$$

where we have applied Lemmas 3.1 and 3.2 in the second inequality. □

4. Theorem 1.2 : Upper bound on non-isolated zeros at criticality

This section is devoted to the proof of Theorem 1.2. Inspired by [18, Lemma 3.1], we first observe that for $x_1,x_2,x_3,x_4\in\mathbb{Z}_+$,

$$\sum_{k=-\infty}^{0}\exp\left(-\beta\sum_{i=1}^{4}|x_i-k|\right)=\exp\left(h_w-\beta\sum_{i=1}^{4}x_i\right).\tag{22}$$

Define a new state space

$$\Omega_N^*:=\{\psi:\ \Lambda_N\to\mathbb{Z}\mid\text{if }\psi(x)\leq -1,\forall y\in\Lambda_N,y\sim x,\psi(y)\geq 1\}.\tag{23}$$

Notice that if $\psi\in\Omega_N^*$, then $\max(\psi,0)\in\Omega_N$ (as defined in (3)). By (22), we have

$$\mathcal{Z}_N^{\beta,h_w}=\sum_{\psi\in\Omega_N^*}\exp\left(-\beta\mathcal{H}_N(\psi)+h_w|q_{2+}(\psi)|\right)$$

where $\mathcal{Z}_N^{\beta,h_w}$ is defined in (5) and $\mathcal{H}_N$ is defined in (1) with zero boundary condition. Define a new probability measure $\widetilde{\mathbb{P}}_N$ on Ω_N^* as follows:

$$\forall \psi \in \Omega_N^*, \quad \widetilde{\mathbb{P}}_N(\psi) := \frac{1}{\mathcal{Z}_N^{\beta,h_w}} \exp\left(-\beta \mathcal{H}_N(\psi) + h_w |q_{2+}(\psi)|\right). \tag{24}$$

Observation (22) yields the following relation between $\widetilde{\mathbb{P}}_N$ and $\mathbb{P}_N^{\beta,h_w}$: for any $\phi \in \Omega_N$,

$$\mathbb{P}_N^{\beta,h_w}(\phi) = \widetilde{\mathbb{P}}_N\left(\left\{\psi \in \Omega_N^* : \max(\psi, 0) = \phi\right\}\right).$$

In particular, we have

$$\mathbb{P}_N^{\beta,h_w}\left(\{\phi \in \Omega_N : \ |q_{2+}(\phi)| \geq CN\}\right) = \widetilde{\mathbb{P}}_N\left(\left\{\psi \in \Omega_N^* : \ |q_{2+}(\psi)| \geq CN\right\}\right), \tag{25}$$

since for any $\psi \in \Omega_N^*$, we have $q_{2+}(\max(\psi, 0)) = q_{2+}(\psi)$.

From now on, we deal with the r.h.s. of (25). For any subset $A \subseteq q_{2+}(\psi)$, we let $\mathcal{N}(A)$ be the edge boundary of A, defined by

$$\mathcal{N}(A) := \left\{\{x, y\} \in E(\mathbb{Z}^2) : \ x \in A, y \in A^{\complement}\right\}, \tag{26}$$

and define $U_A\psi \in \Omega_N^*$ as

$$(U_A\psi)(x) := \begin{cases} \psi(x) + 1 & \text{if } x \notin A, \\ 0 & \text{if } x \in A. \end{cases}$$

For ease of notation, we fix $\psi \in \Omega_N^*$ and write $q_{2+}(A) := q_{2+}(U_A\psi)$ in the sequel. Observing $\mathcal{H}_N(U_A\psi) \leq \mathcal{H}_N(\psi) + 4\beta N + \beta|\mathcal{N}(A)|$, we have by (24):

$$\widetilde{\mathbb{P}}_N(U_A\psi) \geq \widetilde{\mathbb{P}}_N(\psi) \exp\left(-4\beta N - \beta|\mathcal{N}(A)| - h_w\left(|q_{2+}(\psi)| - |q_{2+}(A)|\right)\right).$$

Let $V_1, V_2, \ldots, V_k$ be the connected components of $q_{2+}(\psi)$, and write $A_i = A \cap V_i$. We sum over all subsets $A \subseteq q_{2+}(\psi)$ to obtain

$$\begin{aligned} &\sum_{A \subseteq q_{2+}(\psi)} \widetilde{\mathbb{P}}_N(U_A\psi) \\ &\qquad \geq \widetilde{\mathbb{P}}_N(\psi) \exp\left(-4\beta N - h_w|q_{2+}(\psi)|\right) \sum_{A \subseteq q_{2+}(\psi)} \exp\left(-\beta|\mathcal{N}(A)| + h_w|q_{2+}(A)|\right) \\ &\qquad = \widetilde{\mathbb{P}}_N(\psi) \exp\left(-4\beta N - h_w|q_{2+}(\psi)|\right) \\ &\qquad\quad \times \sum_{A_1,\ldots,A_k} \prod_{i=1}^{k} \exp\left(-\beta|\mathcal{N}(A_i)| + h_w|q_{2+}(A_i)|\right) \\ &\qquad = \widetilde{\mathbb{P}}_N(\psi) \exp\left(-4\beta N\right) \prod_{i=1}^{k} \exp\left(-h_w|V_i|\right) \sum_{A_i \subseteq V_i} \exp\left(-\beta|\mathcal{N}(A_i)| + h_w|q_{2+}(A_i)|\right) \end{aligned} \tag{27}$$

where we have used that $q_{2+}(\psi) = V_1 \cup V_2 \cdots \cup V_k$ is the disjoint union of $V_1, V_2, \ldots, V_k$.

Note that for any finite connected subgraph of $\mathbb{Z}^2$, after deleting some edges (but keeping all the vertices), the graph can be decomposed into a disjoint union of patterns from Fig. 1 (up to rotation and reflection). From now on, we focus on one connected component V_i in the r.h.s. of (27). Denote by E_i the set of edges in this disjoint union of patterns. In the graph (A_i, E_i)

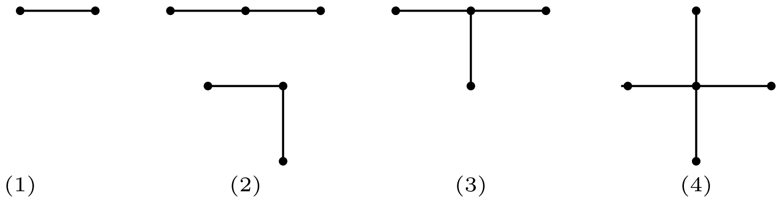

Fig. 1. The vertices of any connected set of size at least 2 can be covered by a disjoint union of these four patterns. (The two configurations in (2) are considered as the same pattern, as both have 3 vertices and 8 boundary edges).

we define the count of non-isolated spikes (similar to (10)) by

$$\widetilde{q}_{2+}(A_i) := \{x \in A_i : \ \psi(x) = 0, \exists y \in A_i, (x, y) \in E_i, \psi(y) = 0\},$$

and the edge boundary, similar to (26), by

$$\widetilde{\mathcal{N}}(A_i) = \{\{x, y\} \notin E_i : x \in A_i \vee y \in A_i\}.$$

Observe that

$$|\widetilde{q}_{2+}(A_i)| \le |q_{2+}(A_i)|, \text{ and } |\widetilde{\mathcal{N}}(A_i)| \ge |\mathcal{N}(A_i)|. \tag{28}$$

The next lemma will therefore provide a lower bound on the r.h.s. in (27).

Lemma 4.1. *If $\beta \ge 1$ and V is the vertex set of one of the patterns shown in Fig. 1, then*

$$\exp(-h_w|V|) \sum_{B \subseteq V} \exp(-\beta|\mathcal{N}(B)| + h_w|q_{2+}(B)|) \ge 1 + \frac{1}{2}e^{-6\beta}. \tag{29}$$

Proof. For simplicity of notation, we write $h = h_w$. We will repeatedly use that $e^{-h} = 1 - e^{-4\beta}$ and $\beta \ge 1$ without further reference. We consider the four patterns in Fig. 1, case by case. If $|V| = 2$ (*i.e.* (1) in Fig. 1), then the l.h.s. of (29) equals

$$\begin{aligned} e^{-2h}\left(1 + 2e^{-4\beta} + e^{-6\beta+2h}\right) 7 &= (1 - e^{-4\beta})^2(1 + 2e^{-4\beta}) + e^{-6\beta} \\ &\ge 1 + e^{-6\beta}(1 - 3e^{-2\beta}) > 1 + \frac{1}{2}e^{-6\beta}. \end{aligned}$$

In the case $|V| = 3$ (pattern (2) in Fig. 1), the l.h.s. of (29) equals

$$\begin{aligned} e^{-3h}\left(1 + 3e^{-4\beta} + e^{-8\beta} + 2e^{-6\beta+2h} + e^{-8\beta+3h}\right) &\ge (1 - e^{-4\beta})^3\left(1 + 3e^{-4\beta}\right) + 2e^{-6\beta-h} \\ &\ge 1 + 2e^{-6\beta}\left(1 - 3e^{-2\beta} - e^{-4\beta}\right) \\ &\ge 1 + e^{-6\beta}. \end{aligned}$$

Consider now the case $|V| = 4$, corresponding to pattern (3) in Fig. 1. By counting connected subsets of size at most two, the l.h.s. of (29) is bounded from below by

$$\begin{aligned} e^{-4h}\left(1 + 4e^{-4\beta} + 3e^{-6\beta+2h}\right) &\ge (1 - e^{-4\beta})^4\left(1 + 4e^{-4\beta}\right) + 3e^{-6\beta-2h} \\ &\ge 1 - 10e^{-8\beta} + 3e^{-6\beta-2h} \ge 1 + e^{-6\beta}. \end{aligned}$$

Lastly, consider the case $|V| = 5$, corresponding to pattern (4) in Fig. 1. By counting connected subsets of size at most two, the l.h.s. of (29) is larger than

$$e^{-5h}\left(1+5e^{-4\beta}+4e^{-6\beta+2h}\right) \geq 1-15e^{-8\beta}+4e^{-6\beta+2h}(1-5e^{-4\beta})$$
$$\geq 1+e^{-6\beta}\left(4\frac{1-5e^{-4\beta}}{(1-e^{-4\beta})^2}-15e^{-2\beta}\right) \geq 1+e^{-6\beta}.$$

This concludes the proof. □

With Lemma 4.1 at hand, we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Using (28) and (29), we may continue the inequality (27) to obtain

$$\sum_{A\subseteq q_{2+}(\psi)} \widetilde{\mathbb{P}}_N(U_A\psi) \geq e^{-4\beta N}\widetilde{\mathbb{P}}_N(\psi)\left(1+\frac{1}{2}e^{-6\beta}\right)^{|q_{2+}(\psi)|/5}, \tag{30}$$

where we have used that the total numbers of patterns covering $q_{2+}(\psi)$ are at least $|q_{2+}(\psi)|/5$. Note that for $A \neq B \subset q_{2+}(\psi)$ we have $U_A\psi \neq U_B\psi$ since $(U_A\psi)|_{A\setminus B} = 0$ and $(U_B\psi)|_{A\setminus B} = 1$. Moreover, for $\psi, \psi' \in \Omega_N^*$ and $A \subset q_{2+}(\psi)$, $A' \subset q_{2+}(\psi')$, we have

$$U_A\psi \neq U_{A'}\psi'.$$

To see this, note that

$$A = \{x \in \Lambda_N : (U_A\psi)(x) = 0,\ \exists y \in \Lambda_N,\ y \sim x,\ (U_A\psi)(y) \in \{0,1\}\}.$$

Thus, given $U_A\psi$, we can first recover the set A and then proceed to recover ψ. Therefore, from (30) we obtain

$$\begin{aligned}
1 &\geq \sum_{\psi\in\Omega_N^*:\ |q_{2+}(\psi)|\geq CN}\ \sum_{A\subset q_{2+}(\psi)} \widetilde{\mathbb{P}}_N\left(U_A\psi\right) \\
&\geq \sum_{\psi\in\Omega_N^*:\ |q_{2+}(\psi)|\geq CN} e^{-4\beta N}\,\widetilde{\mathbb{P}}_N(\psi)\left(1+\tfrac{1}{2}e^{-6\beta}\right)^{|q_{2+}(\psi)|/5} \\
&\geq e^{-4\beta N}\left(1+\tfrac{1}{2}e^{-6\beta}\right)^{CN/5}\widetilde{\mathbb{P}}_N\left(\left\{\psi \in \Omega_N^* :\ |q_{2+}(\psi)| \geq CN\right\}\right) \\
&\geq e^{-4\beta N}e^{\frac{C}{20}e^{-6\beta}N}\,\widetilde{\mathbb{P}}_N\left(\{\psi \in \Omega_N^* :\ |q_{2+}(\psi)| \geq CN\}\right),
\end{aligned}$$

where the last step used the inequality $1+x \geq e^{\frac{1}{2}x}$ for $x \in [0,1]$. By (25), this concludes the proof. □

Now we move to prove Proposition 1.3.

Proof of Proposition 1.3. For $\ell \in [\![1, H_w - m]\!]$, with exactly the same argument as in (20) and (21), setting $m = H_w - \ell$ we have

$$\begin{aligned}
1 &\geq \sum_{\substack{\psi:\ |\psi^{-1}(\ell)|\geq e^{-2\beta m}N^2\\ |\psi^{-1}(0)|\leq CN^{\frac{4}{3}}}}\ \sum_{A\subset\phi^{-1}(\ell)} \mathbb{P}_N^{\beta,h_w}\left(V_A\phi\right) \\
&\geq \exp\left(-4\beta N-4\beta CN^{\frac{4}{3}}+\frac{1}{2}e^{2\beta m}N^{\frac{4}{3}}\right) \\
&\quad\times \mathbb{P}_N^{\beta,h_w}\left(\{|\phi^{-1}(\ell)| \geq e^{-2\beta m}N^2\}\cap\{|\phi^{-1}(0)| \leq CN^{\frac{4}{3}}\}\right)
\end{aligned} \tag{31}$$

where $V_A\phi$ is defined in (19). Moreover, as $(1-e^{-2\beta})^{-1} \le 2$ and

$$\left\{|\phi^{-1}([1, H_w - m])| \ge \frac{e^{-2\beta m}}{1-e^{-2\beta}} N^2\right\} \subset \bigcup_{i=m}^{H_w-1} \left\{|\phi^{-1}(H_w - i)| \ge e^{-2\beta i} N^2\right\},$$

by union bound and (31) we obtain

$$\begin{aligned}
&\mathbb{P}_N^{\beta,h_w}\left(\left\{|\phi^{-1}([1, H_w - m])| \ge \frac{e^{-2\beta m}}{1-e^{-2\beta}} N^2\right\} \bigcap \left\{|\phi^{-1}(0)| \le C N^{\frac{4}{3}}\right\}\right)\\
&\le \sum_{i=m}^{H_w-1} \exp\left(4\beta N + 4\beta C N^{\frac{4}{3}} - \frac{1}{2} e^{2\beta i} N^{\frac{4}{3}}\right)\\
&\le \frac{1}{1-e^{-\beta N^{\frac{4}{3}}}} \exp\left(4\beta N + 4\beta C N^{\frac{4}{3}} - \frac{1}{2} e^{2\beta m} N^{\frac{4}{3}}\right),
\end{aligned}$$

where in the last inequality we have used that for $j \ge 0$,

$$\frac{\exp\left(-\frac{1}{2} e^{2\beta(j+1)} N^{\frac{4}{3}}\right)}{\exp\left(-\frac{1}{2} e^{2\beta j} N^{\frac{4}{3}}\right)} \le \exp\left(-\beta e^{2\beta j} N^{\frac{4}{3}}\right) \le e^{-\beta N^{\frac{4}{3}}}. \quad \square$$

Declaration of competing interest

None.

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