Tel-Aviv University
The Raymond and Beverly Sackler
Faculty of Exact Sciences
School of Mathematics

Zeroes of Gaussian Stationary Functions

Thesis submitted for the degree "Doctor of Philosophy"

by

Naomi Feldheim

Under the supervision of Professor Mikhail Sodin

Submitted to the Senate of Tel-Aviv University

First submission: August 2014

Corrections: March 2015

This work was carried out under the supervision of
Professor Mikhail Sodin

Acknowledgements

It is my pleasant duty to thank all the people who contributed to the creation of this work. My first and foremost thanks go to Professor Mikhail (Misha) Sodin. Misha has known me for many years, ever since I was interested in Mathematics as a child, and has always been caring and encouraging. As a teacher, Misha taught me beautiful mathematics, with brilliant exercises. His enthusiasm and clear presentation were and still are most inspiring to me. As an advisor, he was always available for discussions, made suggestions but left me a great deal of freedom, and was extremely patient and understanding. Misha also suggested and supported my participation in several mathematical events abroad, which were very stimulating for me. Lastly, I thank him for his advice and friendship.

I thank Prof. Boris Tsirelson, who eagerly shared his knowledge and experience and contributed some crucial ideas to this work. Prof. Tsirelson also influenced me through some fascinating courses which he taught in Tel-Aviv University during my graduate studies. v

I am grateful to many colleagues who dedicated time and effort to discuss mathematics with me. These discussions taught me both mathematical ideas and new ways of thinking, and they greatly influenced my taste and work. In particular, I am in debt to Jeremiah Buckley, Amir Dembo, Boris Hanin, Manjunath Krishnapur, Zakhar Kabluchko, Alon Nishry, Jan-Fredrik Olsen, Ron Peled, Yuval Peres, Igor Wigman, and Ofer Zeitouni.

I truly thank my family, for their continuous encouragement and endless devotion. I am especially grateful to my parents, for their enormous effort to help me develop, and for providing me with a loving home. I thank them and my parents-in-law, for taking upon themselves various tasks to allow me to complete this work. I am grateful to my sisters, Sara and Ruti, for their sincere friendship. Above all I thank my husband Ohad, who shares and enriches every aspect of my life, and always stimulates me to think. Last but not least I thank my son, Lahav, for taking everything in good spirit.

I am most grateful to the referee for his careful reading and enlightening comments. His report pointed out some missing or subtle details, which I hope are now clarified, as well as connections to other works and topics.

Naomi

Abstract

In this thesis we study zeroes of stationary Gaussian functions. A *Gaussian function* on a strip $D = D_\Delta = \{z : |\text{Im} z| < \Delta\}$ is a random function $f : D \to \mathbb{C}$ whose finite marginals, that is $(f(t_1), \ldots, f(t_n))$ for any $t_1, \ldots, t_n \in D$, have multi-variate centered Gaussian distribution. The Gaussian functions that we consider will be almost surely continuous, and for most of the results, almost surely analytic. We abbreviate GAF for "Gaussian Analytic Function".

A Gaussian function on D whose distribution is invariant with respect to horizontal shifts (i.e., by any element of $\mathbb{R}$), is called *stationary.* Gaussian functions are a natural model for noise, and in this context stationarity is the natural assumption of "time-invariance". Zeroes of GAFs in various domains attracted growing attention in recent years. One reason for this is that they form new interesting examples of point processes.

In this thesis we address three natural questions regarding a probabilistic model – asymptotic mean, fluctuations, and estimation of some rare events - for the zeroes of stationary Gaussian functions. Usually, we consider functions which are a.s. analytic in some strip (unless stated otherwise).

1. **Horizontal density of zeroes:** We show that almost surely, for all intervals $I \subset (-\Delta, \Delta)$, counting the zeroes in the "long" rectangle $[0, T] \times I$ and dividing by T converges to some (random) number (the "horizontal density" of zeroes). Regarding this limit as a function of the interval I, we get a locally-finite measure, and the convergence holds also in the weak sense. We give a necessary and sufficient condition for the limiting measure to be deterministic (a.s. not random), and provide a simple formula for it in this case. Then we extend this result to a family of "symmetric" GAFs which are real on the real axis, and study some unique properties which appear in this case.

2. **Fluctuations:** We study the variance of the number of zeroes of a stationary GAF in a long rectangle $[0, T] \times I$ (T is large). We prove that

it is always asymptotically between cT and CT^2 with some constants $c, C > 0$, and give necessary and sufficient conditions for achieving each bound.

3. **Gap probability:** We study the probability that there are no zeroes at all in a "long" region. In this part we demonstrate results in the real setting, i.e., for functions $f : \mathbb{R} \to \mathbb{R}$. We do not assume analyticity, but rather a mild mixing condition. We show that for a large family of processes, the probability of having no zeroes at all in a long interval $[0, T]$ is roughly exponentially small in T.

Contents

Introduction

The primary object of study of this work is stationary point processes. In general these are random variables taking values in the space of discrete subsets of a metric space, whose distribution is invariant under some collection of isometries. Each element of the chosen discrete subset is referred to as a point, and hence the name – point process. Point processes have been a primary tool in modeling many physical phenomena, such as the arrangement of particles in gas, the arrangement of electrons in charged matter, and the times of radioactive decay of an array of atoms.

The most important example of a stationary point processes is undoubtedly the *Poisson point process.* This process, which is characterized by independence of the process when restricted to disjoint subsets of the underlying space, has been extensively studied and applied to countless physical models. This is primarily due to its simplicity, and indeed this is often the only point process that an undergraduate student in mathematics or physics encounters.

But the simplicity of the Poisson point process is precisely the cause of its weakness: the points of the process neither tend to attract each other nor to repel from each other. Many physical systems, however, demonstrate attraction (scattering of planets in a galaxy) or repulsion (scattering of electrons in charged material), so it is desirable to reflect these properties in mathematical models. Generating attraction between points in a process is generally an easier task (a well-known example is the Cox process, which is a simple generalization of Poisson process defined via a random underlying intensity). Generating repulsion, in a way which will be convenient to analyze, is much harder.

This was one of the motivations which led researchers to look for other natural point processes. One such family which is extensively studied is determinantal point processes. These have the property that the density of the joint probability of seeing a collection of points $x_1, \dots, x_k$ is given by a determinant of a $k \times k$ matrix whose i, j entry is $F(x_i, x_j)$ for some positive-definite function F. Determinantal processes often demonstrate repulsion, and have been successfully used in many physical models, especially in statistical mechanics.

Few other ways to generate a point process with repulsion are known. One main example, which is not too difficult to analyze, is by considering the zero-set of a random analytic function. This is done, usually, by considering a sum of "basis" functions with stochastically independent Gaussian coefficients (with some mild conditions which ensure convergence). The "basis" could be some set of algebraic polynomials, trigonometric polynomials, translations of a certain function, etc. It is remarkable that repulsion between zeroes of such processes is natural and intrinsic, and is by no mean an artificial aspect in their construction.

Another motivation for studying zeroes of Gaussian analytic functions, is the interest in the random function itself. Some mathematicians view it as representing the "typical" behavior of a function from a certain space, so that randomness is just a tool to investigate the major part of that space. From a more applied point of view, many noise models are either represented or approximated by Gaussian (most likely, analytic) functions. Applications include signal processing (e.g., radio and brain transmissions), statistical mechanics (e.g., heat processes), and statistics (e.g., huge data arrays). For these reasons Gaussian functions, and especially stationary ones, were extensively studied since the middle of the 20-th century, both in academy and industrial research groups.

Much attention was drawn to questions about the behavior of the zero-set (and other level-lines) of stationary Gaussian functions, as they are crucial for applications. Naturally, much discussed were questions about the mean number of zeroes, the fluctuations of this number, and estimating some rare events of interest. Somewhat surprisingly, most of those basic questions remained open, or were not fully settled, even after many years of research.

In this thesis we relate to some questions asked by Wiener, Slepian, Cuzick and more recent works. New tools and ideas enable us to improve their results, or give entirely new ones. Most of the thesis studies the behavior of zeroes of complex Gaussian analytic functions (GAFs), defined in some horizontal strip, and invariant to horizontal shifts. We also study and compare their behavior to the zeroes of some real counterpart processes (defined on the real line or on a strip containing it). We investigate three types of questions about the number of zeroes in a "long" horizontal rectangle $R^{a,b}(T) = [0,T] \times [a,b]$:

- **Limit density:** When does the number of zeroes in $R^{a,b}(T)$ divided by the length T converge (as T rises) to a deterministic limit, and what is this limit?

- **Fluctuations:** What is the order of magnitude of the fluctuations of the number of zeroes in $R^{a,b}(T)$? When is it precisely linear in T, or precisely quadratic?
- **Gap probability:** What is the probability of the rare event that there are no zeroes in $R^{a,b}(T)$?

The rest of this work is divided into three chapters, each addressing a question from the list above. In order to make these chapters self-contained, we provide the relevant notation at the beginning of each chapter. The content of each chapter has appeared as an independent paper or preprint. Partial results from the first chapter also appeared in my M.Sc. thesis; as they have been significantly extended in the present work, we find it appropriate to incorporate them here. The rest of the introduction consists of a description of the content of each chapter.

Definitions

A *Gaussian analytic function* (GAF) over a domain $D \subset \mathbb{C}$ is a random function $f : D \to \mathbb{C}$ which is almost surely analytic in D and whose marginal distribution on any set of n points is Gaussian in $\mathbb{C}^n$ (with zero mean). It is a well-known fact that every GAF can be represented as a random series

$$f(z) = \sum_n \zeta_n \varphi_n(z),$$

where $\varphi_n(z)$ are holomorphic functions in D such that $\sum_n |\varphi_n|^2$ converges uniformly on compacts, and $\zeta_n \sim \mathcal{N}_{\mathbb{C}}(0, 1)$ are i.i.d. complex Gaussian random variables.

A random function on the horizontal strip $D = D_\Delta = \{|\mathrm{Im}\ z| < \Delta\}$ (where $0 < \Delta \leq \infty$) is called *stationary* if its distribution is invariant with respect to horizontal shifts. Stationarity is a natural assumption in noise modeling, and is exemplified by many natural random series, such as

$$\sum_n \zeta_n w_n e^{i\lambda_n z}, \qquad \sum_{n\in\mathbb{Z}} \zeta_n \frac{\sin(\pi(z-n))}{z-n}, \qquad e^{-|z|^2/2}\sum_{n=0}^{\infty} \zeta_n \frac{z^n}{\sqrt{n!}}, \tag{1}$$

where in all examples $\zeta_n \sim \mathcal{N}_{\mathbb{C}}(0, 1)$ are i.i.d., and in the left-most one $w_n, \lambda_n \in \mathbb{R}$ are given (and obey $\sum_n w_n^2 e^{\lambda_n y} < \infty$ for all $|y| < \Delta$, which provides convergence in D_Δ).

A stationary GAF is determined by the covariance function $r(z) = \mathbb{E}\left[f(z)\overline{f(0)}\right]$, or by its inverse Fourier transform – which is a non-negative, finite measure on $\mathbb{R}$ with a finite exponential moment. This measure is called *the spectral measure* of f.

Ch. 1: Horizontal density of zeroes

Result for Gaussian Analytic Functions

Under certain assumptions on the spectral measure, Wiener proved that the zeroes of a stationary GAF in a strip obey the law of large numbers, and computed their horizontal density. This result appears in his classical treatise with Paley [37, chapter X].

In Chapter 1 we prove a stronger result, removing unnecessary assumptions. We show that almost surely, for all intervals $I \subset (-\Delta, \Delta)$, counting the zeroes in the "long" rectangle $[0,T] \times I$ and dividing by T converges to $\nu_f(I)$, where ν_f is some (random) locally-finite measure, which we call *the horizontal density of zeroes*. Secondly, we show that ν_f is a deterministic measure if and only if the spectral measure of f is continuous or consists of exactly one atom. Lastly, we compute the measure ν_f in case it is deterministic, using a version of Edelman-Kostlan (or Kac-Rice) formula; in this case ν_f has the continuous density $\frac{1}{4\pi}\frac{d}{dy}\left\{\frac{\psi'(y)}{\psi(y)}\right\}$ where $\psi(y) = \mathbb{E}\left[|f(iy)|^2\right]$.

Result for Symmetric Gaussian Analytic Functions

A natural counterpart of GAFs are *symmetric GAFs*, that is, Gaussian analytic functions $f : D \to \mathbb{C}$ on a domain $D \subset \mathbb{C}$ which posses a symmetry around the real axis: a.s., $\forall z \in D: \quad \overline{f(z)} = f(\overline{z})$ (notice "Gaussian" this time means that a marginal of n values has Gaussian distribution in $\mathbb{R}^{2n}$). This is a family of models for random analytic functions with a fixed proportion of real zeroes; the zeroes are therefore a "mixed" point process (one and two dimensional). The study of such functions goes back to Kac [23], who was interested in the expected number of real zeroes of random polynomials with real coefficients.

Chapter 1 also treats stationary symmetric GAFs. That is, we prove that almost surely the horizontal density of zeroes ν_f exists, characterize when it is deterministic, and compute this measure in this case. Here, ν_f is the sum of two parts: a measure with continuous density, given by

$\frac{1}{4\pi}\frac{d}{dy}\left\{\frac{\psi'(y)}{\sqrt{\psi(y)^2-\psi(0)^2}}\right\}$, and an atom at zero with weight $\frac{1}{4\pi}\sqrt{\frac{\psi''(0)}{\psi(0)}}$ (once again $\psi(y) = \mathbb{E}\left[|f(iy)|^2\right]$).

For the last computation, we develop a new Edelman-Kostlan (or Kac-Rice) type formula for calculating the mean number of zeroes of a symmetric GAF in any sub-domain of the original domain of definition. This formula does not require stationarity, and extends works by Shepp and Vanderbei [43], Prosen [39] and Macdonald [29].

Using this new formula, we show that in the symmetric GAF case the zeroes repel from the real line at short distances. Moreover, a GAF and a symmetric GAF which share the covariance kernel will have asymptotically the same density of zeroes away from the real line. Special cases of this behavior roused the interest of physicists, as models for condensation; see for instance Schehr-Majumadar [41].

References: The results of this chapter appear in

- N. Feldheim, *Zeroes of Gaussian Analytic Functions with Translation-Invariant Distribution*, Israel Journal of Mathematics 2012.

Ch. 2: Fluctuations of the number of zeroes

After studying convergence to the mean, it is natural to ask about fluctuations of the number of zeroes. Continuing in the setup of the previous chapter, let f be a stationary GAF in the strip D_Δ and let $V^{a,b}(T)$ be the variance of the number of zeroes of f in $[0,T]\times[a,b]$.

In Chapter 2, we show that $V^{a,b}(T)$ is asymptotically between cT and CT^2 with positive constants c and C, and give conditions (in terms of the spectral measure) for the asymptotics to be exactly linear or quadratic in T.

In more detail, first we show quadratic variance holds precisely when the spectral measure contains at least one atom; this is mainly a consequence of the results presented in Chapter 1. Much more effort is dedicated to showing existence and positivity of the limit

$$L(a,b) = \lim_{T\to\infty}\frac{V^{a,b}(T)}{T} \in (0,\infty].$$

Next, we prove that $L(a,b)$ is finite if the spectral measure obeys some L^2 condition. Then we give conditions for the limit to be infinite ("super-linear" variance).

Naturally, fluctuations of zeroes of random functions (as well as fluctuations of eigenvalues of random matrices and other models) have been treated extensively in the literature, but usually these were obtained with much effort. For instance, for the analogous question for real processes (not necessarily real-analytic), an asymptotic formula for the variance was given already in the 1960's by Cramer and Leadbetter in their book [9], but the rate of growth is not apparent from it. Cuzick [10] proved a Central Limit Theorem for the number of zeroes, whose main condition is linear growth of the variance. More than a decade later, Slud [45], using stochastic integration methods, gave a convenient condition for such linearity (which is, in fact, an L^2 condition similar to ours). More recently, a work by Granville and Wigman [19] and an extension by Azaïs and León [5] used similar methods to study the number of zeroes of a Gaussian trigonometric polynomial with integer frequencies $\{-N, -N+1, \dots, N\}$ in the interval $[0, 2\pi]$. They showed the variance of this number is linear in N, and that a Central Limit Theorem holds. Our proof seems more accessible, due to simplifications via harmonic analysis and basic properties of analytic functions. On the other hand, it does not extend directly to real processes.

References: The results of this chapter appear in

- N. Feldheim, *Variance of the Number of Zeroes of Shift-Invariant Gaussian Analytic Functions*, submitted. See: arXiv:1309.2111.

Ch. 3: Gap probability for real stationary processes

In this chapter we consider, an a.s. continuous, stationary Gaussian function $f : \mathbb{R} \to \mathbb{R}$ (definitions are analogous to those given in the beginning of this introduction, omitting analyticity). The *gap probability* is defined as

$$H(T) := \mathbb{P}(f > 0 \text{ on } [0, T])$$

(the name refers to the gap between sign-changes). The gap probability was studied by many authors during the years 1950-1970, including Rice and Slepian. While the behavior for small values of T was extensively explored, the regime of large T is not well-understood. In favorable cases, it is expected that $H(T)$ decays roughly exponentially with T, demonstrating "independent-like" behavior in long distances. In 2012, Antezana, Buckley, Marzo and Olsen [3] gave exponential bounds for the gap probability of a special model, whose correlation function is $\text{sinc}(t) = \frac{\sin(\pi t)}{t}$ (the sinc-kernel model).

In Chapter 3 which is joint work with Ohad Feldheim, we give lower and upper exponential bounds on $H(T)$, valid for a large family of processes. More precisely, we prove the following: Assume the spectral measure of the process obeys some moment condition, and that in some neighborhood of the origin it has density which is bounded away from zero and infinity. Then $e^{-c_1 T} \le H(T) \le e^{-c_2 T}$ for some $c_1, c_2 > 0$ and all large enough T. We also present similar bounds for Gaussian sequences under analogous conditions.

Our main tool is a spectral decomposition of our random function into two parts: a rescaled sinc-kernel process which we are able to analyze, and a second process whose influence on the gap probability we are able to control.

References: The results of this chapter appear in

- N. D. Feldheim, O. N. Feldheim, *Long gaps between sign-changes of Gaussian Stationary Processes.* International Mathematics Research Notices 2014; doi: 10.1093/imrn/rnu020. See also: arXiv:1307.0119.

Chapter 1

Horizontal density of zeroes

1.1 Introduction

Following Wiener, we study zeroes of Gaussian analytic functions with translation-invariant distribution, defined on a strip in the complex plane. Under certain assumptions on the spectral measure, Wiener proved that the zeroes obey the law of large numbers, and computed their horizontal density (limiting measure). This result appears in his classical treatise with Paley [37, chapter X]. Wiener's proof is quite intricate; this may be why it attracted little attention.

In this work, we simplify Wiener's arguments and remove unnecessary assumptions on the spectral measure. We incorporate the result into a theorem that guarantees the existence of the horizontal limiting measure in question, and asserts it is not random if and only if the spectral measure is continuous or consists of a single atom. Then we prove a counterpart of this theorem for a natural class of Gaussian analytic functions which have a symmetry with respect to the real axis.

For this purpose, we developed a general Edelman-Kostlan-type formula for computing the average zero-counting measure of zeroes of a symmetric Gaussian analytic function in some domain (see Theorem 2 below). This result extends those of Shepp and Vanderbei [43], Prosen [39] and Macdonald [29].

1.1.1 Gaussian Analytic Functions

We deal with two classes of random Gaussian analytic functions.

Definition 1.1.1. *Let $D \subset \mathbb{C}$ be a domain, and let $\{\phi_n\}_{n\in\mathbb{N}}$ be analytic functions in D such that the series $\sum_n |\phi_n(z)|^2$ converges uniformly on compact subsets of D.*

1. *Let a_n be independent standard complex Gaussian random variables ($a_n \sim \mathcal{N}_{\mathbb{C}}(0,1)$). The random series $\sum_n a_n\phi_n(z)$ is called a Gaussian Analytic Function (GAF, for short).*

2. *Let b_n be independent standard* real *Gaussian variables ($b_n \sim \mathcal{N}_{\mathbb{R}}(0,1)$). If the domain D and the functions ϕ_n are symmetric w.r.t. the real axis (the latter means that $\phi(\overline{z}) = \overline{\phi(z)}$, $z \in D$) then the random series $\sum_n b_n\phi_n(z)$ is called a symmetric Gaussian Analytic Function.*

Our assumptions on $\{\phi_n\}$ ensure that the sums above a.s. converge to an analytic function in D [22, Chapter 2]. Throughout this thesis we assume that there is no $z_0 \in D$ such that $\phi_n(z_0) = 0$ for all $n \in \mathbb{N}$ (hence the function f has no deterministic zeroes).

The covariance kernel of $f(z)$ is defined by

$$K(z,w) = \mathbb{E}(f(z)\overline{f(w)}) = \sum_n \phi_n(z)\overline{\phi_n(w)}\,. \tag{1.1}$$

The function $K(z,w)$ is positive definite, analytic in z, anti-analytic in w, and obeys the law $K(z,w) = \overline{K(w,z)}$. It turns out that every such function $K(z,w)$ of two variables $z,w \in D$ uniquely defines a GAF in D.

If in addition $K(x,y)$ is real whenever $x,y \in D \cap \mathbb{R}$, then $K(z,w)$ also uniquely defines a symmetric GAF with this kernel. We stress that a GAF and a symmetric GAF with the same kernel are different random processes.

1.1.2 Stationarity

We assume our domain is the Δ-strip $D = D_\Delta = \{|\operatorname{Im} z| < \Delta\}$ with $0 < \Delta \le \infty$.

Definition 1.1.2. *A GAF or a symmetric GAF in a strip D_Δ is called* stationary *if it is distribution-invariant with respect to all horizontal shifts, i.e., for any $t \in \mathbb{R}$, any $n \in \mathbb{N}$, and any $z_1, \dots, z_n \in D$, the random n-tuples*

$$\big(f(z_1), \dots, f(z_n)\big) \qquad \text{and} \qquad \big(f(z_1+t), \dots, f(z_n+t)\big)$$

have the same distribution.

If $f(z)$ is stationary in the Δ-strip, then

$$K(z,w) = r(z - \overline{w})$$

for some analytic function $r : D_{2\Delta} \to \mathbb{C}$. [1]

Since $r(t)$ is continuous and positive-definite, it is the Fourier transform of a positive measure ρ (Bochner's Theorem):

$$r(t) = \int_{\mathbb{R}} e^{2\pi i t\lambda} d\rho(\lambda).$$

The measure ρ is called the spectral measure of the process $f(z)$.

Since $r(t)$ has an analytical extension to the 2Δ-strip, $\rho(\lambda)$ has a finite exponential moment [27, Chapter 2]:

$$\text{for each } \Delta_1 < \Delta, \ \int_{-\infty}^{\infty} e^{2\pi\cdot 2\Delta_1|\lambda|} d\rho(\lambda) < \infty\,. \tag{1.2}$$

In fact, condition (1.2) is also sufficient for $r(t)$ to have an analytic extension to the 2Δ-strip. Therefore, beginning with a finite positive measure ρ obeying (1.2), we can construct a kernel by

$$K(z,w) = \int_{\mathbb{R}} e^{2\pi i(z-\bar{w})\lambda} d\rho(\lambda). \tag{1.3}$$

which defines in its turn a stationary GAF in the Δ-strip.

What measures could be spectral measures of a symmetric GAF? As we mentioned earlier, a kernel $K(z,w)$ defines a symmetric GAF if and only if it is real for $z, w \in \mathbb{R}$; By relation (1.3) this is equivalent to ρ being symmetric with respect to the origin.

Finally, we mention that a random GAF or symmetric GAF may be constructed, as follows, from its spectral measure ρ. If $\{\psi_n(z)\}_n$ comprise an orthonormal basis in $L^2_\rho(\mathbb{R})$, then their weighted Fourier transforms

$$\phi_n(z) = \widehat{\psi_n}(z) = \int_{\mathbb{R}} e^{2\pi i z\lambda} \psi_n(\lambda) d\rho(\lambda)$$

[1] Indeed, define $r(z) = K(z,0)$. Since $K(z,w)$ is assumed to be analytic in z, r is analytic in D_Δ. Now by stationarity, for $x, y \in \mathbb{R}$ the covariance $\mathbb{E}(f(x)\overline{f(y)})$ depends on $(x-y)$ only, so that $K(x,y) = K(x-y,0) = r(x-y)$. Similarly, $K(z,t) = r(z-t)$ for any $z \in D_\Delta$ and $t \in \mathbb{R}$. Now, as $K(z,w)$ is analytic in $\overline{w}$ and agrees with $r(z-\overline{w})$ on an open set, we conclude that r may be extended to $D_{2\Delta}$ and that $K(z,w) = r(z-\overline{w})$ for any $z, w \in D_\Delta$.

comprise a basis in the Hilbert space $\mathcal{F}\{L^2_\rho(\mathbb{R})\}$ (the Fourier image of $L^2_\rho(\mathbb{R})$ with the scalar product transferred from $L^2_\rho(\mathbb{R})$). One easily checks that

$$r(z-\overline{w}) = \sum_n \phi_n(z)\overline{\phi_n(w)}\,.$$

Therefore, when used in Definition 1.1.1, the basis $\{\phi_n\}$ will result in a random function with the desired kernel.

1.2 Results and Discussion

1.2.1 Main Theorem

It will be convenient to introduce some notation:

Notation 1. (zero-set, zero-counting measure) Let $D \subset \mathbb{C}$ be a region, and f a holomorphic function in D. Denote the zero-set of f (counted with multiplicities) by Z_f, and the zero-counting measure by n_f, i.e.,

$$\forall \phi \in C_0^\infty(D), \qquad \int_D \phi(z)dn_f(z) = \sum_{z\in Z_f} \phi(z)\,.$$

We use the abbreviation $n_f(B) = \int_B dn_f(z)$ for the number of zeroes in a Borel subset $B \subset D$.

Notation 2. Let $y \in (-\Delta, \Delta)$. For a stationary GAF or symmetric-GAF in D_Δ with kernel $K(z,w)$, denote

$$\psi(y) = K(iy, iy) = \int_{-\infty}^{\infty} e^{-4\pi y\lambda} d\rho(\lambda)\,.$$

In the case of a GAF, define the function

$$L(y) = \frac{d}{dy}\left(\frac{\psi'(y)}{4\pi\psi(y)}\right) = -\frac{d}{dy}\left(\frac{\int_{-\infty}^{\infty}\lambda e^{-4\pi y\lambda}d\rho(\lambda)}{\int_{-\infty}^{\infty} e^{-4\pi y\lambda}d\rho(\lambda)}\right). \tag{1.4}$$

In the case of a symmetric-GAF, define for $y \neq 0$ the function

$$S(y) = \frac{d}{dy}\left(\frac{\psi'(y)}{4\pi\sqrt{\psi(y)^2-\psi(0)^2}}\right) = -\frac{d}{dy}\left(\frac{\int_{-\infty}^{\infty}\lambda e^{-4\pi y\lambda}d\rho(\lambda)}{\sqrt{\left(\int_{-\infty}^{\infty} e^{-4\pi y\lambda}d\rho(\lambda)\right)^2 - \left(\int_{-\infty}^{\infty} d\rho(\lambda)\right)^2}}\right), \tag{1.5}$$

and the positive number

$$R = \frac{1}{4\pi}\sqrt{\frac{\psi''(0)}{\psi(0)}} = 2\sqrt{\frac{\int_{-\infty}^{\infty} \lambda^2 d\rho(\lambda)}{\int_{-\infty}^{\infty} d\rho(\lambda)}}. \tag{1.6}$$

Finally, a stationary GAF is *degenerate* if its spectral measure ρ_f consists of exactly one atom. Similarly a stationary symmetric GAF is degenerate if ρ_f consists of two symmetric atoms (i.e., $\rho_f = c(\delta_q + \delta_{-q})$ for some $c, q > 0$).

The following theorem is our main result. Denote by m_1 the linear Lebesgue measure.

Theorem 1. *Let f be a stationary non-degenerate GAF or symmetric GAF in the strip D_Δ with $0 < \Delta \le \infty$. Denote by $\nu_{f,T}$ the non-negative locally-finite random measure on $(-\Delta, \Delta)$ defined by*

$$\nu_{f,T}(Y) = \frac{1}{T} n_f([0,T) \times Y), \;\; Y \subset (-\Delta, \Delta).$$

Then:

(i) *Almost surely, the measures $\nu_{f,T}$ converge weakly to a measure ν_f when $T \to \infty$.*

(ii) *The measure ν_f is not random (i.e. $\operatorname{var} \nu_f(I) = 0$ for every interval I) if and only if the spectral measure ρ_f has no atoms.*

(iii) *If the measure ν_f is not random, then:*

$$\begin{aligned} \nu_f &= L\, m_1, && \textit{if } f \textit{ is a GAF},\\ \nu_f &= S\, m_1 + R\,\delta_0, && \textit{if } f \textit{ is a symmetric-GAF}, \end{aligned}$$

where δ_0 is the unit point measure at the origin.

The measure ν_f is referred to as "the horizontal limiting measure of the zeroes of f", or simply "the limiting measure". In the discussion and examples that follow, we assume the normalization $\psi(0) = \int_{\mathbb{R}} d\rho(\lambda) = 1$.

Remark 1.2.1. The limiting measure ν_f might have atoms. Generally speaking, the weak convergence in the theorem guarantees that $\nu_{f,T}([a,b))$ converges to $\nu_f([a,b))$ for all $a, b \in (-\Delta, \Delta)$ with a possible exception of an at most countable set, which corresponds to atoms of the limiting measure ν_f. Yet, due to stationarity, in our case the limit exists *on all intervals.* We prove the following result:

Proposition 1.2.1. *Almost surely, for any $a, b \in (-\Delta, \Delta)$, we have:*

$$\lim_{T\to\infty} \nu_{f,T}([a,b)) = \nu_f([a,b)).$$

The proof is included in Section 1.6, and may be easily modified to apply to any type of interval (i.e., $(a,b]$, (a,b) or $[a,b]$). Notice that in particular for any $a \in (-\Delta, \Delta)$, a is an atom of ν_f if and only if it is an atom of $\nu_{f,T}$ for large enough T.

Remark 1.2.2. The part of the theorem pertaining to GAFs extends the aforementioned Wiener's theorem. In his work, Wiener assumed that the spectral measure ρ has the L^2-density $d\rho(\lambda) = |\phi(\lambda)|^2 d\lambda$, that satisfies convergence conditions:
For any $|y| < \Delta$,

$$\int_{-\infty}^{\infty} (1+x^2)^2\, |\widehat{\phi}(x+iy)|^2 dx < \infty,$$

and

$$\int_{-\infty}^{\infty} (1+x^2)\, |(\widehat{\phi})'(x+iy)|^2 dx < \infty.$$

As above, $\widehat{\phi}$ is the Fourier transform of ϕ. Under these assumptions, Wiener proved that the limiting measure ν_f exists and equals Lm_1, where L is defined by (1.4).

Remark 1.2.3. ***(atomic spectral measure)***
Consider a spectral measure consisting of two atoms:

$$\rho = \frac{1}{2}(\delta_{-q} + \delta_q)\,.$$

The corresponding GAF is $f(z) = (\zeta_1 e^{-2\pi iqz} + \zeta_2 e^{2\pi iqz})/\sqrt{2}$, where $\zeta_1, \zeta_2 \sim \mathcal{N}_{\mathbb{C}}(0,1)$, independently. The zeroes of such a function are

$$z_k = \frac{1}{4\pi q}\left[\arg(\frac{\zeta_2}{\zeta_1}) + 2\pi k - i\log\left|\frac{\zeta_2}{\zeta_1}\right|\right], \qquad k \in \mathbb{N}\,.$$

We see that all zeroes lie on the same (random) horizontal line, equally spaced upon it. The height of this horizontal line is a non-degenerate random variable, and so in this example ν_f is indeed random.

For symmetric GAFs, the spectral measure above is degenerate (all zeroes of the corresponding function are real). We mention that it is possible to construct a random analytic function with continuous spectrum, for which an arbitrarily close to 1 asymptotic proportion of zeroes lie on the real line. For this, choose a continuous symmetric spectral measure, sufficiently close (in the weak sense) to the degenerated measure $\frac{1}{2}(\delta_1+\delta_{-1})$. As a concrete example, consider the symmetric GAF f_ε with spectral density $\frac{1}{4\varepsilon}(1\!\!1_{[-1-\varepsilon,-1+\varepsilon]} + 1\!\!1_{[1-\varepsilon,1+\varepsilon]})$. Using Theorem 1 and Proposition 1.2.1, one may compute that if ε is small enough, f_ε has, in average, 99% of its zeroes in a long rectangle lying on the real line.

Remark 1.2.4. ***(behavior near the boundary and near the real line.)*** We observe that $S(y)$ and $L(y)$ have the same asymptotic behavior as y approaches the boundary $\pm\Delta$. Therefore, zeroes of a GAF and of a symmetric GAF with the same kernel behave similarly near the boundary of the domain of definition.

For a symmetric GAF, we observe a "contraction" of the zeroes to the real line: there are zeroes on the line itself, but they are scarce as we approach it from below or above (see figure 1.3.1 below). This is confirmed by a straightforward computation, which shows that $S(y) = O(y)$, as $y \to 0$.

1.2.2 Expected Zero-Counting Measures

In part (iii) of the theorem, the limit $\nu_f(a,b)$ is actually the expectation $\mathbb{E}n_f([0,1]\times[a,b])$. In order to calculate this quantity in the GAF case, we use the following classical formula, which appeared in Edelman and Kostlan's joint work on random polynomials [13]. Several proofs of this formula are known ([22, Chapter 2]).

Theorem. (Edelman-Kostlan formula) *For a GAF f with covariance kernel $K(z,w)$, the expected zero-counting measure is given by*

$$\mathbb{E}[dn_f(z)] = \frac{1}{4\pi} \triangle \log K(z,z). \tag{1.7}$$

This should be understood as equality of measures in the following sense: for any compactly supported $h \in C^\infty(D)$,

$$\mathbb{E}\int_D h(z)dn_f(z) = \frac{1}{4\pi}\int h(z)\ \triangle \log K(z,z)dm_2(z).$$

Here and throughout this chapter, m_2 denotes the planar Lebesgue measure.

The proofs of this formula depend inherently on the fact that $f(z)$ is a *complex* Gaussian random variable for all z, which fails for the symmetric

GAF. To that end, we prove the following result, that extends previous results by Shepp and Vanderbei [43], Prosen [39] and Macdonald [29].

Theorem 2. *For a symmetric GAF f on some region with covariance kernel $K(z,w)$, the expected zero-counting measure is given by*

$$\mathbb{E}[dn_f(z)] = \frac{1}{4\pi} \triangle \log\left(K(z,z) + \sqrt{K(z,z)^2 - |K(z,\overline{z})|^2}\right), \tag{1.8}$$

where the Laplacian is taken in the distribution sense.

Notice that stationarity is not assumed in the last two theorems. Moreover, this formula combines information about real and complex zeroes.

1.3 Examples

1.3.1 Paley-Wiener Process (Sinc-kernel Process)

Consider the spectrum

$$d\rho_a(\lambda) = \frac{1}{2a}\chi_{[-a,a]}(\lambda)d\lambda\,, \qquad a > 0.$$

Condition (1.2) holds for any $\Delta > 0$, so the sample function f is entire. The kernel is:

$$K(z,w) = \frac{\sin(2\pi a(z-\overline{w}))}{2\pi a(z-\overline{w})} = r(z-\overline{w})$$

A base for construction of the GAF (in the sense of definition 1.1.1) is

$$\phi_n(z) = \frac{\sin(2\pi az)}{2\pi az - n\pi}, \qquad n \in \mathbb{Z}.$$

This example yields a surprising construction of a random series of simple fractions with known poles and stationary zeroes: Take for instance $a = 1$. Our function is

$$f(z) = \sum_{n\in\mathbb{Z}} a_n \frac{\sin(2\pi z)}{2\pi z - n\pi}$$

where $\{a_n\}$ are independent Gaussian random variables. Almost surely, $Z_f \cap \frac{1}{2}\mathbb{Z} = \emptyset$, so we may divide by $\sin(2\pi z)/\pi$ and get the random series

$$g(z) = \sum_{n\in\mathbb{Z}} \frac{a_n}{2z-n}\,.$$

The poles of g are known (and lie on a one-dimensional lattice), but its zeroes are a random set invariant to *all* horizontal shifts!

Using Theorem 1 for ρ_a, we get that the zero-counting measure has the following density of zeroes:

$$L_a\left(\frac{y}{4\pi a}\right) = 4\pi a^2 \frac{d}{dy}\left(\coth y - \frac{1}{y}\right).$$

Similarly, the symmetric GAF with the same spectral measure has the continuous density of zeroes

$$S_a\left(\frac{y}{4\pi a}\right) = 4\pi a^2 \frac{d}{dy}\left(\frac{\cosh y - \frac{\sinh y}{y}}{\sqrt{\sinh^2 y - y^2}}\right)$$

plus an atom at $y = 0$, of size

$$R = \frac{a}{\sqrt{3}}.$$

Figure 1.1(a) represents the graphs of the continuous densities for the parameter $a = \frac{1}{4\pi}$.

1.3.2 Fock-Bargmann Space (Gaussian Spectrum)

Set

$$d\rho_a(\lambda) = \frac{1}{a\sqrt{\pi}} e^{-\lambda^2/a^2} d\lambda\,, \qquad a > 0.$$

Once again, f is entire (i.e., $\Delta = \infty$). The Fourier transform of the measure is

$$r(z) = \frac{1}{a\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\lambda^2/a^2} e^{2\pi i\lambda z} d\lambda = e^{-a^2\pi^2 z^2},$$

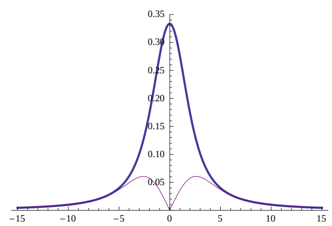
therefore the covariance kernel is:

$$K(z, w) = e^{-a^2\pi^2(z-\overline{w})^2}.$$

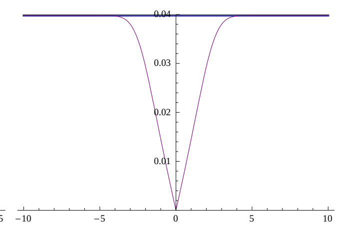
This space has an orthonormal basis of the form $\frac{(bz)^n}{\sqrt{n!}} e^{-cz^2}$, where $b = \sqrt{2}\frac{a}{\pi}$ and $c = -\frac{a^2}{\pi^2}$.

In this model, the density of zeroes is constant:

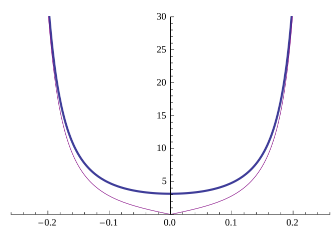
$$L_a\left(\frac{y}{2\pi a}\right) = 2\pi a^2.$$



(a) Paley-Wiener



(b) Fock-Bargmann



(c) exponential spectrum

Figure 1.1: Horizontal density of zeroes for GAF and symmetric GAF models with the same kernel. In each model, the lower graph represents the continuous component of the mean zero-counting measure for the symmetric GAF (the atomic part is an atom at $y = 0$, which is not graphed). The upper graph represents the continuous (and only) part of this measure for the appropriate GAF.

This is the only model with Lebesgue measure as expected counting measure of zeroes. For more information about this model and why the distribution of zeroes determines the GAF, see the paper [46], the book [22, Chapters 2.3, 2.5], or the papers [48], [33] and [32].

However, for the real coefficients case the continuous part of the limiting measure has density

$$S_a\left(\frac{y}{2\pi a}\right) = 2\pi a^2 \frac{d}{dy}\left(\frac{e^{y^2}}{\sqrt{e^{2y^2}-1}}\right)$$

and the atom at $y = 0$ is of size $\sqrt{2}a$.

Both continuous densities are graphed in 1.1(b) for the parameter $a = \frac{1}{4\pi}$.

1.3.3 Exponential Spectrum

Consider a symmetric measure with exponential decay, for instance

$$d\rho(\lambda) = \operatorname{sech}(\pi\lambda)d\lambda = \frac{1}{\cosh(\pi\lambda)}d\lambda\,.$$

Here $r(z) = \operatorname{sech}(\pi z)$ as well. This model is valid in the strip $-\frac{1}{4} < \operatorname{Im}(z) < \frac{1}{4}$. Here

$$L(y) = \frac{\pi}{\cos^2(2\pi y)}.$$

For the symmetric GAF in this model, we have

$$S(y) = \frac{\pi|\sin(2\pi y)|}{\cos^2(2\pi y)}.$$

We see that the zeroes concentrate near the boundaries of the region of convergence (figure 1.1(c)).

1.4 Theorem 2: Zero-Counting Measure for a symmetric GAF

In this section we prove Theorem 2. Similar formulas were proved in specific cases. Our proof follows Macdonald [29], who has considered random polynomials (also in the multi-dimensional case). A novelty is in the extension of his result to arbitrary symmetric GAFs.

Recall that for any analytic function f (not necessarily random) in a domain D we have

$$n_f = \frac{1}{2\pi} \triangle \log|f|.$$

This is understood in the distribution sense.

Using this for our random f, we would like to take expectation of both sides, to get:

$$\mathbb{E}\left[\int_X h(z)dn_f(z)\right] = \mathbb{E}\left[\frac{1}{2\pi}\int_X \triangle h(z)\log|f(z)|dm_2(z)\right] = \frac{1}{2\pi}\int_X \triangle h(z)\mathbb{E}\left[\log|f(z)|\right]dm_2(z), \tag{1.9}$$

where m_2 denotes the Lebesgue measure in $\mathbb{C}$. The last equality is justified by Fubini's Theorem, as we show at the end of this section. Thus we can conclude that (in the weak sense):

$$\mathbb{E}(dn_f) = \frac{1}{2\pi}\triangle\,\mathbb{E}\log|f|. \tag{1.10}$$

Let us return to our setup: f is a random function generated by a basis $\phi_k(z)$ of holomorphic functions, each real on $\mathbb{R}$, and such that the sum $\sum_k |\phi_k(z)|^2$ converges locally-uniformly. Denote $\phi_k(z) = u_k(z) + iv_k(z)$ where u_k, v_k are real functions. Our random function is decomposed thus:

$$f(z) = \sum b_k\phi_k(z) = \sum b_k u_k(z) + i\sum b_k v_k(z) = u(z) + iv(z),$$

where $b_k \sim \mathcal{N}_{\mathbb{R}}(0,1)$ are *real* Gaussian standard variables. $(u(z), v(z))$ have a joint Gaussian distribution, with mean (0,0) and covariance matrix

$$\Sigma = \begin{pmatrix} \sum u_k^2 & \sum u_k v_k \\ \sum u_k v_k & \sum v_k^2 \end{pmatrix}.$$

Lemma 1.4.1. *The above matrix Σ has two positive eigenvalues $\lambda_2 \geq \lambda_1$ obeying:*

$$\lambda_{2,1} = \frac{K(z,z) \pm |K(z,\overline{z})|}{2}$$

where $K(z,w) = \sum \phi_k(z)\overline{\phi_k(w)} = \sum \phi_k(z)\phi_k(\overline{w})$.

Proof. For any complex number $\phi = u + iv$, we have:

$$u^2 = \frac{1}{2}\left(|\phi|^2 + \mathrm{Re}(\phi^2)\right),\ \ v^2 = \frac{1}{2}\left(|\phi|^2 - \mathrm{Re}(\phi^2)\right),\ \ uv = \frac{1}{2}\mathrm{Im}(\phi^2).$$

Applying this, we can rewrite Σ as

$$\Sigma = \begin{pmatrix} \frac{1}{2}\left(\sum|\phi_k|^2 + \mathrm{Re}\sum\phi_k^2\right) & \frac{1}{2}\mathrm{Im}\sum\phi_k^2 \\ \frac{1}{2}\mathrm{Im}\sum\phi_k^2 & \frac{1}{2}\left(\sum|\phi_k|^2 - \mathrm{Re}\sum\phi_k^2\right) \end{pmatrix},$$

and then calculate its determinant and trace:

$$\begin{aligned}\lambda_1\lambda_2 = \det\Sigma &= \frac{1}{4}\Big((\sum|\phi_k|^2)^2 - (\mathrm{Re}\sum(\phi_k^2))^2 - (\mathrm{Im}\sum(\phi_k^2))^2\Big)\\ &= \frac{1}{4}\left(K(z,z)^2 - |K(z,\overline{z})|^2\right),\\ \lambda_1+\lambda_2 = \mathrm{trace}\,\Sigma &= \sum|\phi_k|^2 = K(z,z).\end{aligned} \tag{1.11}$$

The lemma follows. □

Using the law of bivariate-normal distribution, we get:

$$\mathbb{E}[\log|f(z)|] = \frac{1}{2\pi\sqrt{\det\Sigma}}\iint_{\mathbb{R}^2}\log(\sqrt{x^2+y^2})e^{-\frac{1}{2}(x,y)\Sigma^{-1}(x,y)^T}dx\,dy \tag{1.12}$$

$$= \frac{1}{2\pi\sqrt{\det\Sigma}}\iint_{\mathbb{R}^2}\log(\sqrt{x^2+y^2})e^{-\frac{1}{2}(\lambda_1^{-1}x^2+\lambda_2^{-1}y^2)}dx\,dy.$$

Applying to the last integral the change of variables $x = u\sqrt{\lambda_1}$, $y = w\sqrt{\lambda_2}$, with Jacobian $\sqrt{\lambda_1\lambda_2} = \sqrt{\det\Sigma}$, we have

$$\mathbb{E}[\log|f(z)|] = \frac{1}{2\pi}\iint_{\mathbb{R}^2}\log(\sqrt{\lambda_1u^2+\lambda_2w^2})e^{-(u^2+w^2)/2}du\,dw.$$

Now, changing to polar coordinates $u = r\cos\theta$, $w = r\sin\theta$, we get:

$$\mathbb{E}[\log|f(z)|] = \frac{1}{2\pi}\int_0^{2\pi}\log(\sqrt{\lambda_1\cos^2\theta+\lambda_2\sin^2\theta})\,d\theta + C,$$

where C is a constant which does not depend on the point z (i.e., is independent of λ_1 and λ_2). In the following, we write C for any such constant (which may be different each time we use this symbol). These constants will vanish when we apply Laplacian (recall (1.10)).

So, the integral we should compute is:

$$\begin{aligned}&\int_0^{2\pi}\log\left|\sqrt{\lambda_1}\cos\theta + i\sqrt{\lambda_2}\sin\theta\right|\frac{d\theta}{2\pi}\\ &= \log(\sqrt{\lambda_1}+\sqrt{\lambda_2}) + \int_0^{2\pi}\log\left|e^{2i\theta}+\frac{\sqrt{\lambda_1}-\sqrt{\lambda_2}}{\sqrt{\lambda_1}+\sqrt{\lambda_2}}\right|\frac{d\theta}{2\pi} + C.\end{aligned}$$

The remaining integral is computed easily by Jensen's formula for the function $g(z) = z^2 + c$, where $c = \frac{\sqrt{\lambda_1}-\sqrt{\lambda_2}}{\sqrt{\lambda_1}+\sqrt{\lambda_2}} < 1$. Indeed, it has two zeroes in the unit circle, denoted a_1 and a_2, and so:

$$\int_0^{2\pi}\log\left|e^{2i\theta}+c\right|\frac{d\theta}{2\pi} = \log|g(0)| - \log|a_1| - \log|a_2| = \log|c| - 2\log\sqrt{|c|} = 0.$$

Recalling (1.10) and using the relations (1.11), we arrive at

$$\begin{aligned}\mathbb{E}[dn_f(z)] &= \frac{\triangle}{2\pi}\frac{1}{2}\log\left(\lambda_1+\lambda_2+\sqrt{4\lambda_1\lambda_2}\right)\\ &= \frac{1}{4\pi}\triangle\log\left(K(z,z)+\sqrt{K(z,z)^2-|K(z,\overline{z})|^2}\right).\end{aligned}$$

1.4.1 Justification of (1.9)

We must show that the following integral converges:

$$\frac{1}{2\pi}\int_X |\triangle h(z)|\cdot\mathbb{E}|\log|f(z)||\,dm_2(z).$$

It is enough to prove that $\mathbb{E}|\log|f(z)||$ is bounded on a compact subset S of the plane. Given z, $f(z)$ is a 2-dimensional real Gaussian variable with parameters noted above, so we get

$$\mathbb{E}|\log|f(z)|| = \frac{1}{2\pi\sqrt{\det\Sigma}}\iint_{\mathbb{R}^2}|\log(\sqrt{x^2+y^2})|e^{-\frac{1}{2}(\lambda_1^{-1}x^2+\lambda_2^{-1}y^2)}dxdy.$$

As before, λ_1,λ_2 are the eigenvalues of Σ, dependent on z. By another change of variables ($x=u\sqrt{\lambda_1},\ y=w\sqrt{\lambda_2}$) we get:

$$\mathbb{E}|\log|f(z)|| = \frac{1}{4\pi}\iint_{\mathbb{R}^2}|\log(\lambda_1u^2+\lambda_2w^2)|e^{-(u^2+w^2)/2}du\,dw.$$

Fix z, and assume $\lambda_1\le\lambda_2$. Let us split the integral into two domains: $\Omega_+=\{(u,w)\in\mathbb{R}^2:\log(\lambda_1u^2+\lambda_2w^2)\ge 0\}$ and $\Omega_-=\{(u,w)\in\mathbb{R}^2:\log(\lambda_1u^2+\lambda_2w^2)<0\}$.

Then, on Ω_+ we estimate $0<\log(\lambda_1u^2+\lambda_2w^2)<\log(\lambda_2)+\log(u^2+w^2)$. From here clearly the integral on Ω_+ is bounded by $C_0+C_1\log\lambda_2$. By lemma 1.4.1, $\lambda_2=\frac{1}{2}(K(z,z)+|K(z,\overline{z})|)$ is a continuous function of z, and therefore is bounded on our compact set S.

For $(u,w)\in\Omega_-$ notice that $0>\log(\lambda_1u^2+\lambda_2w^2)>\log(\lambda_1u^2)$, therefore:

$$\begin{aligned}&\iint_{\Omega_-}|\log(\lambda_1u^2+\lambda_2w^2)|e^{-(u^2+w^2)/2}du\,dw\le\\ &\iint_{\Omega_-}(|\log(\lambda_1)+\log(u^2)|)e^{-(u^2+w^2)/2}du\,dw\le C_0+C_1|\log\lambda_1|.\end{aligned}$$

Denote $m=\min\{\lambda_1(z):z\in S\}$. If $m=0$, this leads to $K(z_0,z_0)=0$ for some $z_0\in K$, but this means z_0 is a deterministic zero. Therefore $m>0$ and $|\log\lambda_1|$ is bounded from above.

1.5 Theorem 1: Horizontal Limiting Measure

1.5.1 Preliminaries

We present the probability space of our interest, equipped with a measure-preserving transformation. We explain the notion of ergodicity in this setup.

The probability space Ω is a countable product of copies of $\mathbb{C}$, with $\mathbb{P}$ being the product of complex Gaussian measures (one on each copy). These copies represent the random coefficients in the construction of f: each $\omega = \{a_n\}_n \in \Omega$ corresponds to a function $f_\omega(z) = \sum a_n \phi_n(z)$. $\mathcal{F}_f$ is the Borel σ-algebra generated by the basic sets $\{\omega \in \Omega : f_\omega(z) \in B(w,r)\}$, where $z \in D, r > 0$. Here $B(w,r) = \{p \in \mathbb{C} : |p - w| < r\}$. The group of automorphisms S_t shall be defined via the correspondence $\omega \leftrightarrow f_\omega$:

$$f_{S_t\omega}(z) = f_\omega(z+t).$$

The map S_t is measure-preserving, since we assumed that f is stationary. Thus, we will say the random process $f(z)$ is ergodic, if any measurable set $A \in \mathcal{F}_f$ which is invariant to all translations ($S_tA = A$, $\forall t \in \mathbb{R}$) is in fact trivial ($\mathbb{P}A \in \{0,1\}$).

In a similar way, one can define when is the zero-set Z_f ergodic (it is itself a random point-process in the plane). The space Ω, the measure $\mathbb{P}$ on it and the automorphisms $\{S_t\}$ are just as before. Now, the σ-algebra $\mathcal{F}_{Z_f}$ is generated by the basic sets $\{\omega \in \Omega : Z_{f_\omega} \cap B(z,r) \neq \emptyset\}$ with $z \in D, r > 0, B(z,r) \subset D$. Regarding this definition and other basic notions on point processes, see for instance Chapter 1.2 in the book [22].

Corollary 1.5.1. *Ergodicity of f implies ergodicity of Z_f.*

Proof. It is enough to prove $\mathcal{F}_{Z_f} \subset \mathcal{F}_f$. Let A be a countable dense set in $\mathbb{C}$. Basic sets of $\mathcal{F}_{Z_f}$ can be written as

$$\{Z_{f_\omega} \cap B(z,r) \neq \emptyset\} = \bigcup_{m\in\mathbb{N}} \bigcap_{n\in\mathbb{N}} \bigcup_{p\in A\cap B(z,r-\frac{1}{m})} \left\{ f_\omega(p) \in B\left(0, \frac{1}{n}\right)\right\},$$

which is indeed in $\mathcal{F}_f$. □

We will use the following classical result:

Theorem 3. (Fomin, Grenander, Maruyama) *A stationary GAF (symmetric or not) is ergodic w.r.t. horizontal shifts $\{f(z) \to f(z+t)\}_{t\in\mathbb{R}}$ if and only if its spectral measure ρ has no atoms.*

This theorem was originally proved for real processes over $\mathbb{R}$ (see for instance Grenander [20]), but small modifications extend it to a strip in the complex plane for both types of functions (GAFs and symmetric GAFs).

1.5.2 Existence of the horizontal limiting measure (statement (i))

As above, for $T \geq 1$, let ν_T be the random locally-finite measure on $(-\Delta, \Delta)$ defined by:

$$\nu_T(Y) = \nu_{f,T}(Y) := \frac{n_f([0,T) \times Y)}{T}, \qquad Y \subset (-\Delta, \Delta). \tag{1.13}$$

In this section we show that a.s. the measures ν_T converge weakly as T tends to infinity. First, we assume that T tends to infinity along positive integers.

In this case, we use the subscript N instead of T. By a known theorem (see for instance, [21, section 2.1]), a sequence of measures ν_N converges weakly to some measure if and only if the sequence of real numbers $\nu_N(h)$ is convergent for every $h \in C_0^\infty(-\Delta, \Delta)$. It suffices to check whether $\nu_N(h)$ is convergent for all $h \in M$, where $M \subset C_0^\infty(-\Delta, \Delta)$ is a dense set of test-functions, and we may choose M to be countable. Given a test-function $h \in M$, denote by A_h the event that $\nu_N(h)$ is a convergent sequence of numbers. To prove our claim it suffices to show $\mathbb{P}(A_h) = 1$ for every $h \in M$. Note that $\nu_N(h) = \frac{1}{N}(X_1 + X_2 + \cdots + X_N)$, where

$$X_k = X_k(h) = \int \mathbb{1}_{[k,k+1)}(x) h(y) \, dn_f(x,y) \tag{1.14}$$

is a stationary sequence of random variables.

The random variables X_k are integrable. This follows at once from an Offord-type large deviations estimate [22, Theorem 3.2.1]:

Theorem 4 (Offord-type estimate)**.** *Let f be a GAF or symmetric GAF on a domain D. Then for any compact set $K \subset D$, the number $n_f(K)$ of zeroes of f on K has exponential tail: there exist positive constants C and c depending on the covariance function of f and on K such that, for each $\lambda \geq 1$,*

$$\mathbb{P}\big\{n_f(K) > \lambda\big\} < Ce^{-c\lambda}.$$

(This theorem is stated and proved in [22] for GAFs, but minor modifications are needed to verify it for symmetric GAFs.)

Therefore, we can apply the Birkhoff theorem [9, chapter 7]. It yields that the limit $\frac{1}{N}(X_1+X_2+\cdots+X_N)$ almost surely exists, and so $\mathbb{P}(A_h)=1$. This completes the proof of the weak convergence of the sequence ν_N.

Now, we consider the general case in statement (i). Let $T \ge 1$, and let $N=[T]$ be the integer part of T. Then

$$\nu_T(h) = \frac{N}{T}\,\nu_N(h) + \underbrace{\frac{1}{T}\int \mathbb{1}_{[N,T)}(x)h(y)\,dn_f(x,y)}_{=:R_T(h)} .$$

We show that a.s. the second term on the right-hand side converges to zero for all bounded compactly supported test functions h. It suffices to prove this for all bounded test functions supported by an interval $[-\Delta_1,\Delta_1]$ with an arbitrary $0<\Delta_1<\Delta$. We have

$$|R_T(h)| \le \frac{\|h\|_\infty}{T}\, n_f\big([N,N+1)\times[-\Delta_1,\Delta_1]\big) .$$

Employing the Offord-type estimate with $K=[0,1]\times[-\Delta_1,\Delta_1]$ and using translation-invariance of the zero distribution of f, we see that for each $\varepsilon>0$,

$$\begin{aligned} &\mathbb{P}\big\{\, n_f([N,N+1]\times[-\Delta_1,\Delta_1]) \ge \varepsilon T \,\big\} \\ &\quad = \mathbb{P}\big\{\, n_f([0,1]\times[-\Delta_1,\Delta_1]) \ge \varepsilon T \,\big\} < Ce^{-c\varepsilon N} . \end{aligned}$$

Hence, for each $M\in\mathbb{N}$,

$$\mathbb{P}\big\{\limsup_{T\to\infty} |R_T(h)| \ge \varepsilon\,\|h\|_\infty\big\} \le \sum_M^\infty Ce^{-c\varepsilon N} = C(1-e^{-c\varepsilon})^{-1}e^{-c\varepsilon M} ,$$

and we conclude that a.s.

$$\lim_{T\to\infty} R_T(h) = 0$$

for all smooth compactly supported test functions h. This completes the proof of statement (i) in Theorem 1. □

1.5.3 Non-random limiting measure (statement (iia),(iii))

Here we will prove that if the spectral measure ρ_f has no atoms, then the horizontal mean zero-counting measure ν_f is not random, which is half of statement (ii). We then compute the limit ν_f, which is statement (iii).

Assume the spectral measure ρ is continuous. By Theorem 3 (Fomin-Maruyama-Grenander) we get that f is ergodic, and by corollary 1.5.1 so is Z_f. Using the notation introduced in the proof of statement (i), we get that for any smooth test function h, the stationary sequence of random variables $X_k(h)$ introduced in (1.14) is ergodic. In this case the Birkhoff ergodic theorem asserts that, a.s.,

$$\lim_{N\to\infty} \nu_N(h) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} X_k(h) = \mathbb{E}X_0(h)\,.$$

Therefore, the horizontal mean zero-counting measure is non-random, and equals

$$\mathbb{E}X_0(h) = \mathbb{E}\int \mathbb{1}_{[0,1)}(x)h(y)\,dn_f(x,y) = \int \mathbb{1}_{[0,1)}(x)h(y)\,\mathbb{E}dn_f(x,y)\,,$$

where $\mathbb{E}dn_f(x,y)$ is the mean zero-counting measure. For a GAF, we compute it directly by Edelman-Kostlan formula (1.7); while for a symmetric GAF we use the formula (1.8) (in Theorem 2). As before, denote $\psi(y) = \int_{-\infty}^{\infty} e^{-4\pi y\lambda} d\rho(\lambda)$. Note that

$$K(z,z) = \int e^{2\pi i\cdot 2yt} d\rho(t) = \psi(y), \text{ and}$$
$$K(z,\overline{z}) = \int e^{2\pi i(z-\overline{z})t} d\rho(t) = \int d\rho(t) = \psi(0),$$

where $z = x+iy$. Putting this into (1.8), we get the first intensity of zeroes:

$$\mathbb{E}n_f = \frac{1}{4\pi}\frac{d^2}{dy^2}\log\left(\psi(y) + \sqrt{\psi(y)^2-\psi(0)^2}\right) = \frac{1}{4\pi}\frac{d}{dy}\frac{\psi'(y)}{\sqrt{\psi(y)^2-\psi(0)^2}}.$$

For any $y \neq 0$, this is a derivative in the functional sense, which equals $S(y)$. At $y = 0$, the function is not defined; but the limits

$$\lim_{y\to 0+} \frac{\psi'(y)}{4\pi\sqrt{\psi(y)^2-\psi(0)^2}} = -\lim_{y\to 0-} \frac{\psi'(y)}{4\pi\sqrt{\psi(y)^2-\psi(0)^2}} = A$$

exist. This follows from $\frac{\psi'(y)}{\sqrt{\psi(y)^2-\psi(0)^2}}$ being an odd function, increasing in $y \in (0,\Delta)$. So, in order to compute $\mathbb{E}n_f$ we take the required derivative in the distribution sense, which yields the continuous point-wise derivative $S(y)$ (for $y \neq 0$) plus an atom of size $2A$ at $y = 0$.

In order to compute A let us write this limit again, and apply L'Hôpital's rule:

$$4\pi A = \lim_{y\to 0+} \frac{\psi'(y)}{\sqrt{\psi(y)^2 - \psi(0)^2}}$$
$$= \lim_{y\to 0+} \frac{\psi''(y)\cdot\sqrt{\psi(y)^2-\psi(0)^2}}{\psi(y)\cdot\psi'(y)} = (4\pi)^2\mathcal{E}_2\frac{1}{4\pi A},$$

where $\mathcal{E}_2 = \dfrac{\int_{-\infty}^{\infty}\lambda^2 d\rho(\lambda)}{\int_{-\infty}^{\infty} d\rho(\lambda)}$ is the ratio between the second and the zero moments of the spectral measure. We conclude that $A = \sqrt{\mathcal{E}_2}$, and therefore the atom has twice this size.

1.5.4 Random limiting measure (statement (iib))

In this section we prove the second half of (ii). We present the proof for symmetric GAFs, since it is slightly more involved. We assume that the spectral measure has the form

$$\rho_f = c\delta_q + c\delta_{-q} + \mu,$$

where μ is a non-trivial measure. Our goal is to show that the horizontal mean zero-counting measure of some segment $\nu_f(a,b)$ is a non-constant random variable. We may assume that $c=1$ and $q=1$ (if $q=0$ the analysis is easier).

Since $L^2_\rho(\mathbb{R})$ is the direct sum of $L^2_{\delta_1+\delta_{-1}}(\mathbb{R})$ and $L^2_\mu(\mathbb{R})$, a union of any orthonormal bases in these subspaces is an orthonormal basis in $L^2_\rho(\mathbb{R})$. By the remark at the end of section 1.1.2, after applying Fourier transform on this union we get a basis $\phi_n(z)$ from which a GAF with spectral measure ρ can be constructed. This gives the representation

$$f(z) = g(z) + \alpha\cos(2\pi z) + \beta\sin(2\pi z),$$

where $g(z)$ is a symmetric GAF with spectral measure μ and $\alpha,\ \beta \sim \mathcal{N}_{\mathbb{R}}(0,1)$ are real Gaussians, independent of each other and of g. We write for short $\eta(z) = \eta_{\alpha,\beta}(z) = \alpha\cos(2\pi z) + \beta\sin(2\pi z)$.

Fix $a,b \in (-\Delta,\Delta)$. Denote the number of zeroes of f in $[0,T]\times[a,b)$ by $N_T(g,\alpha,\beta) = \#\{z\in[0,T]\times[a,b) : g(z) = -\eta_{\alpha,\beta}(z)\}$.

Assume to the contrary that there is some constant C (depending on a and b) such that

$$\text{a.s. in } \alpha,\beta,\ \exists \lim_{T\to\infty}\frac{N_T(g,\alpha,\beta)}{T} = C. \tag{1.15}$$

Here we denote by $\mathbb{P}_g$ and $\mathbb{E}_g$ the probability and expectation (respectively) conditioned on α, β. We claim that:

$$\mathbb{E}_g \lim_{T\to\infty} \frac{N_T(g,\alpha,\beta)}{T} = \lim_{T\to\infty} \frac{\mathbb{E}_g N_T(g,\alpha,\beta)}{T}. \tag{1.16}$$

This exchange is justified by the dominated convergence principle, as seen by the following Offord-type estimate:

Proposition 1.5.1. *Let g be a symmetric stationary GAF on a horizontal strip, α and β are fixed real numbers. There exist positive constants C and c such that:*

$$\sup_{T\geq 1} \mathbb{P}_g \left(\frac{N_T(g,\alpha,\beta)}{T} > s \right) < Ce^{-cs},$$

This fact is proved in Section 1.7 below.

Next we claim that the right-hand side of (1.16) is just $\mathbb{E}_g N_1(g,\alpha,\beta)$. To see this, notice that for integer T, $N_T(g,\alpha,\beta)$ is the sum of T identically distributed random variables, all distributed like $N_1(g,\alpha,\beta)$. This follows immediately from the stationarity of g and 1-periodicity of $\eta_{\alpha,\beta}(z)$. Therefore, for integer T,

$$\frac{1}{T}\,\mathbb{E}_g N_T(g,\alpha,\beta) = \mathbb{E}_g N_1(g,\alpha,\beta).$$

For non-integer T, denote $M = \lfloor T \rfloor$. Since

$$\mathbb{E}_g N_{[M,T]} := \mathbb{E}\#\{z \in [M,T]\times[a,b) : g(z) = -\eta_{\alpha,\beta}(z)\} \leq \mathbb{E}_g N_1 < \infty,$$

it follows that for non-integer T,

$$\lim_{T\to\infty} \frac{\mathbb{E}_g N_T(g,\alpha,\beta)}{T} = \lim_{T\to\infty} \left(\frac{\mathbb{E}_g N_M}{M} \cdot \frac{M}{T} + \frac{\mathbb{E}_g N_{[M,T]}}{T} \right) = \mathbb{E}_g N_1(g,\alpha,\beta). \tag{1.17}$$

Combining (1.15), (1.16) and (1.17) we have:

$$\text{a.s. in } \alpha,\beta,\ \ \mathbb{E}_g N_1(g,\alpha,\beta) = C. \tag{1.18}$$

We divide the rest of our argument into three claims.

Claim 1.5.1. *The function $(\alpha,\beta) \mapsto \mathbb{E}_g N_1(g,\alpha,\beta)$ is continuous at the point $(0,0)$.*

Claim 1.5.2. *For any compact set $K \subset D$, let $N(g,\alpha,\beta;K)$ be the number of solutions to $g(z) = -\eta_{\alpha,\beta}(z)$ with $z \in K$. Then*

$$\mathbb{E}_g N(g,\alpha,\beta;K) \to n_{\cos(2\pi z)}(K) \quad as \quad |\alpha| \to \infty.$$

Here $n_{\cos(2\pi z)}$ is the zero-counting measure of $\cos(2\pi z)$.

Relying on the last claim and (1.18), we get that

$$\mathbb{E}_g N_1(g,\alpha,\beta) = 2\delta_0([a,b)) \tag{1.19}$$

for almost all α, β. Since $\mathbb{E}_g N_1(g,\alpha,\beta)$ is continuous at $(\alpha,\beta) = (0,0)$ (Claim 1.5.1), equation (1.19) is true for $(\alpha,\beta) = (0,0)$. The following claim asserts this happens only for one family of symmetric GAFs:

Claim 1.5.3. *If for $-\Delta < a < 0 < b < \Delta$,*

$$\mathbb{E}_g N_1(g,0,0) = \mathbb{E} n_g([0,1) \times [a,b)) = 2\delta_0([a,b)),$$

then the spectral measure of g is $\frac{1}{2}(\delta_1 + \delta_{-1})$, up to a constant multiplier.

From this last claim it follows that the spectral measure of f consists only of symmetric atoms at ± 1, which contradicts our assumption.

It remains now to prove the claims. In the course of their proof, we justify the exchange of limits and expectations by the following

Proposition 1.5.2 (Offord-type estimate for sine-like levels)**.** *Let g be a symmetric GAF on a domain D, and let α and β be fixed complex numbers. Then for any compact $K \subset D$, the number $N(g,\alpha,\beta;K)$ of solutions to $g(z) = -\eta_{\alpha,\beta}(z)$ with $z \in K$ has exponential tail: There exist positive constants C and c such that*

$$\mathbb{P}(N(g,\alpha,\beta;K) > s) \le Ce^{-cs}.$$

The proof of the last proposition is similar to that of Proposition 1.5.1, and is also included in Section 1.7.

Proof of Claim 1.5.1. First, observe that the event of g having a zero on the boundary of $[0,1] \times [a,b]$ is negligible, since the expectation of the number such zeroes is zero (this expectation, computed using Theorem 2, is zero on any line except the real line).

Using this, it follows that almost surely in g, $N_1(g,\alpha,\beta)$ approaches $N_1(g,0,0)$ as (α,β) approaches $(0,0)$. By Proposition 1.5.2, we may pass to the limit:

$$\lim_{(\alpha,\beta)\to(0,0)} \mathbb{E}_g N_1(g,\alpha,\beta_0) = \lim_{(\alpha,\beta)\to(0,0)} \int \mathbb{P}_g(N_1(g,\alpha,\beta_0) > s)ds =$$
$$\int \lim_{\alpha\to\alpha_0} \mathbb{P}_g(N_1(g,\alpha,\beta_0) > s)ds = \int \mathbb{P}_g(N_1(g,\alpha_0,\beta_0) > s)ds = \mathbb{E}_g(N_1(g,\alpha_0,\beta_0)).$$

□

Proof of Claim 1.5.2. Fix β and g. For any $\alpha \neq 0$, the zeroes of

$$h_\alpha(z) = \frac{g(z) + \beta\sin(2\pi z)}{\alpha} + \cos(2\pi z)$$

and of $f(z) = g(z) + \eta_{\alpha,\beta}(z)$ are identical. Now notice that $h_\alpha(z)$ converges locally uniformly to $\cos(2\pi z)$ as $\alpha \to \infty$ (i.e., uniformly on any compact set). By Hurwitz's Theorem, this implies that the zero-counting measures also converge locally uniformly, in the sense that for any compact $K \subset D$,

$$\lim_{\alpha\to\infty} n_{h_\alpha}(K) = n_{\cos(2\pi z)}(K).$$

By the bound in Proposition 1.5.2, this almost sure convergence in g yields moment convergence:

$$\mathbb{E}_g n_{h_\alpha}(K) \to n_{\cos(2\pi z)}(K), \text{ as } \alpha \to \infty.$$

□

Proof of Claim 1.5.3. Suppose the spectral measure is normalized, so that $\psi(0) = \int_{\mathbb{R}} d\rho(\lambda) = 1$ (else, multiply it by a constant). The premise and Theorem 1 give two conditions on $\psi(y) = K(iy,iy)$:

$$\frac{\psi'(y)}{\sqrt{\psi(y)^2 - 1}} = c\,, \qquad\qquad R = 2\sqrt{\int_{\mathbb{R}} \lambda^2 d\rho(\lambda)} = 2\,,$$

for some constant $c \in \mathbb{R}$. Solving the left-hand side ordinary differential equation, and using $\psi(0) = 1$, we get $\psi(y) = \cosh(cy)$. Since ψ is a Laplace transform of ρ, we get $\rho = \frac{1}{2}(\delta_{c/2\pi} + \delta_{-c/2\pi})$. But the right-hand side equation is satisfied only if $c = 2\pi$. □

1.6 Convergence on all intervals

In this section we prove Proposition 1.2.1. We use the notation developed in section 1.5.1. For any point in the probability space $\omega \in \Omega$, let ν_N^ω be the sequence of measures introduced in (1.13), for integer $T = N$ (the non-integer case follows just as in the proof of part (i) of Theorem 1, and will not be discussed).

Define the set

$$C = \{\omega \in \Omega : (\nu_N^\omega)_N \text{ converges weakly}\}$$

Notice that by part (i) of Theorem 1, $\mathbb{P}(C) = 1$. From general measure theory, one can deduce that almost surely, $\nu_N^\omega([a,b))$ converges to $\nu^\omega([a,b))$ for all a, b out of a countable exceptional set. This exceptional set is the set of atoms of ν^ω (which might be random). We thus turn to define

$$A = \{\omega \in \Omega : \lim_{N\to\infty} \nu_N^\omega\{a\} = \nu^\omega\{a\}, \text{ for each atom } a \text{ of } \nu^\omega\} \subset C.$$

Claim 1.6.1. *A is measurable with respect to $\mathcal{F}_f$.*

The proof of this claim will be presented in the end of this section. Our next goal would be to prove:

Claim 1.6.2. $\mathbb{P}(A) = 1$.

Our main tool will be the Ergodic Decomposition Theorem (proved, for instance, in [1, chapter 2.2.8]):

Theorem 5 (Ergodic Decomposition). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a standard Borel-space, equipped with a measure preserving transformation $S : \Omega \to \Omega$. Then the set $E^S(\Omega)$ of ergodic probability measures on Ω is not empty, and there exists a map $\beta : \Omega \to E^S(\Omega)$ such that for any measurable set $A \in \mathcal{F}$ the following holds:*

1. *the map* $\begin{cases} \Omega \to [0,1] \\ \omega \mapsto \beta_\omega(A) \end{cases}$ *is measurable.*

2. $\mathbb{P}(A) = \int_\Omega \beta_\omega(A) d\mathbb{P}(\omega)$.

Proof of Claim 1.6.2. The stationary system $(\Omega, \mathcal{F}_{Z_f}, \mathbb{P}, S)$ defined in section 1.5.1 and the set A defined above meet the requirements of the Ergodic Decomposition Theorem. Therefore, in order to prove our claim it is enough to show that

$$\forall \gamma \in E^S(\Omega), \gamma(A) = 1.$$

Fix an S-ergodic measure γ. Since A is an S-invariant set, we get $\gamma(A) \in \{0, 1\}$. Moreover, the event $\{\nu^\omega$ has an atom in the interval $I\}$ is also invariant, for any interval $I \subset (-\Delta, \Delta)$. Therefore, γ-a.s. the limiting measure ν^ω has atoms at some known points $(a_n)_{n\in\mathbb{N}} \subset (-\Delta, \Delta)$.

For a certain atom $a = a_n$, define the stationary sequence:

$$X_k(a) = n_f([k, k+1) \times \{a\}),$$

and notice that

$$\nu_N\{a\} = \frac{1}{N}\sum_{k=0}^{N-1} X_k(a).$$

As γ is ergodic, we have by Birkohff's ergodic Theorem:

$$\gamma\text{-a.s. } \nu_N^\omega\{a\} \text{ converges to } \mathbb{E}_\gamma n_f([0,1) \times \{a\}) = \nu^\omega\{a\}, \text{ as } N \to \infty$$

Since there are at most countably many atoms, we get $\gamma(A) = 1$. □

We now know that $\mathbb{P}$-a.s., the sequence ν_N is weak convergent and converges on any atom of the limiting measure (to the desired limit). A general claim from measure theory will assure us that in this case, ν_N converge on any interval:

Claim 1.6.3. *Suppose $(\nu_N)_N$ is a weak-converging sequence of measures on some interval I, and let ν be the limiting measure. If $\lim_{N\to\infty} \nu_N\{a\} = \nu\{a\}$ for every atom a of ν, then $\lim_{N\to\infty} \nu_N(J) = \nu(J)$ for* every *interval $J \subset I$.*

Proof. We demonstrate the case $J = [a, b)$, where ν has no atom at b (other cases are similar).

Given $\varepsilon > 0$, one can construct piecewise linear functions ϕ^+, $\phi^- \in C(I)$ such that:

$$\forall x,\ \phi^-(x) \le \mathbb{1}_{(a,b)} \le \mathbb{1}_{[a,b)}(x) \le \phi^+(x), \tag{1.20}$$

and additionally

$$0 < \nu(\phi^+) - \big(\nu(\phi^-) + \nu\{a\}\big) < \epsilon.$$

(For instance, for large enough parameter n, take ϕ^+ supported on $[a - \frac{1}{n}, b + \frac{1}{n}]$, equals 1 on $[a, b]$ and linear otherwise; ϕ^- supported on $[a, b]$, equals 1 on $[a + \frac{1}{n}, b - \frac{1}{n}]$, and linear otherwise).

By applying the measure ν_N to relation (1.20), we get:

$$\nu_N(\phi^-) + \nu_N\{a\} \leq \nu_N([a, b)) \leq \nu_N(\phi^+)$$

But, from our assumptions, for large enough N we have

$$\nu(\phi^-) + \nu\{a\} - \epsilon \leq \nu_N([a, b)) \leq \nu(\phi^+) + \epsilon$$

As the difference between those bounds does not exceed 3ϵ, we see the limit $\lim_{N\to\infty} \nu_N([a, b))$ exists. Since $\nu(\phi^+)$ is as close as we want to $\nu([a, b))$, we are done.

□

It remains only to prove the measurability of A.

Proof of Claim 1.6.1. We first investigate some underlying objects. Denote by $P = P(-\Delta, \Delta)$ the space of all locally finite measures on $(-\Delta, \Delta)$ induced with the Lévy - Prokhorov metric (for which convergence in metric is equivalent to weak convergence):

$$\pi(\mu, \nu) := \inf\{\varepsilon > 0 \,|\, \forall Y \in \mathcal{B}\, \mu(Y) \leq \nu(Y^\varepsilon) + \varepsilon \text{ and } \nu(Y) \leq \mu(Y^\varepsilon) + \varepsilon\},$$

where $\mathcal{B}$ is the sigma-algebra of Borel subsets of $(-\Delta, \Delta)$, and $Y^\varepsilon = \cup_{p\in Y} B(p, \varepsilon)$ is an ε-neighborhood of Y.

We claim that the map

$$\omega \mapsto \nu_1^\omega(\cdot) = n_{f_\omega}([0, 1) \times \cdot) \in P$$

is measurable; in fact, it is continuous (A small change of the coefficients $\omega = (a_1, a_2, \dots)$ in l^2 sense will yield a small change in the counting measure of zeroes ν_1^ω in Lévy - Prokhorov sense).

Now consider the space $X = P^{\mathbb{N}}$ of sequences of measures with the product topology. Notice that the map $\Omega \to X$ defined by $\omega \mapsto \{\nu_N^\omega\}$ is measurable, as each coordinate is measurable; Moreover, its image lies almost surely in the (measurable subset) of weak converging sequences. The map $C \to P$ which takes a weak converging sequence $(\nu_N) \in C$ to its limit $\nu \in P$ is also measurable. We arrive at

Observation 1. *Any measurable set $M \in P$ induces a measurable set $\widetilde{M} = \{\omega : \nu^\omega \in M\} \subset C \subset \Omega$.*

Consider the event:

$$B = \{\omega \in \Omega : \text{ the limiting measure } \nu^\omega \text{ has at least one atom}\} \subset C \subset X$$

By the last observation, B is measurable w.r.t. $\mathcal{F}$.

We construct a measurable function $h : B \to (-\Delta, \Delta)^{\mathbb{N}}$ which maps $\omega \in B$ to a list of all atoms of the limiting measure ν^ω, as follows. Let $h_1 : B \to (-\Delta, \Delta)$ be the map which maps some $\omega \in B$ to the largest atom among those of ν^ω (if some (finite) number of atoms share this property, return the left-most one). Again by observation 1, h_1 is a measurable map. In a similar manner we construct h_2, which gives the second (left-most) largest atom; and so forth. Our list of atoms is simply $h = (h_1, h_2, \dots)$. We notice that

$$A = \cap_{i\in\mathbb{N}}\{\omega : (\nu_N^\omega\{h_i\omega\})_N \text{ is a convergent sequence of numbers}\} =: \cap_{i\in\mathbb{N}} E_i.$$

All that remains is to prove measurability of E_1.

Indeed, the map $H : X \times (-\Delta, \Delta) \to \{0, 1\}$ which matches $(\{\nu_N\}, a)$ to the indicator of the event $\{(\nu_N\{a\})_N$ is a convergent sequence$\}$ is measurable; by composition of measurable maps $\mathbb{1}_{E_1} = H((\nu_N^\omega), h_1\omega)$ is a measurable function, as anticipated. □

1.7 Exponential decay of some tail events

In the course of the proof of the main theorem, we used several times exponential estimates on certain probabilities: Theorem 4, Propositions 1.5.2 and 1.5.1, and similar propositions for GAFs (which were not stated explicitly). Such estimates are sometimes referred to as "Offord-type large deviations estimates". In this section we prove Propositions 1.5.1 and 1.5.2 (the proofs are very similar). We adopt the proof of Sodin [46], presented also in [22, chapter 7].

We first present our key-lemma, which deals with 2-dimensional Gaussian random variables.

Lemma 1.7.1. *If $\eta \sim \mathcal{N}_{\mathbb{R}^2}(\mu, \Sigma)$, and E is an event in the probability space with $\mathbb{P}(E) = p$, then:*

$$|\mathbb{E}(\chi_E \log|\eta|)| \le p\left[-(1 + \frac{1}{2\lambda_1})\log p + \frac{p}{4\lambda_1} + \frac{1}{2}\log(\mathit{trace}\ \Sigma + |\mu|^2)\right],$$

where λ_1 is the biggest eigenvalue of Σ.

Proof. Upper bound: by Jensen's inequality,

$$\frac{1}{p}\mathbb{E}(\chi_E \log|\eta|^2) \le \log\left(\frac{\mathbb{E}(|\eta|^2\chi_E)}{p}\right) \le \log \mathbb{E}|\eta|^2 - \log p.$$

If $\eta = u + iv$, then

$$\mathbb{E}|\eta|^2 = \mathbb{E}u^2 + Ev^2 = \text{var }u + (\mathbb{E}u)^2 + \text{var v} + (\mathbb{E}v)^2 = \text{trace }\Sigma + |\mu|^2$$

Putting this in the previous equation, we get:

$$\mathbb{E}(\chi_E \log|\eta|) \le \frac{p}{2}[\log(\text{trace }\Sigma + |\mu|^2) - \log p]. \tag{1.21}$$

Lower bound:

$$\begin{aligned}\mathbb{E}(\chi_E \log|\eta|) &\ge -\mathbb{E}(\chi_E \log^-|\eta|)\\ &= -\mathbb{E}(\log^-|\eta|\chi_{E\cap\{|\eta|<p\}}) - \mathbb{E}(\log^-|\eta|\chi_{E\cap\{|\eta|>p\}})\end{aligned}$$

The second term may be bounded below by

$$-\mathbb{E}(\log^-|\eta|\chi_{E\cap\{|\eta|>p\}}) \ge p\log p \tag{1.22}$$

For the first term, we begin with some general manipulations:

$$\begin{aligned}-\mathbb{E}(\log^-|\eta|\chi_{E\cap\{|\eta|<p\}}) \ge -\mathbb{E}(\log^-|\eta|\chi_{\{|\eta|<p\}}) &= -\mathbb{E}\left[\chi_{|\eta|\le p}\int_0^1 \chi_{s>|\eta|}\frac{ds}{s}\right]\\ &= -\int_0^1 \mathbb{P}[|\eta|<\min(p,s)]\frac{ds}{s}\end{aligned}$$

Let us therefore bound from above the probability $\mathbb{P}(|\eta|<R)$. Denote by λ_1, λ_2 the eigenvalues of Σ, where $\lambda_1 \ge \lambda_2 \ge 0$.

$$\begin{aligned}\mathbb{P}(|\eta|<R) &= \frac{1}{2\pi\sqrt{|\Sigma|}}\int_{|x|<R}\exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right)dm_2(x)\\ &\le \frac{1}{2\pi\sqrt{|\Sigma|}}\int_{|x|<R}\exp\left(-\frac{1}{2}x^T\Sigma^{-1}x\right)dm_2(x)\\ &\le \frac{1}{2\pi}\int_{|y|<R}\exp\left(-\frac{1}{2}(\lambda_1^{-1}y_1+\lambda_2^{-1}y_2)\right)dm_2(y)\\ &\le \int_0^{R/\sqrt{\lambda_1}} e^{-\frac{1}{2}r^2}rdr = 1 - e^{-\frac{R^2}{2\lambda_1}}\end{aligned}$$

We have used the changes of variables $y = Ux$ where U is the orthogonal matrix diagonalizing Σ, and later $y_i = \sqrt{\lambda_i} w_i$ for $i = 1, 2$.

Continuing, we have:

$$-\mathbb{E}(\log^- |\eta| \chi_{E\cap\{|\eta|<p\}}) \geq -\int_0^p \frac{1-e^{-s^2/2\lambda_1}}{s} ds - \int_p^1 \frac{1-e^{-p^2/2\lambda_1}}{s} ds$$
$$= \frac{1}{2}\int_0^{p^2/2\lambda_1} e^{-t}\log(2\lambda_1 t)dt$$
$$\geq \frac{1}{2}\int_0^{p^2/2\lambda_1} \log(t)dt + \frac{p^2}{4\lambda_1}\log(2\lambda_1) = \frac{p^2}{2\lambda_1}(\log p - \frac{1}{2})$$

Therefore our lower bound is

$$\mathbb{E}(\chi_E \log|\eta|) \geq \frac{p^2}{2\lambda_1}(\log p - \frac{1}{2}) - p\log p \geq -\frac{p^2}{4\lambda_1} + (1 - \frac{1}{2\lambda_1})p\log p$$

Combining the two bounds we get the desired result. □

We now turn to the proof of our propositions.

Proof of Proposition 1.5.1. Take $\phi(z) = \phi_T(z)$ a real C^2 function, whose support is $[-\frac{1}{2}, T+\frac{1}{2}] \times [a', b']$ with $-\Delta < a' < a < b < b' < \Delta$, and which takes the value 1 on $[0,T]\times[a,b)$. We may build such $\phi_T(z)$ that will obey also the bound $\|\Delta\phi\|_{L^1} < 10(T + b - a)$. Assume α and β are fixed for now, and fix also $s > 0$. We are interested in dominating the probability of the event $A_T = \{N_T > sT\}$. Write $p = p_T = \mathbb{P}(A_T)$.

We have

$$N_T < \frac{1}{2\pi}\int \Delta\phi_T(z)\log|f(z)|dm_2(z)\,,$$

and therefore,

$$sT\cdot p \leq \mathbb{E}_g(\chi_{A_T} N_T) \leq \mathbb{E}_g\left(\chi_{A_T}\frac{1}{2\pi}\int \Delta\phi(z)\log|f(z)|dm_2(z)\right)$$
$$= \frac{1}{2\pi}\int \Delta\phi\,\mathbb{E}_g\left(\chi_{A_T}\log|f(z)|\right)dm_2(z)$$
$$\leq \frac{1}{2\pi}\|\Delta\phi\|_{L^1}\sup_{z\in D}\mathbb{E}_g\left(\chi_{A_T}\log|f(z)|\right)$$

Before we continue, let us justify the exchange of expectation and integral. Recall $f(z) = g(z) + \eta_{\alpha,\beta}(z)$; so in order to use Fubini's theorem we need

$$\int_D \mathbb{E}_g|\Delta\phi(z)\cdot \log|g(z)+\eta_{\alpha,\beta}(z)||dm_2(z)$$
$$= \int_D |\Delta\phi(z)|\,\mathbb{E}_g|\log|g(z)+\eta_{\alpha,\beta}(z)||<\infty \qquad (1.23)$$

For each $z\in D$, $f(z)=g(z)+\eta_{\alpha,\beta}(z)$ is a 2 dimensional Gaussian random variable, with mean $\mu(z)=\eta_{\alpha,\beta}(z)$, and the same covariance matrix $\Sigma(z)$ as the 2 dimensional Gaussian r.v. $g(z)$. By lemma 1.4.1, we see that both $\mu(z)$ and $\Sigma(z)$ depend continuously on the paremeter z. So, the function $\mathbb{E}_g|\log|g(z)+\eta_{\alpha,\beta}(z)|$ is bounded above for $z\in \text{support}(\phi)$, which ends this argument.

Notice further, that in our stationary case $\lambda_1(z)$, $\lambda_2(z)$, the eigenvalues of $\Sigma(z)$, depend on y only, where $z=x+iy$. Therefore they have lower and upper bounds on $\mathbb{R}\times[a',b']$. Notice that also $\mu(z)$, being a trigonometric function, has such bounds. By applying lemma 1.7.1 with $\eta(z)=g(z)+\eta_{\alpha,\beta}(z)$, we get:

$$\sup_{z\in\mathbb{R}\times[a',b']} \mathbb{E}(\chi_{A_T}\log|g(z)+\zeta|)<p(c_1-c_2\log p).$$

where c_1,c_2 are positive constants (c_1 depending on α,β, the horizontal lines a,b, and the kernel of g). Putting all this together, we get:

$$sT\cdot p\le \frac{5}{\pi}(T+b-a)p(c_1-c_2\log p),$$

which leads to the exponential bound we strived for:

$$\exists c,C>0\,\text{such that}\; p_T=\mathbb{P}_g(N_T>T\,s)\le Ce^{-cs},\ \forall T\ge 1\,.$$

□

Proof of Proposition 1.5.2. We follow the outline of the previous proof. Let $\phi(z)$ be a real C^2 function supported on a compact K' such that $K\subset K$, and $\phi(z)=1$ for all $z\in K$. Denoting $N=N(g,\alpha,\beta;K)$ and $A=\{N>s\}$, we have

$$N<\frac{1}{2\pi}\int \Delta\phi(z)\log|f(z)|dm_2(z),$$

and so

$$
\begin{aligned}
s\mathbb{P}(A) &\leq \mathbb{E}_g(\chi_A N) \leq \mathbb{E}_g\left(\chi_A \frac{1}{2\pi}\int \Delta\phi(z)\log|f(z)|dm_2(z)\right)\\
&= \frac{1}{2\pi}\int \Delta\phi\,\mathbb{E}_g\left(\chi_A \log|f(z)|\right)dm_2(z)\\
&\leq \frac{1}{2\pi}\left\|\Delta\phi\right\|_{L^1}\sup_{z\in K'}\mathbb{E}_g\left(\chi_A\log|f(z)|\right)\\
&\leq c_3\cdot\mathbb{P}(A)(c_1 - c_2\log\mathbb{P}(A)).
\end{aligned}
$$

The change of integral and expectation is justified by (1.23) on the domain K' (no change in the arguments), and in the last line we used Lemma 1.7.1 with uniform bounds on $\lambda_1(z)$, $\lambda_2(z)$ and $\mu(z)$ for $z \in K'$. The constant c_3 stands for $\frac{1}{2\pi}\left\|\Delta\phi\right\|_{L^1}$. The last inequality clearly leads to an exponential bound in s on the probability $\mathbb{P}(A)$. □

1.8 Directions of further research

1.8.1 Random Trigonometric Series

The results described in this chapter do not give much information about the measure ν_f when the spectral measure contains atoms (i.e., when ν_f is random). It would be interesting to know, for instance, under what conditions is ν_f a.s. free of atoms or absolutely continuous.

These questions have the following concrete form, when the spectral measure is purely atomic: The function is described by a random trigonometric series

$$f(z) = \sum_n a_n w_n e^{i\lambda_n z}$$

where $a_n \sim \mathcal{N}_{\mathbb{C}}(0,1)$ are i.i.d., and $w_n, \lambda_n \in \mathbb{R}$ obey $\sum_n w_n^2 e^{\lambda_n y} < \infty$ for all $|y| < \Delta$. Continuity and other traits of ν_f might be effected by the choice of parameters $\{\lambda_n\}$ and $\{w_n\}$ (in fact, arithmetic properties of the frequencies $\{\lambda_n\}$ are expected to play a role in the answer).

By the argument principle, these questions relate to the mean increment of the argument of $f(z)$ on some horizontal line (often called "mean motion"). For deterministic functions it is a well-known problem, posed by Lagrange, to determine whether the mean motion always exists. The answer was proved to be positive in a sequence of works ranging between 1916-1945 (including authors such as Weyl, Wintner, Jessen and Tornehave), but hardly anything could be said about the limit itself or its dependence on the parameters. I believe the stochastic version will be easier to analyze and will shed light on the deterministic questions as well.

1.8.2 Universality

In related models, such as random polynomials, random matrices and some lattice models, replacing the probabilistic distribution (in some range of well-behaved distributions) will not affect the limiting object. This phenomenon is often called *universality*. In our context, it is interesting to consider random stationary series, aside from Gaussian. For an explicit example, consider the Paley-Wiener process

$$f(z) = \sum_{n\in\mathbb{Z}} a_n \frac{\sin(\pi(z-n))}{z-n},$$

where $\{a_n\}_{n\in\mathbb{Z}}$ are i.i.d. random variables, with $\mathbb{E}a_0 = 0$ and $\mathbb{E}|a_0|^2 = 1$. The resulting sum is a stationary function with covariance $\mathbb{E}\left[f(z)\overline{f(w)}\right] = \frac{\sin(\pi(z-\overline{w})}{z-\overline{w}}$, so in particular the values of points lying on the lattice $\mathbb{Z}$ are independent (distributed like a_0).

It would be interesting to investigate limiting properties of f and their dependence on the distribution of the coefficients. In particular, we ask when is the horizontal density of zeroes non-random, and when is it the same as in the Gaussian case.

Chapter 2

Fluctuations of the number of zeroes

2.1 Introduction

In Chapter 1, we gave a law of large numbers for the zeroes in a long horizontal rectangle $[0,T]\times[a,b]$ (Theorem 1), which extends a result of Wiener [37, chapter X]. Here we go further to study the variance of the number of zeroes in such a rectangle. In Theorems 6 and 7 we show that this number is asymptotically between cT and CT^2 with positive constants c and C, and give conditions (in terms of the spectral measure) for the asymptotics to be exactly linear or quadratic in T. In Theorem 8 we give some conditions for intermediate variance.

2.1.1 Recapitulation of Definitions

We recall some definitions and notation from the previous chapter. Let f be a stationary GAF in the strip D_Δ. In other words, f is a random variable taking values in the space of analytic functions on D_Δ, whose finite marginals have a mean zero complex Gaussian distribution, and whose distribution is invariant to real shifts. As before, denote the covariance function by

$$r(t) = \mathcal{F}[\rho](x) = \int_{\mathbb{R}} e^{-2\pi i t\lambda} d\rho(\lambda),$$

where ρ is a non-negative finite measure on $\mathbb{R}$, which we call *the spectral measure* of f. We recall that, since $r(t)$ has an analytic continuation to the strip $D_{2\Delta}$, ρ must have a finite exponential moment:

$$\text{for each } |\Delta_1| < \Delta, \ \int_{-\infty}^{\infty} e^{2\pi\cdot 2\Delta_1|\lambda|} d\rho(\lambda) < \infty\,. \tag{2.1}$$

2.1.2 Results

In Chapter 1, we have studied convergence to the mean of the number of zeroes of a stationary analytic function in a long rectangle (recall Theorem 1). The next natural question is, how big are the fluctuations of the number of zeroes in a long rectangle? More rigorously, define

$$R_T^{a,b} = [-T,T) \times [a,b], \ V_f^{a,b}(T) = \operatorname{var}\left[n_f(R_T^{a,b})\right],$$

where for a random variable X the variance is defined by

$$\operatorname{var}(X) = \mathbb{E}\left(X - \mathbb{E}X\right)^2.$$

We are interested at the asymptotic behavior of $V_f^{a,b}(T)$ as T approaches infinity. The next two theorems show that $V_f^{a,b}(T)$ is asymptotically bounded between cT and CT^2 for some $c, C > 0$, and give conditions under which each of the bounds is achieved. We begin by stating the upper bound result, a relatively easy consequence of Theorem 1.

Theorem 6. *Let f be a non-degenerate stationary GAF in a strip D_Δ. Then for all $-\Delta < a < b < \Delta$ the limit*

$$L_2 = L_2(a,b) := \lim_{T\to\infty} \frac{V_f^{a,b}(T)}{T^2} \in [0,\infty)$$

exists. This limit is positive if and only if the spectral measure of f has a non-zero discrete component.

The lower bound result, which is our main result, is stated in the following theorem.

Theorem 7. *Let f be a non-degenerate stationary GAF in a strip D_Δ. Then for all $-\Delta < a < b < \Delta$ the limit*

$$L_1 = L_1(a,b) := \lim_{T\to\infty} \frac{V_f^{a,b}(T)}{T} \in (0,\infty]$$

exists. Moreover, the limit $L_1(a,b)$ is finite if ρ is absolutely continuous with density $d\rho(\lambda) = p(\lambda)d\lambda$, such that

$$(1+\lambda^2)e^{2\pi\cdot 2y\lambda}p(\lambda) \in L^2(\mathbb{R}), \ \text{for } y \in \{a,b\}. \tag{2.2}$$

Remark 2.1.1. Another form of condition (2.2) is the following: For $y \in \{2a, 2b\}$,

$$\int_{\mathbb{R}} |r(x+iy)|^2 dx, \quad \int_{\mathbb{R}} |r''(x+iy)|^2 dx < \infty.$$

This implies also that $\int_{\mathbb{R}} |r'(x+iy)|^2 dx < \infty$. Moreover, since the set $\{c : \ e^{2\pi \cdot c\lambda} p(\lambda) \in L^2(\mathbb{R})\}$ is convex, it implies the same condition for all $y \in [2a, 2b]$.

The next theorem deals with conditions under which $L_1(a,b)$ is infinite.

Theorem 8. *Let f be a non-degenerate stationary GAF in a strip D_Δ.*

(i) *Suppose $J \subset (-\Delta, \Delta)$ is a closed interval such that for every $y \in J$, the function $\lambda \mapsto (1+\lambda^2)e^{2\pi \cdot 2y\lambda}p(\lambda)$ does not belong to $L^2(\mathbb{R})$. Then for every $\alpha \in J$ the set $\{\beta \in J : \ L_1(\alpha, \beta) < \infty\}$ is at most finite.*

(ii) *The limit $L_1(a,b)$ is infinite for particular a, b if either ρ does not have density, or, if it has density p and for any two points $\lambda_1, \lambda_2 \in \mathbb{R}$ there exists intervals I_1, I_2 such that I_j contains λ_j ($j = 1, 2$) and*

$$(1+\lambda)e^{2\pi \cdot 2y\lambda}p(\lambda) \notin L^2(\mathbb{R} \setminus (I_1 \cup I_2)), \tag{2.3}$$

for at least one of the values $y = a$ or $y = b$.

Remark 2.1.2. There is a gap between the conditions given for linear variance (in Theorem 7) and those for super-linear variance (in Theorem 8). For instance, the theorems do not decide about all the suitable pairs (a, b) in case the spectral measure has density $\frac{1}{\sqrt{|\lambda|}} 1\!\!1_{[-1,1]}(\lambda)$. On the other hand, we are ensured to have super-linear variance in case ρ has a singular part. If ρ has density $p \in L^1(\mathbb{R})$ which is bounded on any compact set, then $(1+\lambda^2)p(\lambda) \in L^2(\mathbb{R})$ implies asymptotically linear variance, and $(1+\lambda)p(\lambda) \notin L^2(\mathbb{R})$ implies asymptotically super-linear variance.

Remark 2.1.3. Minor changes to the developments in this chapter may be made in order to prove analogous results regarding the increment of the argument of a stationary GAF f along a horizontal line. Namely, let $V^{a,a}(T)$ denote the variance of the increment of the argument of f along the line $[0, T] \times \{a\}$ (for some $-\Delta < a < \Delta$). Then:

- the limit $L_2(a) = \lim_{T\to\infty} \frac{V^{a,a}(T)}{T^2}$ exists, belongs to $[0, \infty)$, and is positive if and only if the spectral measure contains an atom.

- the limit $L_1(a) = \lim_{T\to\infty} \frac{V^{a,a}(T)}{T}$ exists, belongs to $(0,\infty]$, and is finite if ρ has density $p(\lambda)$ such that $(1+\lambda^2)e^{2\pi\cdot 2a\lambda}p(\lambda) \in L^2(\mathbb{R})$. Moreover, $L_1(a)$ is infinite if for any $\lambda_0 \in \mathbb{R}$ there is an interval I containing λ_0 such that the measure $(1+\lambda)e^{2\pi\cdot 2a\lambda}d\rho(\lambda)$ restricted to $\mathbb{R}\setminus I$ is not in $L^2(\mathbb{R})$.

In fact, the first item is essentially proved in this chapter (Claim 2.3.11 below).

The rest of the chapter is organized as follows: Theorem 6 concerning quadratic growth of variance is proved in Section 2.2, and is mainly a consequence of Theorem 1. For theorems 7 and 8 we develop in Section 2.3 an asymptotic formula for $V_f^{a,b}(T)/T$ (Proposition 2.3.1 below). Then we prove Theorem 7 by analyzing this formula and using tools from harmonic analysis. We end by proving Theorem 8 in Section 2.5.

2.1.3 Discussion

We mention here some related results in the literature (though they do not seem to apply directly to our case). The question for real processes (not necessarily real-analytic) was treated by many authors. An asymptotic formula for the variance was given in Cramer and Leadbetter [9], but the rate of growth is not apparent from it. Cuzick [10] proved a Central Limit Theorem (CLT) for the number of zeroes, whose main condition is linear growth of the variance. Later, Slud [45], using stochastic integration methods he developed earlier with Chambers [8], proved that in case the spectral measure has density which is in $L^2(\mathbb{R})$, this condition is satisfied. It is interesting to note that the condition for linear variance in the present theorem (condition (2.2)) is the main assumption in the work by Slud for real (non-analytic) processes.

More recently, Granville and Wigman [19] studied the number of zeroes of a Gaussian trigonometric polynomial of large degree N in the interval $[0,2\pi]$, and showed the variance of this number is linear in N. This work was extended to other level-lines by Azaïs and León [5].

Sodin and Tsirelson [48] and Nazarov and Sodin [34] studied fluctuations of the number of zeroes of a planar GAF (a special model which is invariant to plane isometries), proving linear growth of variance and a CLT for the zeroes in large balls (as the radius approaches infinity).

2.2 Theorem 6: Quadratic Variance

Recall the notation $R_T = R_T^{a,b} = [-T, T] \times [a, b]$. From Proposition 1.2.1 we know that

$$\lim_{T\to\infty} \frac{n_f(R_T)}{T} = Z,$$

where Z is some random variable and the limit is in the almost sure sense. Moreover, $\operatorname{var} Z > 0$ if and only if the spectral measure of f contains an atom. Clearly

$$\operatorname{var}\left(\lim_{T\to\infty} \frac{n_f(R_T)}{T}\right) = \operatorname{var} Z$$

Theorem 6 would be proved if we could change the limit with the variance on the left-hand side. By dominant convergence, it is enough to find an integrable majorant for the tails of

$$X_T = \frac{n_f(R_T)}{T} \text{ and } X_T^2 = \frac{n_f(R_T)^2}{T^2}.$$

To this end we refer to an Offord-type estimate, which provides exponential bounds on tails of X_T:

Proposition 2.2.1. *Let f be a stationary GAF in some horizontal strip, then using the notation above we have*

$$\exists C, c > 0 : \sup_{T\geq 1} \mathbb{P}(X_T > s) < Ce^{-cs} = h(s).$$

The statement is very similar to that of Proposition 1.5.1, where $\alpha = \beta = 0$ and g is a GAF (not a symmetric GAF). We omit the proof as it follows directly from the proof of the latter proposition, changing every Gaussian distribution in $\mathbb{R}^2$ to one in $\mathbb{C}$ (a particular case).

We may then conclude that

$$\sup_{T\geq 1} \mathbb{P}(X_T^2 > s) < Ce^{-c\sqrt{s}} = h(\sqrt{s}).$$

Since both $h(s)$ and $h(\sqrt{s})$ are integrable on $\mathbb{R}$, we have the desired majorants. Exchanging limit and variance then yields:

$$\lim_{T\to\infty} \frac{\operatorname{var}\,(n_f(R_T))}{T^2} = \lim_{T\to\infty} \operatorname{var}\left(\frac{n_f(R_T)}{T}\right) = \operatorname{var}\left(\lim_{T\to\infty} \frac{n_f(R_T)}{T}\right) = \operatorname{var} Z,$$

and the result is proved.

2.3 An Asymptotic Formula for the Variance

This section is devoted to the derivation of a formula for the variance $V_f^{a,b}(T) = \operatorname{var} n_f([-T,T]\times[a,b])$ where T is large. We prove the following:

Proposition 2.3.1. *Let f be a stationary GAF in D_Δ with spectral measure ρ. Suppose ρ has no discrete component. Then for any $-\Delta < a < b < \Delta$, and any $T \in \mathbb{R}$, the series*

$$v^{a,b}(T) = \frac{1}{4\pi^2}\sum_{k\geq 1}\frac{1}{k^2}\int_{\mathbb{R}}\int_{\mathbb{R}} T\operatorname{sinc}^2\left(2\pi T(\lambda-\tau)\right) h_k^{a,b}(\lambda+\tau)d\rho^{*k}(\lambda)d\rho^{*k}(\tau)$$

converges, and

$$\lim_{T\to\infty}\left(\frac{V^{a,b}(T)}{2T} - v^{a,b}(T)\right) = 0.$$

Here ρ^{*k} is the k-fold convolution of ρ, $\operatorname{sinc}(x) = \frac{\sin x}{x}$, *and*

$$h_k^{a,b}(\lambda) = \left(l_k^a(\lambda)e^{2\pi a\lambda} - l_k^b(\lambda)e^{2\pi b\lambda}\right)^2,$$

where

$$l_k^y(\lambda) = \frac{\partial}{\partial y}\left(\frac{1}{r^k(2iy)}\right) + \frac{2\pi}{r^k(2iy)}\lambda,\ \textit{for } y\in(-\Delta,\Delta), k\in\mathbb{N}.$$

2.3.1 Integrals on significant edges.

The boundary of the rectangle $R_T = [-T,T]\times[a,b]$ is composed of four segments $\partial R_T = \bigcup_{1\leq i\leq 4} I_j$ with induced orientation from the counter-clockwise orientation of ∂R_T, where $I_1 = [-T,T]\times\{a\}$ and $I_3 = [T,-T]\times\{b\}$. By the argument principle,

$$n_f(R_T) = \sum_{1\leq i\leq 4}\frac{1}{2\pi}\triangle_i^T \arg f,$$

where $\triangle_i^T \arg f$ is the increment of the argument of f along the segment I_i (a.s. f has no zeroes on the boundary of the rectangle R_T [1]).

[1] To see this, first notice that the distribution of $n_f(I_j)$ for $j = 2,4$ (the number of zeroes in a "short" vertical segments) does not depend on T. If it were not a.s. zero, then $\mathbb{E}n_f(I_2) > 0$. Now for any finite set of points $\{t_j\}_{j=1}^N \subset [0,1]$, we have $\mathbb{E}n_f([0,1]\times[a,b]) \geq \sum_{j=1}^N \mathbb{E}n_f(\{t_j\}\times[a,b]) = N\mathbb{E}n_f(I_2)$, yielding $\mathbb{E}n_f([0,1]\times[a,b]) = \infty$ - which is false. For

Then, by the argument principle,

$$V_f^{a,b}(T) = \text{var}\,[n_f(R_T)] = \frac{1}{4\pi^2} \sum_{1\le i,j\le 4} \text{cov}\,\left(\triangle_i^T \arg f,\ \triangle_j^T \arg f\right), \qquad (2.4)$$

where

$$\text{cov}\,(X,Y) = \mathbb{E}[XY] - \mathbb{E}X \cdot \mathbb{E}Y.$$

Our first claim is that asymptotically when T is large, the terms involving the (short) vertical segments are negligible in this sum.

Claim 2.3.1. *As $T \to \infty$,*

$$V_f^{a,b}(T) = \frac{1}{4\pi^2} \sum_{i,j\in\{1,3\}} \text{cov}\,\left(\triangle_i^T \arg f,\ \triangle_j^T \arg f\right)$$
$$+ O\left(1 + \sqrt{\text{var}\,(\triangle_1^T \arg f)} + \sqrt{\text{var}\,(\triangle_3^T \arg f)}\right).$$

Proof. We demonstrate how to bound one of the terms in (2.4) involving a "short" vertical segment (corresponding, say, to $i = 2$). We have by stationarity:

$$\text{var}\,(\triangle_2^T \arg f) = \text{var}\,(\triangle_2^0 \arg f) =: c^2$$

Therefore by Cauchy-Schwarz,

$$\begin{aligned}\text{cov}\,\left(\triangle_1^T \arg f,\ \triangle_2^T \arg f\right) &\le \sqrt{\text{var}\,(\triangle_1^T \arg f)}\sqrt{\text{var}\,(\triangle_2^T \arg f)} \\ &= c \cdot \sqrt{\text{var}\,(\triangle_1^T \arg f)}.\end{aligned}$$

□

We now give an alternative formulation of Claim 2.3.1. Using Cauchy-Riemann equations we write:

$$\triangle_1^T \arg f = \int_{-T}^{T} \left(\frac{\partial}{\partial x} \arg f(x+ia)\right) dx = -\int_{-T}^{T} \frac{\partial}{\partial a} \log|f(x+ia)|\, dx =: -X^a(T)$$
$$\triangle_3^T \arg f = -\int_{-T}^{T} \left(\frac{\partial}{\partial x} \arg f(x+ib)\right) dx = \int_{-T}^{T} \frac{\partial}{\partial b} \log|f(x+ib)|\, dx = X^b(T)$$

$j = 1, 3$, recall that since there are no atoms in the spectral measure, f is ergodic with respect to horizontal shifts (this is Fomin-Grenander-Maruyama Theorem, see explanation and references within [15]). This implies that each horizontal line (such as $L_a = \mathbb{R} \times \{a\}$) either a.s. contains a zero or a.s. contains no zeroes. If the former holds, then also $\mathbb{E}n_f([0,1]\times\{a\}) > 0$, and the measure ν_f from Theorem 1 has an atom at a - contradiction to part (iii) of that Theorem.

Denoting $C^{a,b}(T) = \operatorname{cov}(X^a(T), X^b(T))$ we may rewrite Claim 2.3.1 as

$$V_f^{a,b}(T) = \frac{1}{4\pi^2}\left(C^{a,a}(T) - 2C^{a,b}(T) + C^{b,b}(T)\right) + O\left(1 + \sqrt{C^{a,a}(T)} + \sqrt{C^{b,b}(T)}\right),$$

and so we arrive at:

Claim 2.3.1a. *As $T \to \infty$, we have:*

$$\frac{V^{a,b}(T)}{2T} = \frac{C^{a,a}(T) - 2C^{a,b}(T) + C^{b,b}(T)}{4\pi^2 \cdot 2T} + O\left(\frac{1 + \sqrt{C^{a,a}(T)} + \sqrt{C^{b,b}(T)}}{T}\right).$$

Later on we shall prove that $\lim_{T\to\infty} \frac{\sqrt{C^{a,a}(T)}}{T} = 0$ if no atom is present in the spectral measure (Claim 2.3.11 below). This may be viewed as a one-dimensional counterpart of Theorem 6 (though the methods of proof are different). In the mean time, we turn to find an expression for $C^{a,b}(T)$, which will be refined through most of the section.

2.3.2 Passing to covariance of logarithms

Our first step is a technical change of order of operations.

Claim 2.3.2.

$$C^{a,b}(T) = \int_{-T}^{T}\int_{-T}^{T} \frac{\partial^2}{\partial a\,\partial b}\operatorname{cov}\left(\log|f(t+ia)|,\ \log|f(s+ib)|\right) dt\ ds.$$

We comment that the right-hand-side (RHS) of the equation contains a mixed partial derivative, so for $C^{a,a}(T)$ the computation is as follows: take the prescribed mixed derivative (as if $a \neq b$) and then substitute $b = a$.

Proof. Following the definition of $C^{a,b}(T)$, we shall first prove that

$$\mathbb{E}\left\{\int_{-T}^{T} dt \int_{-T}^{T} ds \left(\frac{\partial}{\partial a}\log|f(t+ia)|\ \frac{\partial}{\partial b}\log|f(s+ib)|\right)\right\} - \mathbb{E}\left\{\int_{-T}^{T}\frac{\partial}{\partial a}\log|f(t+ia)|\,dt\right\}\mathbb{E}\left\{\int_{-T}^{T}\frac{\partial}{\partial b}\log|f(s+ib)|\,ds\right\}$$

coincides with

$$\int_{-T}^{T}\int_{-T}^{T}\Big[\mathbb{E}\frac{\partial^2}{\partial a\,\partial b}\left\{\log|f(t+ia)|\ \log|f(s+ib)|\right\} - \mathbb{E}\frac{\partial}{\partial a}\log|f(t+ia)|\,\mathbb{E}\frac{\partial}{\partial b}\log|f(s+ib)|\ \Big]dt\ ds. \tag{2.5}$$

Notice that
$$\left|\frac{\partial}{\partial a}\log|f(x+ia)|\right| \le \left|\frac{f'(x+ia)}{f(x+ia)}\right|.$$
Therefore, by Fubini's theorem, it is enough to prove the following two statements:

(I) $\mathbb{E}\left|\frac{f'(t+ia)}{f(t+ia)}\right| < \infty$ for all $t\in\mathbb{R}$, $a\in(-\Delta,\Delta)$, and
$$\int_{-T}^{T}\mathbb{E}\left|\frac{f'(t+ia)}{f(t+ia)}\right| dt < \infty.$$

(II) $\mathbb{E}\left|\frac{f'(t+ia)}{f(t+ia)}\frac{f'(s+ib)}{f(s+ib)}\right| < \infty$ for all $t,s\in\mathbb{R}, t\neq s$ and $(a,b)\in(-\Delta,\Delta)^2$, and
$$\int_{-T}^{T}\int_{-T}^{T}\mathbb{E}\left|\frac{f'(t+ia)}{f(t+ia)}\frac{f'(s+ib)}{f(s+ib)}\right| dt\ ds < \infty.$$

After proving this, it will also follow that (2.5) coincides with
$$\int_{-T}^{T}\int_{-T}^{T}\frac{\partial^2}{\partial a\ \partial b}\Big[\mathbb{E}\{\log|f(t+ia)|\ \log|f(s+ib)|\} \\ - \mathbb{E}\log|f(t+ia)|\,\mathbb{E}\log|f(s+ib)|\ \Big]dt\ ds,$$
thus ending the proof of our claim. Indeed, it is enough to see that for any fixed t,s, $t\neq s$ and given a_0,b_0 we have
$$\frac{\partial}{\partial a}\mathbb{E}\log|f(t+ia)|\,\big|_{a=a_0} = \mathbb{E}\frac{\partial}{\partial a}\log|f(t+ia)|\,\big|_{a=a_0}, \tag{2.6}$$
$$\frac{\partial^2}{\partial a\ \partial b}\mathbb{E}\{\log|f(t+ia)|\ \log|f(s+ib)|\}\,\big|_{a=a_0,\ b=b_0} \tag{2.7}$$
$$= \mathbb{E}\frac{\partial^2}{\partial a\ \partial b}\{\log|f(t+ia)|\ \log|f(s+ib)|\}\,\big|_{a=a_0,\ b=b_0}.$$
To see this, fix $\varepsilon>0$ and define the event
$$G_\varepsilon = \{\omega\in\Omega:\ f_\omega(z)\neq 0\,\forall z\in B(t+ia_0,\varepsilon)\cup B(s+ib_0,\varepsilon)\},$$
where $B(w,\varepsilon)=\{z\in\mathbb{C}:\ |z-w|<\varepsilon\}$. Under G_ε, each derivative ($\frac{\partial}{\partial a}$ or $\frac{\partial^2}{\partial a\ \partial b}$) is in fact a limit of a sequence of random variables, which are dominated by $\left|\frac{f'(t+ia)}{f(t+ia)}\right|+1$ or $\left|\frac{f'(t+ia)}{f(t+ia)}\frac{f'(t+ib)}{f(t+ib)}\right|+1$ respectively. By items I and II,

these majorants have finite expectations, so by dominated convergence we get that

$$\frac{\partial}{\partial a}\Big|_{a=a_0}\mathbb{E}\left(\log|f(t+ia)|\,1\!\!1_{G_\varepsilon}\right)=\mathbb{E}\left(\frac{\partial}{\partial a}\Big|_{a=a_0}\log|f(t+ia)|\,1\!\!1_{G_\varepsilon}\right),\qquad(2.8)$$

and a similar statement for the double derivative. Now,

$$\begin{aligned}\mathbb{E}\left(\frac{\partial}{\partial a}\log|f(t+ia)|\right) &= \lim_{\varepsilon\to 0+}\mathbb{E}\left(\frac{\partial}{\partial a}\log|f(t+ia)|\,1\!\!1_{G_\varepsilon}\right)\\ &= \lim_{\varepsilon\to 0+}\frac{\partial}{\partial a}\mathbb{E}\left(\log|f(t+ia)|\,1\!\!1_{G_\varepsilon}\right)\\ &= \frac{\partial}{\partial a}\lim_{\varepsilon\to 0+}\mathbb{E}\left(\log|f(t+ia)|\,1\!\!1_{G_\varepsilon}\right)\\ &= \frac{\partial}{\partial a}\mathbb{E}\left(\log|f(t+ia)|\right).\end{aligned}$$

The first and last equalities are due to $\lim_{\varepsilon\to 0}=\mathbb{P}(G_\varepsilon)=1$. The second is precisely (2.8), and the third follows from monotonicity of the limit in ε (Dini's theorem). This establishes (2.6). An analogous argument establishes (2.7).

We now turn to prove (I). Let $z=t+ia$ be fixed. The vector $(f(z),f'(z))$ is jointly Gaussian, in fact we may write

$$f(z)=\rho f'(z)+Y(z)$$

where ρ is a number and $Y(z)$ is a Gaussian random variable independent of $f'(z)$. Therefore,

$$\big(f(z)|f'(z)\big)\sim\mathcal{N}_{\mathbb{C}}(0,\sigma^2)+\mu(f'(z));$$

that is, $f(z)$ conditioned on the value of $f'(z)$ is Gaussian, with mean depending on $f'(z)$ and variance not depending on it (equal to $\sigma^2=\mathrm{var}\,(Y(z))$). The following is a straightforward computation.

Lemma 2.3.1. *Let $\sigma>0$ and $\zeta\sim\mathcal{N}_{\mathbb{C}}(0,\sigma^2)$. Then there is a constant $C_\sigma>0$ such that for any $\mu\in\mathbb{C}$, $\mathbb{E}\frac{1}{|\zeta+\mu|}<C_\sigma$.*

Using this lemma, we have

$$\mathbb{E}\left|\frac{f'(z)}{f(z)}\right|=\mathbb{E}\;\mathbb{E}\left(\left|\frac{f'(z)}{f(z)}\right|\;\Big|\;f'(z)\right)\le\mathbb{E}\left(|f'(z)|\cdot C_{\mathrm{Im}(z)}\right),$$

where $C_{\mathrm{Im}(z)}$ is a constant which depends only on $\mathrm{Im}(z)$. The notation $\mathbb{E}\;\mathbb{E}(X|Y)$ for random variables X,Y means first taking the conditional expectation of X given Y (which results in a function of Y), then taking expectation of this function. Now (I) follows easily.

We now turn to prove (II). We use the notation $f \lesssim g$ to stand for the inequality $f \leq C \cdot g$, where $C > 0$ is a constant (which may vary from line to line). Similarly, $f \eqsim g$ stands for $f = C \cdot g$ with some $C > 0$.

Let $q > 2$, and let $1 < p < 2$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. By Hölder inequality we have

$$\begin{aligned}
&\int_{-T}^{T}\int_{-T}^{T}\mathbb{E}\left|\frac{f'(t+ia)\ f'(s+ib)}{f(t+ia)\ f(s+ib)}\right|dtds \\
&\leq \int_{-T}^{T}\int_{-T}^{T}\left[\mathbb{E}\left|f'(t+ia)f'(s+ib)\right|^{q}\right]^{1/q}\left[\mathbb{E}\left|f(t+ia)f(s+ib)\right|^{-p}\right]^{1/p} dt\ ds \\
&\lesssim \int_{-T}^{T}\int_{-T}^{T}\mathbb{E}\left[\left|f(t+ia)f(s+ib)\right|^{-p}\right]^{1/p} dt\ ds. \qquad (2.9)
\end{aligned}$$

The last inequality is an application of Cauchy-Schwarz inequality and stationarity, as follows:

$$\begin{aligned}
&\left[\mathbb{E}\left|f'(t+ia)f'(s+ib)\right|^{q}\right]^{\frac{1}{q}} \\
&\leq \left(\sqrt{\mathbb{E}|f'(t+ia)|^{2q}\ \mathbb{E}|f'(s+ib)|^{2q}}\right)^{\frac{1}{q}} = \left(\mathbb{E}[|f'(ia)|^{2q}]\mathbb{E}[|f'(ib)|^{2q}]\right)^{\frac{1}{2q}} < \infty
\end{aligned}$$

(finiteness follows from the fact that $f'(ia)$ and $f'(ib)$ are both Gaussian random variables, thus have finite moments of any order).

Let

$$A = \left\{(t,s) \in [-T,T]^2 :\ |r(t-s+ia+ib)|^2 \leq \frac{2}{3}\right\}.$$

We split the last integral in (2.9) into two parts: on A and on $A^c = [-T,T]^2 \setminus A$. For the integral on A, we use the following lemma (to be proved later in this subsection).

Lemma 2.3.2. *Let Z_1, Z_2 be $\mathcal{N}_{\mathbb{C}}(0,1)$ random variables with $\mathbb{E}[Z_1\overline{Z_2}] = \alpha$, and let $1 < p < 2$. Then*

$$\mathbb{E}\left[\ |Z_1Z_2|^{-p}\ \right] \leq (1-|\alpha|^2)^{-\frac{p}{2}}\Gamma\left(1-\frac{p}{2}\right)^2$$

Using lemma 2.3.2, we have:

$$\iint_A \left(\mathbb{E}\left|f(t+ia)f(s+ib)\right|^{-p}\right)^{1/p} dt\ ds \lesssim \iint_A \left(1-|r(t-s+ia+ib)|^2\right)^{-\frac{1}{2}} \lesssim T^2,$$

which is bounded. In order to bound the integration on A^c, we use another lemma (to be proved in the end of this subsection).

Lemma 2.3.3. *Suppose ξ_1, ξ_2 are independent $\mathcal{N}_{\mathbb{C}}(0,1)$ random variables, and let $Z_1 = \alpha\xi_1$ and $Z_2 = \beta\xi_1 + \gamma\xi_2$ where $\alpha, \beta, \gamma \in \mathbb{C} \setminus \{0\}$. Let $1 < p < 2$. Then there exists a constant $c > 0$ such that*

$$\mathbb{E}\left[\frac{1}{|Z_1 Z_2|^p}\right] \le \frac{c}{|\alpha\beta|^p}\left|\frac{\gamma}{\beta}\right|^{2-2p}.$$

Moreover, given $1 < p < 2$ and a number $M > 0$, the constant c may be chosen uniformly for all parameters α, β, γ such that $|\frac{\gamma}{\beta}| < M$.

We would like to apply this lemma with $Z_1 = f(t + ia)$ and $Z_2 = f(s + ib)$. This yields the choice of parameters α, β, γ so that $\alpha = \sqrt{r(2ia)}$, $\alpha\overline{\beta} = \overline{r(t - s + ia + ib)}$ and $|\beta|^2 + |\gamma|^2 = r(2ib)$. Thus

$$\left|\frac{\gamma}{\beta}\right|^2 = \frac{r(2ia)r(2ib) - |r(t - s + ia + ib)|^2}{|r(t - s + ia + ib)|^2}$$

is uniformly bounded for $(t, s) \in A^c$. Now, applying Lemma 2.3.3 we get that for some $c > 0$,

$$\begin{aligned}
&\iint_{A^c} \left(\mathbb{E}\Big|f(t + ia)f(s + ib)\Big|^{-p}\right)^{1/p} dt\ ds \\
&\quad \lesssim \iint_{A^c} \left(\frac{c}{|r(t - s + ia + ib)|^p}\ \frac{(r(2ia)r(2ib) - |r(t - s + ia + ib)|^2)^{1-p}}{|r(t - s + ia + ib)|^{2-2p}}\right)^{\frac{1}{p}} dt\ ds \\
&\quad \lesssim \iint_{A^c} \left(r(2ia)r(2ib) - |r(t - s + ia + ib)|^2\right)^{-\frac{p-1}{p}} dt\ ds \\
&\quad \lesssim \int_{\{x:|r(x+ia+ib)|\ge 2/3\}} \left(r(2ia)r(2ib) - |r(x + ia + ib)|^2\right)^{-\frac{p-1}{p}} dt\ ds,
\end{aligned}$$

where the last inequality is obtained by a simple change of varibales (similar to Claim 2.3.4 below). If $a \neq b$, then $|r(x + ia + ib)|^2 \le r(ia + ib)^2 < r(2ia)r(2ib)$, and the integral is finite. In case $a = b$, there may be isolated points x_0 for which $|r(x_0 + 2ia)|^2 = r(2ia)^2$. Taylor expansion near any of those points gives $|r(x + 2ia)|^2 = r(2ia)^2 - C(x - x_0)^2 + o((x - x_0)^2)$ as x tends to x_0. Notice $C > 0$ since

$$|r(x + 2ia)| = \left|\int_{\mathbb{R}} e^{-2\pi ix\lambda}e^{2\pi\cdot 2a\lambda}d\rho(\lambda)\right| \le \int_{\mathbb{R}} e^{2\pi\cdot 2a\lambda}d\rho(\lambda) = r(2ia).$$

Thus, the finiteness of the integral is equivalent to that of $\int_{|x-x_0|<\delta}(x-x_0)^{-2(p-1)/p}dx$ (with some $\delta > 0$), which is indeed finite for $1 < p < 2$.

The proof of Claim 2.3.2 is complete. □

It remains now to prove the lemmata.

Proof of Lemma 2.3.2. Let ξ_1, ξ_2 be independent $\mathbb{N}_{\mathbb{C}}(0,1)$ random variables, write $Z_1 = \xi_1$ and $Z_2 = \overline{\alpha}\xi_1 + \sqrt{1-|\alpha|^2}\xi_2$. Then:

$$\mathbb{E}\left[|Z_1Z_2|^{-p}\right] = \frac{1}{\pi^2}\iint_{\mathbb{C}^2} |\xi_1(\overline{\alpha}\xi_1 + \sqrt{1-|\alpha|^2}\xi_2)|^{-p}e^{-|\xi_1|^2-|\xi_2|^2}dm(\xi_1)dm(\xi_2)$$

Now, by the Hardy-Littlewood re-arrangement inequality, we have:

$$\frac{1}{\pi}\int_{\mathbb{C}}|\overline{\alpha}\xi_1 + \sqrt{1-|\alpha|^2}\xi_2|^{-p}e^{-|\xi_2|^2}dm(\xi_2) \le (1-|\alpha|^2)^{-\frac{p}{2}}\cdot\frac{1}{\pi}\int_{\mathbb{C}}|\xi_2|^{-p}e^{-|\xi_2|^2}dm(\xi_2)$$
$$= (1-|\alpha|^2)^{-\frac{p}{2}}\Gamma\left(1-\frac{p}{2}\right).$$

So,

$$\begin{aligned}\mathbb{E}\left[|Z_1Z_2|^{-p}\right] &\le (1-|\alpha|^2)^{-\frac{p}{2}}\Gamma\left(1-\frac{p}{2}\right)\cdot\frac{1}{\pi}\int_{\mathbb{C}}|\xi_1|^{-p}e^{-|\xi_1|^2}dm(\xi_1)\\ &= (1-|\alpha|^2)^{-\frac{p}{2}}\Gamma\left(1-\frac{p}{2}\right)^2\end{aligned}$$

□

Proof of Lemma 2.3.3. In this proof we shall use the notation $f \lesssim g$ to denote $f \le Cg$, were $C > 0$ is a constant which may depend on p and M only (M is an upper bound on $|\frac{\gamma}{\beta}|$), and may vary from line to line.

We begin by writing-out the desired expectation explicitly.

$$\begin{aligned}\mathbb{E}\left[\ |Z_1Z_2|^{-p}\ \right] &= |\alpha\beta|^{-p}\ \mathbb{E}\left[\ \left|\xi_1^2 + \frac{\gamma}{\beta}\xi_1\xi_2\right|^{-p}\ \right]\\ &= |\alpha\beta|^{-p}\cdot\frac{1}{\pi^2}\iint_{\mathbb{C}^2}\left|z^2+\frac{\gamma}{\beta}zw\right|^{-p}e^{-|z|^2-|w|^2}\ dm(z)\ dm(w)\\ &= |\alpha\beta|^{-p}\pi^{-2}\int_{\mathbb{C}}|z|^{-p}\left(\int_{\mathbb{C}}\left|z+\frac{\gamma}{\beta}w\right|^{-p}e^{-|w|^2}dm(w)\right)e^{-|z|^2}dm(z)\end{aligned} \tag{2.10}$$

we bound the inner integral as follows:

$$\begin{aligned}
&\int_{\mathbb{C}} \left|z+\frac{\gamma}{\beta}w\right|^{-p} e^{-|w|^2} dm(w) \\
&\lesssim \int_{|w|\le\frac{1}{2}\left|\frac{\beta}{\gamma}z\right|} |z|^{-p} e^{-|w|^2} dm(w) + |z|^{-p} e^{-\frac{1}{4}\left|\frac{\beta}{\gamma}z\right|^2} \left|\frac{\beta}{\gamma}z\right|^2 + \int_{|w|>2\left|\frac{\beta}{\gamma}z\right|} \left|\frac{\gamma}{\beta}w\right|^{-p} e^{-|w|^2} dm(w) \\
&= |z|^{-p}\left(1-e^{-\frac{1}{4}\left|\frac{\beta}{\gamma}z\right|^2}\right) + \left|\frac{\beta}{\gamma}\right|^2 |z|^{2-p} e^{-\frac{1}{4}\left|\frac{\beta}{\gamma}z\right|^2} + \left|\frac{\beta}{\gamma}\right|^p I\left(\left|\frac{\beta}{\gamma}z\right|\right),
\end{aligned}$$

where

$$I(s) = \int_{|w|>2s} |w|^{-p} e^{-|w|^2} dm(w) \lesssim \begin{cases} 1, & 0 < s \le 1, \\ s^{-p} e^{-4s^2}, & s > 1. \end{cases}$$

The last bound is achieved by changing to polar coordinates, as follows:

$$I(s) = \int_{2s}^{\infty} r^{-p+1} e^{-r^2} dr \lesssim s^{-p+1} \int_{2s}^{\infty} e^{-r^2} dr \lesssim s^{-p+1} \frac{1}{2s} e^{-4s^2}.$$

Returning to the double integral in (2.10), we have:

$$\begin{aligned}
&\mathbb{E}\left[\left|\xi_1^2 + \frac{\gamma}{\beta}\xi_1\xi_2\right|^{-p}\right] \\
&\lesssim \int_{\mathbb{C}} \left\{ |z|^{-2p}\left(1-e^{-\frac{1}{4}\left|\frac{\beta}{\gamma}z\right|^2}\right) + \left|\frac{\beta}{\gamma}\right|^2 |z|^{2-2p} e^{-\frac{1}{4}\left|\frac{\beta}{\gamma}z\right|^2} + |z|^p \left|\frac{\beta}{\gamma}\right|^p I\left(\left|\frac{\beta}{\gamma}z\right|\right)\right\} e^{-|z|^2} dz
\end{aligned}$$

This is the sum of three integrals, which we bound separately. For the first, we have:

$$\begin{aligned}
&\int_{\mathbb{C}} |z|^{-2p}\left(1-e^{-\frac{1}{4}\left|\frac{\beta}{\gamma}z\right|^2}\right) e^{-|z|^2} dz \\
&\lesssim \int_{|z|\le\left|\frac{\gamma}{\beta}\right|} \left|\frac{\beta}{\gamma}\right|^2 |z|^{2-2p} dz + \int_{|z|>\left|\frac{\gamma}{\beta}\right|} |z|^{-2p} e^{-|z|^2} dz \\
&\eqsim \left|\frac{\beta}{\gamma}\right|^2 \left|\frac{\gamma}{\beta}\right|^{4-2p} + O(1) \eqsim \left|\frac{\gamma}{\beta}\right|^{2-2p}
\end{aligned}$$

Next, denote $A = 1 + \frac{1}{4}\left|\frac{\beta}{\gamma}\right|^2$ and compute

$$\begin{aligned}
&\int_{\mathbb{C}} |z|^{2-2p} e^{-A|z|^2} dz \\
&\quad \approx \int_0^\infty r^{2-2p} e^{-Ar^2} r dr && [r = |z|\ ] \\
&\quad = \frac{1}{2A}\int_0^\infty \left(\frac{s}{A}\right)^{1-p} e^{-s} ds && [s = Ar^2] \\
&\quad \approx A^{-(2-p)}.
\end{aligned}$$

Thus, the second integral is

$$\left|\frac{\beta}{\gamma}\right|^2 \int_{\mathbb{C}} |z|^{2-2p} e^{-\left(1+\frac{1}{4}\left|\frac{\beta}{\gamma}\right|^2\right)|z|^2} dz \approx \left|\frac{\beta}{\gamma}\right|^2 \left(1 + \frac{1}{4}\left|\frac{\beta}{\gamma}\right|^2\right)^{-(2-p)} \approx \left|\frac{\gamma}{\beta}\right|^{2-2p}.$$

Finally, the last integral is

$$\begin{aligned}
&\left|\frac{\beta}{\gamma}\right|^p \int_{\mathbb{C}} |z|^{-p} I\left(\left|\frac{\beta}{\gamma}z\right|\right) e^{-|z|^2} dz \\
&\quad \approx \left|\frac{\beta}{\gamma}\right|^p \left\{ \int_{|z|<\left|\frac{\gamma}{\beta}\right|} |z|^{-p} e^{-|z|^2} dz + \left|\frac{\gamma}{\beta}\right|^p \int_{|z|>\left|\frac{\gamma}{\beta}\right|} |z|^{2p} e^{-(1+4\left|\frac{\beta}{\gamma}\right|^2)|z|^2} dz \right\} \\
&\quad \approx \left|\frac{\beta}{\gamma}\right|^p \left|\frac{\gamma}{\beta}\right|^{2-p} + \left(1 + 4\left|\frac{\beta}{\gamma}\right|^2\right)^{p-1} \approx \left|\frac{\gamma}{\beta}\right|^{2-2p}.
\end{aligned}$$

The proof is complete.

□

2.3.3 Expansion in terms of the original covariance function.

The covariance between logarithms of two Gaussians can be expressed as a power series, using the following claim.

Claim 2.3.3. *Let $\xi^*, \eta^* \sim \mathcal{N}_{\mathbb{C}}(0,1)$ be standard complex Gaussian random variables. Then*

$$\operatorname{cov}\left(\log|\xi^*|, \log|\eta^*|\right) = \frac{1}{4}\sum_{k=1}^\infty \frac{|\mathbb{E}\xi^*\overline{\eta^*}|^{2k}}{k^2}.$$

A proof is included in the book [22, Lemma 3.5.2], or in a slightly different language in the paper by Nazarov and Sodin [34, Lemma 2.2].

For any centered complex Gaussian random variable $\xi \sim \mathcal{N}_{\mathbb{C}}(0, \sigma^2)$ we may write $\xi = \sigma\xi^*$ where $\xi^* \sim N_{\mathbb{C}}(0,1)$, and thus get

$$\log|\xi| - \mathbb{E}\log|\xi| = \log|\xi^*| - \mathbb{E}\log|\xi^*|.$$

Therefore Claim 2.3.3 implies that for any centered complex Gaussians ξ and η we have:

$$\operatorname{cov}(\log|\xi|, \log|\eta|) = \frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k^2}\left(\frac{|\mathbb{E}(\xi\overline{\eta})|^2}{\mathbb{E}|\xi|^2\mathbb{E}|\eta|^2}\right)^k.$$

We now apply this formula for $\xi = f(t+ia)$ and $\eta = f(s+ib)$: By stationarity and our notation, we have

$$\frac{|\mathbb{E}(f(t+ia)\overline{f(s+ib)})|^2}{\mathbb{E}|f(t+ia)|^2\,\mathbb{E}|f(s+ib)|^2} = \frac{|r(t-s+ia+ib)|^2}{r(2ia)\,r(2ib)} =: q(t-s,a,b),$$

so that Claim 2.3.2 gives:

$$C^{a,b}(T) = \frac{1}{4}\int_{-T}^{T}dt\int_{-T}^{T}ds\ \frac{\partial^2}{\partial a\,\partial b}\sum_{k=1}^{\infty}\frac{1}{k^2}q(t-s,a,b)^k. \tag{2.11}$$

2.3.4 From double to single integral

Next, we pass to a one-dimensional integral using a simple change of variables:

Claim 2.3.4. *For any function $Q \in L^1([-2T,2T])$, the following equality holds:*

$$\int_{-T}^{T}\int_{-T}^{T}Q(t-s)dt\,ds = 2\int_{-2T}^{2T}(2T-|x|)Q(x)dx$$

Applying Claim 2.3.4 to (2.11) we get:

$$\frac{C^{a,b}(T)}{2T} = \frac{1}{2}\int_{-2T}^{2T}\left(1-\frac{|x|}{2T}\right)\frac{\partial^2}{\partial a\,\partial b}\sum_{k\geq 1}\frac{q^k(x,a,b)}{k^2}\ dx \tag{2.12}$$

2.3.5 Some properties of q

We digress shortly to summarize some properties of q, which we will use later in our proofs. In the following, when we do not specify the variables we mean the statements holds on all the domain of definition. We use the subscript notation for partial derivatives (such as q_a for $\frac{\partial}{\partial a}q$).

Claim 2.3.5. *The function*

$$q(x,a,b) = \frac{|r(x+ia+ib)|^2}{r(2ia)\,r(2ib)} \tag{2.13}$$

is well-defined, infinitely differentiable on $\mathbb{R} \times (-\Delta, \Delta)^2$, *and satisfies the following properties:*

1. $q(x, y_1, y_2) \in [0, 1]$.
   $q(x, y_1, y_2) = 1$ *if and only if* ($x = 0$ *and* $y_1 = y_2$).
2. $\sup_{x\in\mathbb{R}} q(x, y_1, y_2) < 1$ *for any* $y_1 \neq y_2$ *in* $(-\Delta, \Delta)$.
3. *For fixed* y_1 *and* y_2 *let* $g_{y_1,y_2}(x)$ *be one of the functions* q, q_a, q_b, q_{ab} *evaluated on the line* $\{(x, y_1, y_2) : x \in \mathbb{R}\}$. *Then* $g_{y_1,y_2} \in L^\infty(\mathbb{R})$. *If condition* (2.2) *holds, then for any* $y_1, y_2 \in [a, b]$ *we have also* $g_{y_1,y_2} \in L^1(\mathbb{R}) \cap C_0(\mathbb{R})$ *(i.e., is integrable and tends to zero as* $x \to \pm\infty$*).*
4. $q_a(0, t, t) = 0$, *for any* $t \in (-\Delta, \Delta)$.

Proof. Since $r(2iy) > 0$ for all $y \in \mathbb{R}$, the function q is indeed well-defined; differentiability follows from that of $r(z)$.

For item 1, notice that

$$q(x,a,b) = \frac{\left(\int e^{2\pi(a+b)\lambda} e^{-2\pi i x\lambda} d\rho(\lambda)\right)^2}{\int e^{2\pi\cdot 2a\lambda} d\rho(\lambda) \int e^{2\pi\cdot 2a\lambda} d\rho(\lambda)}$$

and so, by Cauchy-Schwarz, is in $[0, 1]$. Equality $q(x, a, b) = 1$ holds only if the function $\lambda \mapsto e^{2\pi\cdot a\lambda} e^{-2\pi i x\lambda}$ is a constant times the function $\lambda \mapsto e^{2\pi\cdot b\lambda}$, ρ-a.e., but, if ρ is non-atomic, this is impossible unless $x = 0$ and $a = b$.

Further, we notice that

$$|r(x+ia+ib)| = \left|\int e^{2\pi(a+b)\lambda} e^{-2\pi i x\lambda} d\rho(\lambda)\right| \leq \int e^{2\pi(a+b)\lambda} d\rho(\lambda) = r(ia+ib),$$

so that $q(x, a, b) \leq q(0, a, b) < 1$ (the right-most inequality is by item 1). Taking the supremum yields item 2.

For item 3, notice any one of the functions q, q_a, q_b, q_{ab} is the sum of summands of the form

$$C(a,b)\; r^{(j)}(x+ia+ib)\; r^{(m)}(-x+ia+ib), \tag{2.14}$$

where $0 \le j, m \le 2$ are integers. It is enough therefore to explain why $r^{(j)}(x+ia+ib)$ is bounded and approaches zero as $x \to \pm\infty$, for any integer $0 \le j \le 2$. Recall that

$$r^{(j)}(x+iy) = c_j \mathcal{F}_\lambda[\lambda^j e^{2\pi y\lambda}\, d\rho(\lambda)](x),$$

where c_j is some constant. As a function of x, this is a Fourier transform of a non-atomic measure, therefore has the desired properties.

If condition (2.2) holds, then $d\rho(\lambda) = p(\lambda)d\lambda$, and the function $\lambda \mapsto \lambda^j e^{2\pi(y_1+y_2)\lambda} p(\lambda)$ is in $L^2(\mathbb{R})$. Then, its Fourier transform $r^{(j)}(x+iy_1+iy_2)$ is also in $L^2(\mathbb{R})$, and each summand of the form (2.14) is in $L^1(\mathbb{R})$, as anticipated.

For item 4, notice that for all $x \in \mathbb{R}$ and all $a, b \in (-\Delta, \Delta)$ we have the symmetry $q(x,a,b) = q(x,b,a)$, and therefore for all $t \in \mathbb{R}$: $q_a(x,t,t) = q_b(x,t,t)$. On the other hand, for all $t \in (-\Delta, \Delta)$ it holds that $q(0,t,t) = 1$, so taking derivative by t we get $q_a(0,t,t)\cdot 1 + q_b(0,t,t)\cdot 1 = 0$. This proves the result. □

2.3.6 Change of derivative and sum

We would like to take derivative term-by-term in (2.12).

Claim 2.3.6. *For all $x \neq 0$,*

$$\frac{\partial^2}{\partial a\, \partial b} \sum_{k\ge 1} \frac{q^k(x,a,b)}{k^2} = \sum_{k\ge 1} \frac{\partial^2}{\partial a\, \partial b} \frac{q^k(x,a,b)}{k^2} \tag{2.15}$$

Proof. Fix $x \neq 0$. For shortness, we do not write the variables (x,a,b), and use again the subscript notation for partial derivatives. We compute:

$$S_k^{a,b}(x) := \frac{\partial^2}{\partial a\, \partial b}\left\{q^k\right\} = \begin{cases} q_{ab}, & k=1 \\ k(k-1)q^{k-2}q_a q_b, + kq^{k-1}q_{ab} & k>1. \end{cases}$$

Therefore,

$$\left|\frac{S_k^{a,b}(x)}{k^2}\right| \le q^{k-2}|q_a q_b| + \frac{1}{k} q^{k-1}|q_{ab}|. \tag{2.16}$$

By part 1 of Claim 2.3.5, $q(x,a,b) < 1$ (notice this holds also if $a = b$). We deduce that $\sum \left| \frac{S_k^{a,b}}{k^2} \right| < \infty$. i.e., the RHS of (2.15) converges in absolute value. By standard arguments, this is enough to prove equality (2.15). □

Thus, continuing (2.12), we arrive at

$$\frac{C^{a,b}(T)}{2T} = \frac{1}{2} \int_{-2T}^{2T} \left(1 - \frac{|x|}{2T}\right) \sum_{k \geq 1} \frac{\partial^2}{\partial a \, \partial b} \frac{q^k(x,a,b)}{k^2} \, dx \tag{2.17}$$

2.3.7 Parseval's identity

The next claim is a special case of Parseval's identity for measures (see Katznelson [24, VI.2.2]):

Claim 2.3.7. *For any finite measure γ on $\mathbb{R}$,*

$$\int_{-2T}^{2T} \left(1 - \frac{|x|}{2T}\right) \mathcal{F}[\gamma](x) dx = \int_{\mathbb{R}} 2T \mathrm{sinc}^2(2\pi T\xi) d\gamma(\xi).$$

where $\mathrm{sinc}(\xi) = \frac{\sin \xi}{\xi}$ *and $\mathcal{F}[\gamma]$ is the Fourier transform of γ.*

In order to apply this claim to simplify equation (2.17), we shall first find a finite measure $\gamma_k^{a,b}$ such that $\mathcal{F}[\gamma_k^{a,b}](x) = q(x,a,b)^k$. This is done in the next step.

2.3.8 The search for an inverse Fourier transform

For now, we keep a, b and k fixed. Our goal is to find a measure whose Fourier transform results in $q^k(x,a,b)$ (or, instead, in $|r(x + ia + ib)|^{2k}$). This measure is given in Claim 2.3.8 in the end of this subsection. In order to present it we must first discuss some definitions and relations between operations on measures.

Denote by $\mathcal{M}(\mathbb{R})$ the space of all finite measures on $\mathbb{R}$, similarly $\mathcal{M}^+(\mathbb{R})$ denotes all finite non-negative measures on $\mathbb{R}$. For two measure $\mu, \nu \in \mathcal{M}(\mathbb{R})$ the *convolution* $\mu * \nu \in \mathcal{M}(\mathbb{R})$ is a measure defined by:

$$\forall \varphi \in C_0(\mathbb{R}) : \ (\mu * \nu)(\varphi) = \iint \varphi(\lambda + \tau) d\mu(\lambda) d\nu(\tau).$$

When both measures have density, this definition agrees with the standard convolution of functions. We write μ^{*k} for the iterated convolution of μ with itself k times.

Next recall that

$$r(z) = \int_{\mathbb{R}} e^{-2\pi i z\lambda} d\rho(\lambda) =: \mathcal{F}[\rho](z).$$

By properties of Fourier transform,

$$r^k(z) = \mathcal{F}[\rho^{*k}](z),$$

or, writing $z = x + iy$ we have

$$r^k(x+iy) = \int_{\mathbb{R}} e^{-2\pi i x\lambda} e^{2\pi y\lambda} d\rho^{*k}(\lambda). \tag{2.18}$$

This gives rise to the following notation: for a measure $\mu \in \mathcal{M}^+(\mathbb{R})$ having exponential moments up to 2Δ (i.e., obeying condition (2.1)), and a number $y \in (-2\Delta, 2\Delta)$, we define the *exponentially rescaled measure* $\mu_y \in \mathcal{M}^+(\mathbb{R})$ by

$$\forall \varphi \in C_0(\mathbb{R}) : \ \mu_y(\varphi) = \mu(e^{2\pi y\lambda}\varphi(\lambda)) = \int_{\mathbb{R}} e^{2\pi y\lambda}\varphi(\lambda) d\mu(\lambda)$$

Observation. *For any $\mu, \nu \in \mathcal{M}(\mathbb{R})$ and any $|y| < 2\Delta$,*

$$(\mu * \nu)_y = \mu_y * \nu_y.$$

Proof. for any test function $\varphi \in C_0(\mathbb{R})$ we have:

$$\begin{aligned}\int \varphi \ d(\mu_y * \nu_y) &= \iint \varphi(\lambda+\tau) \ d\mu_y(\lambda) d\nu_y(\tau) \\ &= \iint \varphi(\lambda+\tau) e^{2\pi y(\lambda+\tau)} \ d\mu(\lambda) d\nu(\tau) = \int \varphi \ d(\mu * \nu)_y\end{aligned}$$

□

As a corollary, we get that for any $|y| < 2\Delta$ and any $k \in \mathbb{N}$,

$$(\rho_y)^{*k} = (\rho^{*k})_y.$$

Therefore there will be no ambiguity in the notation ρ_y^{*k}.

Next, we define for $\mu \in \mathcal{M}(\mathbb{R})$ the *flipped measure* $\text{flip}\{\mu\} \in \mathcal{M}(\mathbb{R})$ by:

$$\text{flip}\{\mu\}(I) = \mu(-I) \text{ for any interval } I \subset \mathbb{R},$$

and the *cross-correlation* of measures $\mu, \nu \in \mathcal{M}(\mathbb{R})$ by:

$$\mu \star \nu := \mu * \text{flip}\{\nu\}.$$

An alternative definition, via actions on test-functions, would be:

$$\forall\varphi\in C_0(\mathbb{R}):\ (\mu\star\nu)(\varphi)=\iint\varphi(\lambda-\tau)d\mu(\lambda)d\nu(\tau).$$

Notice that the cross-correlation operator is bi-linear, but not commutative.

Now relation (2.18) easily implies:

- $r^k(x+iy)=\mathcal{F}[\rho_y^{*k}](x)$
- $\overline{r^k(x+iy)}=\mathcal{F}[\rho_y^{*k}](-x)=\mathcal{F}[\text{flip}\{\rho_y^{*k}\}](x)$,

which leads at last to the end of our investigation:

Claim 2.3.8. *For any $x\in\mathbb{R}$, $|y|<2\Delta$ and $k\in\mathbb{N}$, we have:*

$$|r^k(x+iy)|^2=\mathcal{F}\left[(\rho_y^{*k})\star(\rho_y^{*k})\right](x).$$

This measure acts on a test-function $\varphi\in C_0(\mathbb{R})$ in the following way:

$$(\rho_y^{*k}\star\rho_y^{*k})(\varphi)=\iint\varphi(\lambda-\tau)e^{2\pi y(\lambda+\tau)}d\rho^{*k}(\lambda)d\rho^{*k}(\tau).$$

2.3.9 Taking the double derivative

Using Claim 2.3.8, we rewrite equation (2.17):

$$\frac{C^{a,b}(T)}{2T}=\int_{-2T}^{2T}\left(1-\frac{|x|}{2T}\right)\sum_{k\geq 1}\frac{1}{2k^2}\frac{\partial^2}{\partial a\ \partial b}\left\{\frac{\mathcal{F}[\rho_{a+b}^{*k}\star\rho_{a+b}^{*k}](x)}{r^k(2ia)r^k(2ib)}\right\}dx \quad (2.19)$$

The double derivative in this expression may be rewritten using the following claim.

Claim 2.3.9.

$$S_k^{a,b}(x):=\frac{\partial^2}{\partial a\ \partial b}\frac{\mathcal{F}[\rho_{a+b}^{*k}\star\rho_{a+b}^{*k}](x)}{r^k(2ia)r^k(2ib)}=\mathcal{F}\left[l_k^a(\lambda)\rho_{a+b}^{*k}\star l_k^b(\lambda)\rho_{a+b}^{*k}\right](x).$$

where $l_k^a(\lambda),l_k^b(\lambda)$ are linear functions in λ, given by

$$l_k^a(\lambda)=\frac{\partial}{\partial a}\left(\frac{1}{r^k(2ia)}\right)+\frac{2\pi}{r^k(2ia)}\lambda=\frac{2}{r^k(2ia)}\left(-ik\frac{r'(2ia)}{r(2ia)}+\pi\lambda\right).$$

Proof. Recall that

$$\mathcal{F}[\rho_{a+b}^{*k} \star \rho_{a+b}^{*k}](x) = \iint e^{-ix(\lambda-\tau)} e^{2\pi i(a+b)(\lambda+\tau)} d\rho^{*k}(\lambda) d\rho^{*k}(\tau),$$

and notice we may differentiate by a and b under the integral, as the result would be continuous and integrable w.r.t. ρ^{*k}. From here, the proof is a straightforward computation. □

Futher, using Claim 2.3.7 and the definitions from Section 2.3.8, we have for fixed k,

$$\begin{aligned}\int_{-2T}^{2T} \left(1 - \frac{|x|}{2T}\right) S_k^{a,b}(x)\ dx &= \int_{-2T}^{2T} \left(1 - \frac{|x|}{2T}\right) \mathcal{F}\left[l_k^a(\lambda)\rho_{a+b}^{*k} \star l_k^b(\lambda)\rho_{a+b}^{*k}\right](x) \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} 2T\mathrm{sinc}^2(2\pi T(\lambda-\tau))\ l_k^a(\lambda) l_k^b(\tau) e^{2\pi(a+b)(\lambda+\tau)} d\rho^{*k}(\lambda) d\rho^{*k}(\tau).\end{aligned} \tag{2.20}$$

Now, recalling Claim 2.3.1a, we use the form (2.19) and Claim 2.3.9, we write the expression which, up to an error term, is asymptotically equivalent to $\frac{V_f^{a,b}(T)}{2T}$:

$$\begin{aligned}&\frac{1}{4\pi^2}\int_{-2T}^{2T}\left(1-\frac{|x|}{2T}\right)\sum_{k\geq 1}\frac{1}{2k^2}(S_k^{a,a}(x) - 2S_k^{a,b}(x) + S_k^{b,b}(x))dx \\ &= \frac{1}{4\pi^2}\sum_{k\geq 1}\frac{1}{2k^2}\int_{-2T}^{2T}\left(1-\frac{|x|}{2T}\right)\ (S_k^{a,a}(x) - 2S_k^{a,b}(x) + S_k^{b,b}(x))dx \\ &= \frac{1}{4\pi^2}\sum_{k\geq 1}\frac{1}{k^2}\int_{\mathbb{R}}\int_{\mathbb{R}} T\mathrm{sinc}^2(T(\lambda-\tau)) h_k^{a,b}(\lambda+\tau) d\rho^{*k}(\lambda) d\rho^{*k}(\tau),\end{aligned} \tag{2.21}$$

where

$$h_k^{a,b}(\lambda) = \left(l_k^a(\lambda)e^{2\pi a\lambda} - l_k^b(\lambda)e^{2\pi b\lambda}\right)^2. \tag{2.22}$$

The exchange of sum and integral in the first equality of (2.21) is justified by the monotone convergence theorem, as each term in the series is non-negative. The second equality follows from (2.20).

We summarize the result in the following claim.

Claim 2.3.10.

$$\begin{aligned}&\frac{C^{a,a}(T) - 2C^{a,b}(T) + C^{b,b}(T)}{4\pi^2\cdot 2T} \\ &= \sum_{k\geq 1}\frac{1}{4\pi^2 k^2}\int_{\mathbb{R}}\int_{\mathbb{R}} T\mathrm{sinc}^2(2\pi T(\lambda-\tau)) h_k^{a,b}(\lambda+\tau) d\rho^{*k}(\lambda) d\rho^{*k}(\tau),\end{aligned}$$

where $h_k^{a,b}$ *is given by* (2.22).

One more step is required in order to establish Proposition 2.3.1.

2.3.10 The error term

At last, we show that the error term in Claim 2.3.1a approaches zero as T tends to infinity.

Claim 2.3.11. *If* ρ *contains no atoms, then for any* $a \in (-\Delta, \Delta)$*:*

$$\lim_{T\to\infty} \frac{C^{a,a}(T)}{T^2} = 0.$$

Proof. By Theorem 3, since ρ has no atoms, f is an ergodic process. Thus, by the ergodic theorem,

$$\lim_{T\to\infty} \frac{1}{T} X^a(T) = \mathbb{E} X^a(1) \tag{2.23}$$

converges almost surely and in L^1 to a constant. Recall $X^a(T)$ has finite second moment (this is precisely relation II, which was proved for Claim 2.3.2). Thus, the convergence in (2.23) is also in the L^2 sense (see [12, Exercise 7.2.1]). This yields the convergence

$$\lim_{T\to\infty} \frac{1}{T^2} \operatorname{var}\left(X^a(T)\right) = \lim_{T\to\infty} \frac{1}{T^2} C^{a,a}(T) \to 0,$$

which is our claim. □

2.4 Theorem 7: Linear and Intermediate Variance

The proof is divided into two parts. First we prove the existence of the limit L_1 and its positivity, and later we prove that it is finite under condition (2.2).

2.4.1 Existence and Positivity.

In this section we prove that L_1 exists and belongs to $(0, \infty]$. If ρ has at least one atom, Theorem 6 implies that $\lim_{T\to\infty} \frac{V_f^{a,b}(T)}{T^2} > 0$, and therefore $L_1 = \lim_{T\to\infty} \frac{V_f^{a,b}(T)}{T} = \infty$. We thus assume that ρ has no atoms.

Using the formula for the variance obtained in Proposition 2.3.1, and recalling the functions $h_k^{a,b}$ are non-negative, we see that the limit L_1 exists and is in $[0,\infty]$. More effort is needed in order to establish that $L_1 > 0$. We begin with a simple bound arising from Proposition 2.3.1:

$$\begin{aligned}\liminf_{T\to\infty}\frac{V_f^{a,b}(T)}{2T} &= \frac{1}{4\pi^2}\liminf_{T\to\infty}\sum_{k\geq 1}\frac{1}{k^2}\int_{\mathbb{R}}\int_{\mathbb{R}} T\mathrm{sinc}^2(2\pi T(\lambda-\tau))h_k^{a,b}(\lambda+\tau)d\rho^{*k}(\lambda)d\rho^{*k}(\tau)\\ &\geq \frac{1}{4\pi^2}\liminf_{T\to\infty}\int_{\mathbb{R}}\int_{\mathbb{R}} T\ \mathrm{sinc}^2(2\pi T(\lambda-\tau))\ h_1^{a,b}(\lambda+\tau)\ d\rho(\lambda)d\rho(\tau)\\ &\geq C_0\ \liminf_{\varepsilon\to 0+}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{1}{2\varepsilon}\mathbb{I}\{(\lambda,\tau):\,|\lambda-\tau|<\varepsilon\}h_1^{a,b}(\lambda+\tau)\ d\rho(\lambda)d\rho(\tau),\end{aligned} \tag{2.24}$$

where $C_0 > 0$ is an absolute constant. The last step follows from ignoring the integration outside

$$\mathrm{Diag}_\varepsilon = \mathbb{I}\{(\lambda,\tau):\,|\lambda-\tau|<\varepsilon\},$$

for $\varepsilon < \frac{1}{4T}$. Next we turn to investigate $h_1^{a,b}$. Recall its form is given in Proposition 2.3.1 or more recently in (2.22).

Claim 2.4.1. *The function $h_1^{a,b}$ has exactly two real zeroes.*

Proof. By the form of $h_1^{a,b}$, $h_1^{a,b}(\lambda) = 0$ if and only if

$$e^{2\pi(b-a)\lambda} = \frac{l_1^a(\lambda)}{l_1^b(\lambda)} = \frac{\frac{1}{r(2ia)}\left(\pi\lambda - i\frac{r'}{r}(2ia)\right)}{\frac{1}{r(2ib)}\left(\pi\lambda - i\frac{r'}{r}(2ib)\right)} = C\cdot\frac{\lambda-\psi(a)}{\lambda-\psi(b)},$$

where $C > 0$ is a positive constant and $\psi(y) = \frac{1}{2\pi}\frac{d}{dy}\left[\log r(2iy)\right]$. Since $y\mapsto \log r(2iy)$ is a convex function, for $a < b$ we have $\psi(a) < \psi(b)$. Therefore, $\lambda\mapsto C\,\frac{\lambda-\psi(a)}{\lambda-\psi(b)}$ is a strictly decreasing function, with a pole at $\psi(b)$ and with the same positive limit at $\pm\infty$. Thus, it crosses exactly twice the increasing exponential function $e^{2\pi(b-a)\lambda}$. □

The next claim will enable us to bound $h_1^{a,b}$ from below, on most of the real line. Denote by $z_1, z_2 \in \mathbb{R}$ $(z_1 < z_2)$ the two real zeroes of $h_1^{a,b}$ whose existence is guaranteed by Claim 2.4.1. We also use the notation $B(x,\delta)$ for the interval of radius $\delta > 0$ around $x\in\mathbb{R}$.

Claim 2.4.2. *For all $\delta_0 > 0$, there exists $c_\delta > 0$ such that for all $\lambda \in \mathbb{R}\setminus(B(z_1,\delta_0)\cup B(z_2,\delta_0))$:*

$$h_1^{a,b}(\lambda) > c_\delta(1+\lambda^2)\max(e^{2a\cdot 2\pi\lambda}, e^{2b\cdot 2\pi\lambda}).$$

Proof. Since the function $\frac{h_1^{a,b}(\lambda)}{(1+\lambda^2)e^{2a\cdot 2\pi\lambda}} = \left(\frac{l_1^a(\lambda)-l_1^b(\lambda)e^{2\pi(b-a)\lambda}}{\sqrt{1+\lambda^2}}\right)^2$ approaches strictly positive limits as $|\lambda|\to\infty$, there exist $M_a, c_a > 0$ such that

$$\forall|\lambda|\geq M_a : h_1^{a,b}(\lambda) \geq c_a(1+\lambda^2)e^{2a\cdot 2\pi\lambda}.$$

Similarly, there exist some $M_b, c_b > 0$ such that $\forall|\lambda| \geq M_b : h_1^{a,b}(\lambda) \geq c_b(1+\lambda^2)e^{2b\cdot 2\pi\lambda}$. Take $M = \max(M_a, M_b)$. Since $h(\lambda)$ attains a positive minimum on $[-M,M]\setminus(B(z_1,\delta_0)\cup B(z_2,\delta_0))$, there exists some $c>0$ such that for all λ in this set, $h(\lambda)\geq c(1+\lambda^2)\max(e^{2a\cdot 2\pi\lambda}, e^{2b\cdot 2\pi\lambda})$. Choosing now $c_\delta = \min(c, c_a, c_b)$ will yield the result. □

The next claim is a slight modification of the previous one, in order to fit our specific need.

Claim 2.4.3. *For every $\delta>0$ there exist a set $F = F_\delta = \mathbb{R}\setminus(I_1\cup I_2)$ such that I_j is an interval containing z_j and of length at most δ ($j=1,2$), $\rho(F)>0$, and there exists $c_\delta>0$ such that for all small enough ε,*

$$h(\lambda+\tau)\geq c_\delta(1+(\lambda+\tau)^2)\max\left(e^{2a\cdot 2\pi(\lambda+\tau)}, e^{2b\cdot 2\pi(\lambda+\tau)}\right),$$

for all $\lambda,\tau\in(F\times F)\cap Diag_\varepsilon$.

Proof. Choose $F = \mathbb{R}\setminus\left(B\left(\frac{z_1}{2},\delta_0\right)\cup B\left(\frac{z_2}{2},\delta_0\right)\right)$, where $\delta_0\leq\delta$ is small enough so that $\rho(F)>0$. Then, for $\varepsilon\leq\delta_0$ and $(\lambda,\tau)\in(F\times F)\cap\mathrm{Diag}_\varepsilon$, we have

$$|\lambda+\tau-z_j|\geq|2\tau-z_j|-|\lambda-\tau|\geq 2\delta_0-\varepsilon\geq\delta_0.$$

Choosing the constant $c_\delta>0$ which is the consequence of applying Claim 2.4.2 will end our proof. □

Fix a parameter $\delta>0$, and fix $F=F_\delta$ to be the set provided by Claim 2.4.3. Continuing from equation (2.24), we have

$$\begin{aligned}\liminf_{T\to\infty}\frac{V_f^{a,b}(T)}{2T} &\geq c_\delta\liminf_{\varepsilon\to 0}\iint_{F\times F}\frac{1}{2\varepsilon}1\!\!1_{\mathrm{Diag}_\varepsilon}(\lambda,\tau)\ e^{2\pi\cdot 2a(\lambda+\tau)}d\rho(\lambda)d\rho(\tau)\\ &= c_\delta\liminf_{\varepsilon\to 0}\int_F\frac{1}{2\varepsilon}\rho_{2a}\left((\tau-\varepsilon,\tau+\varepsilon)\cap F\right)d\rho_{2a}(\tau)\\ &= c_\delta\liminf_{\varepsilon\to 0}\int_{\mathbb{R}}\frac{1}{2\varepsilon}\mu\left(\tau-\varepsilon,\tau+\varepsilon\right)d\mu(\tau),\end{aligned}$$

where μ is the restriction of ρ_{2a} to F, i.e. $\mu(\varphi)=\rho_{2a}(1\!\!1_F\cdot\varphi)$ for any test-function φ. Notice that by the choice of F, $\mu(\mathbb{R})=\rho_{2a}(F)>0$. The next lemma characterizes the limit we are investigating.

Lemma 2.4.1. *Let $\mu \in \mathcal{M}^+(\mathbb{R})$ ($\mu \not\equiv 0$). Then the following limit exists (finite or infinite):*

$$\lim_{\varepsilon\to 0+} \int_{\mathbb{R}} \frac{1}{2\varepsilon}\mu\left(\tau-\varepsilon,\tau+\varepsilon\right)d\mu(\tau) = \int_{\mathbb{R}} |\mathcal{F}[\mu]|^2(x)dx.$$

Positivity of the lower bound which we gave for the limit L_1 is now clear.

Proof of Lemma 2.4.1. Denote $\varphi_\varepsilon = \frac{1}{2\varepsilon}1\!\!1_{(-\varepsilon,\varepsilon)}$ for $\varepsilon > 0$. Rewriting the integral and using Parseval's identity, we get:

$$\begin{aligned} I_\mu(\varepsilon) &:= \frac{1}{2\varepsilon}\int_{\mathbb{R}} \mu\left(\tau-\varepsilon,\tau+\varepsilon\right)d\mu(\tau) \\ &= \int_{\mathbb{R}} (\mu * \varphi_\varepsilon)(\tau)d\mu(\tau) \\ &= \int_{\mathbb{R}} (\mathcal{F}[\mu]\cdot\mathcal{F}[\varphi_\varepsilon])(x)\mathcal{F}[\mu](-x)dx \\ &= \int_{\mathbb{R}} \operatorname{sinc}(2\pi\varepsilon x)|\mathcal{F}[\mu]|^2(x)dx \end{aligned}$$

Since $|\operatorname{sinc}(2\pi\varepsilon x)| \le 1$, we have the upper bound $I_\mu(\varepsilon) \le \int_{\mathbb{R}} |\mathcal{F}[\mu]|^2(x)dx$.

For a lower bound we shall use the following general fact:

Observation. *For any $\psi_1,\psi_2 \in C_0(\mathbb{R})$ and $\mu \in \mathcal{M}^+(\mathbb{R})$,*

$$\int \psi_1\, d(\mu*\psi_2) = \int\, (\psi_1 * \operatorname{flip}\{\psi_2\})\, d\mu.$$

Proof.

$$\begin{aligned} \int \psi_1\, d(\mu*\psi_2) &= \int \psi_1(x+y)\, d\mu(x)\psi_2(y)dy \\ &= \int\left(\int \psi_1(x+y)\operatorname{flip}\{\psi_2\}(-y)\,dy\right)d\mu(x) = \int (\psi_1*\operatorname{flip}\{\psi_2\})(x)d\mu(x). \end{aligned}$$

□

Using the last observation and the fact that $\varphi_\varepsilon * \varphi_\varepsilon \le 2\varphi_{2\varepsilon}$ we get:

$$\int_{\mathbb{R}} (\mu*\varphi_\varepsilon)\ d(\mu*\varphi_\varepsilon) = \int_{\mathbb{R}} \mu*(\varphi_\varepsilon*\varphi_\varepsilon)d\mu \le 2\int_{\mathbb{R}} \mu*\varphi_{2\varepsilon}\ d\mu = 2I_\mu(2\varepsilon)$$

On the other hand,

$$\int_{\mathbb{R}} (\mu * \varphi_\varepsilon)\ d(\mu * \varphi_\varepsilon) = \int |\mathcal{F}[\mu * \varphi_\varepsilon]|^2 = \int |\mathcal{F}[\mu]|^2 \cdot \operatorname{sinc}^2(2\pi\varepsilon x)\ dx$$
$$\geq \int_K |\mathcal{F}[\mu]|^2 \cdot \operatorname{sinc}^2(2\pi\varepsilon x)\ dx$$

for any compact set $K \subset \mathbb{R}$. Sin ce the limit $\lim_{\varepsilon\to 0+} \operatorname{sinc}(2\pi\varepsilon x) = 1$ is uniform in $x \in K$, the last expression approaches $\int_K |\mathcal{F}[\mu]|^2$ as $\varepsilon \to 0+$. Thus, by choosing K and then $\varepsilon > 0$ properly, the lower bound may be made arbitrarily close to $\int_{\mathbb{R}} |\mathcal{F}[\mu]|^2$. This concludes the proof. □

2.4.2 Linear Variance

Recall that, combining (2.21) and Claims 2.3.1a and 2.3.11, we obtained the formula

$$\frac{V^{a,b}(T)}{2T} \tag{2.25}$$
$$= \frac{1}{8\pi^2} \sum_{k\geq 1} \frac{1}{k^2} \int_{-2T}^{2T} \left(1 - \frac{|x|}{2T}\right) \left(S_k^{a,a}(x) - 2S_k^{a,b}(x) + S_k^{b,b}(x)\right)\ dx + o(1),$$

where

$$S_k^{a,b}(x) := \frac{\partial^2}{\partial a\ \partial b} \left\{ q^k(x, a, b) \right\},$$

and $S_k^{a,a}(x)$ denotes the evaluation of the same mixed partial derivative at the point (x, a, a). In the next claim we prove strong convergence properties of similar sums, provided that condition (2.2) holds.

Claim 2.4.4. *If condition* (2.2) *is satisfied, then for every* $k \in \mathbb{N}$ *the functions* $S_k^{a,a}(x)$, $S_k^{a,b}(x)$ *and* $S_k^{b,b}(x)$ *are in* $L^1(\mathbb{R})$ *with respect to the variable* x. *Moreover,*

$$\sum_{k\geq 1} \frac{1}{k^2} \int_{\mathbb{R}} S_k(x) dx \text{ converges,}$$

with any of the three possible superscripts on the letter S.

Let us first see how to finish the proof of linear variance using this claim. Again, as we saw in section 2.3.9, each term of the series in the RHS of (2.25) is non-negative. Therefore, by the monotone convergence theorem:

$$\lim_{T\to\infty}\frac{V_f^{a,b}(T)}{2T}$$
$$=\frac{1}{8\pi^2}\sum_{k\geq 1}\frac{1}{k^2}\lim_{T\to\infty}\int_{-2T}^{2T}\left(1-\frac{|x|}{2T}\right)\,(S_k^{a,a}(x)-2S_k^{a,b}(x)+S_k^{b,b}(x))\,dx.$$

The limit in each term can be computed using the following:

Claim 2.4.5. *If $Q:\mathbb{R}\to[0,\infty)$ is integrable on $\mathbb{R}$, then*

$$\lim_{T\to\infty}\int_{-T}^{T}\left(1-\frac{|x|}{T}\right)Q(x)dx=\int_{\mathbb{R}}Q.$$

Proof. Notice that:

$$\int_{-\sqrt{T}}^{\sqrt{T}}\left(1-\frac{1}{\sqrt{T}}\right)Q(x)dx\leq\int_{-T}^{T}\left(1-\frac{|x|}{T}\right)Q(x)dx\leq\int_{-T}^{T}Q(x)dx,$$

and both ends of the inequality approach the limit $\int_{\mathbb{R}}Q$. □

We conclude that

$$\lim_{T\to\infty}\frac{V_f^{a,b}(T)}{2T}=\frac{1}{8\pi^2}\sum_{k\geq 1}\frac{1}{k^2}\int_{\mathbb{R}}(S_k^{a,a}(x)-2S_k^{a,b}(x)+S_k^{b,b}(x))\,dx,$$

which is finite by Claim 2.4.4.

Lastly, once we know the limit is finite we may obtain another formula of it using Proposition 2.3.1. We may take term-by-term limit of $T\to\infty$, again by monotone convergence, and get an alternative form for the asymptotic varaince:

$$\lim_{T\to\infty}\frac{V_f^{a,b}(T)}{2T}=\frac{1}{8\pi^3}\sum_{k\geq 1}\frac{1}{k^2}\int_{\mathbb{R}}\left(p^{*k}(\lambda)\right)^2h_k^{a,b}(2\lambda)d\lambda\in(0,\infty).$$

All that remains now is to prove Claim 2.4.4.

Proof of Claim 2.4.4. We recall that $S_k^{a,b}$ was computed in the proof of Claim 2.3.11, to be:

$$S_k=\frac{\partial^2}{\partial a\;\partial b}\left\{q^k\right\}=\begin{cases}q_{ab}, & k=1\\ k(k-1)q^{k-2}q_aq_b,+kq^{k-1}q_{ab} & k>1.\end{cases}\tag{2.26}$$

Step 1: Let g be one of the functions q, q_a, q_b or q_{ab}. Then $g(x,a,a)$, $g(x,a,b)$ and $g(x,b,b)$ are all in $\left(L^1 \cap L^\infty\right)(\mathbb{R})$ with respect to the variable x.

This is, in fact, part 3 of Claim 2.3.5. This step ensures that $S_k^{a,a}$, $S_k^{a,b}$ and $S_k^{b,b}$ are in $L^1(\mathbb{R})$ with respect to x.

We turn now to prove the "moreover" part of the claim. We use (2.26) in order to rewrite the desired series:

$$\sum_{k\geq 1} \frac{1}{k^2} \int_{\mathbb{R}} S_k(x)dx \tag{2.27}$$
$$= \int_{\mathbb{R}} q_{ab}\ dx + \sum_{k\geq 2} \int_{\mathbb{R}} q^{k-2} q_a q_b\ dx + \sum_{k\geq 2} \frac{1}{k} \int_{\mathbb{R}} q^{k-2} \left(q q_{ab} - q_a q_b\right)\ dx.$$

Once again, all functions are evaluated at (x,a,a), (x,a,b) or (x,b,b) and what follows holds for each of the three options. By step 1,

$$\int_{\mathbb{R}} |q_{ab}|\ dx < \infty, \quad \int_{\mathbb{R}} |q_a q_b|\ dx < \infty. \tag{2.28}$$

For the middle sum in (2.27), it is therefore enough to show that:

Step 2: The sum $\sum_{m\geq 1} \int_{\mathbb{R}} q^m q_a q_b\ dx$ converges.

Proof. We will show, in fact, that the positive series $\sum_{m\geq 1} \int_{\mathbb{R}} q^m |q_a q_b|\ dx$ converges.

First, in case we are evaluating at (x,a,b) $(a<b)$, our series converges due to (2.28) and the bound in part 2 of Claim 2.3.5. Now assume we are evaluating at (x,t,t) (where $t \in \{a,b\}$). As we deal with a positive series, it is enough to show that both

(I) $\sum_{m\geq 1} \int_{-1}^{1} q^m |q_a q_b|\ dx < \infty$, and

(II) $\sum_{m\geq 1} \int_{|x|\geq 1} q^m |q_a q_b|\ dx < \infty$.

Denote by $C = \sup_{x\in\mathbb{R}} |q_a q_b(x,t,t)| \in (0,\infty)$. The sum in (II) is bounded by

$$C \sum_{m\geq 1} \int_{|x|\geq 1} q^m(x,t,t)\ dx = C \int_{|x|\geq 1} \frac{q}{1-q}(x,t,t)\ dx \leq C' \int_{\mathbb{R}} q(x,t,t)\ dx,$$

where $C' \in (0,\infty)$ is another constant. C, C' and $\int_{\mathbb{R}} q(x,t,t)dx$ are all finite by part 3 of Claim 2.3.5.

We turn to show (I). By parts 1 and 4 of Claim 2.3.5, the sum

$$\sum_{m\geq 1} q^m |q_a q_b| \ dx = \frac{|q_a q_b|}{1-q}$$

is well-defined for all x (including $x=0$). By the monotone convergence theorem, item (I) is then equivalent to

$$\int_{-1}^{1} \frac{|q_a q_b|}{1-q}(x,t,t) \ dx < \infty,$$

which is indeed finite as an integral of a continuous function on $[-1,1]$. □

At last, only the right-most sum in (2.27) remains. Using the boundedness and integrability guaranteed in Step 1, it is enough to show:

Step 3: The sum $\sum_{m\geq 1} \frac{1}{m+2} \int_{\mathbb{R}} q^m \ dx$ converges.

Proof. We use a fact which is the basis for on of the standard proofs of the Central Limit Theorem (CLT). For completeness, we include a proof in the end of this subsection.

Lemma 2.4.2. *Let $g \in L^1(\mathbb{R})$ be a probability density, i.e., $g \geq 0$ and $\int_{\mathbb{R}} g = 1$. Suppose further that*

(a) $\int_{\mathbb{R}} |\lambda|^k g(\lambda) d\lambda < \infty$ for $k = 1,2,3$ and

(b) $\int_{\mathbb{R}} |\mathcal{F}[g](x)|^\nu \ dx < \infty$ for some $\nu \geq 1$.

Then there exists $C > 0$ such that for all $m \geq \nu$,

$$\int_{\mathbb{R}} |\mathcal{F}[g](x)|^m dx < \frac{C}{\sqrt{m}}.$$

We would like to apply the lemma to

$$g^{a,b}(\lambda) = \frac{e^{2\pi(a+b)\lambda} p(\lambda)}{r(ia+ib)}.$$

Notice that indeed this is probability measure, as by equation (2.18) with $k=1$:

$$\mathcal{F}[g^{a,b}](x) = \frac{r(x+ia+ib)}{r(ia+ib)},$$

and in particular $\mathcal{F}[g^{a,b}](0) = \int_{\mathbb{R}} g^{a,b} = 1$. This choice also obeys the extra integrability conditions in the lemma (as condition (2.1) implies (a) and (2.2) implies (b) with $\nu = 2$). We see now that

$$q(x,a,b) = \frac{r(ia+ib)^2}{r(2ia)r(2ib)} \cdot |\mathcal{F}[g^{a,b}](x)|^2 \leq |\mathcal{F}[g^{a,b}](x)|^2,$$

the last inequality following from the log-convexity of $y \mapsto r(iy)$. Similarly we define $g^{a,a}$ and have $q(x,a,a) = |\mathcal{F}[g^{a,a}](x)|^2$. Thus in all three cases of evaluation, using the lemma with the appropriate function g yields:

$$\sum_{m\geq 1} \frac{1}{m+2}\int_{\mathbb{R}} q^m \ dx \leq \sum_{m\geq 1} \frac{1}{m+2}\int_{\mathbb{R}} |\mathcal{F}[g](x)|^{2m} dx$$
$$< C \sum_{m\geq} \frac{1}{(m+2)\sqrt{2m}} < \infty,$$

as required. □

Combining all three steps with (2.27), we end the proof of Claim 2.4.4. □

Our last debt now is to prove Lemma 2.4.2. The proof is a minor variation of the proof for CLT appearing in Feller [18, Chapter XV.5].

Proof of Lemma 2.4.2. Write $G(x) = \mathcal{F}[g](x)$. We may assume that $\int_{\mathbb{R}} \lambda g(\lambda) = 0$ (otherwise we shall consider, instead of g, the function $g_\mu(\lambda) = g(\lambda + \mu)$ where $\mu := \int_{\mathbb{R}} \lambda g(\lambda) d\lambda$. There is no penalty since $|\mathcal{F}[g_\mu](x)| = |\mathcal{F}[g](x)|$ for all $x \in \mathbb{R}$). By assumption (a), $G(x)$ is thrice differentiable, and by the above assumptions $G(0) = 1$ and $G'(0) = 0$.

To prove the lemma, it is enough to show that

$$\lim_{m\to\infty} \sqrt{m} \int_{\mathbb{R}} |G(x)|^m dx \text{ exists and is finite.}$$

Notice that $\sqrt{m}\int_{\mathbb{R}} |G(x)|^m dx = \int_{\mathbb{R}} |G(x/\sqrt{m})|^m dx$, and so it is enough to show that

$$\lim_{m\to\infty} \int_{\mathbb{R}} \left| \left| G\left(\frac{x}{\sqrt{m}}\right)\right|^m - e^{-\frac{\alpha x^2}{2}} \right| dx = 0, \tag{2.29}$$

for some value of $\alpha > 0$, which in fact is $\alpha := G''(0)$.

We shall achieve (2.29) by splitting the integral into three parts, and showing each could be made less than a given $\varepsilon > 0$ if $m \geq \nu$ is chosen large enough.

Fix $R>0$ (to be determined later). By Taylor expansion,

$$G(x)=G(0)+xG'(0)+\frac{x^2}{2}G''(0)+o(x^2)=1+\frac{\alpha x^2}{2}+o(x^2),\ x\to 0 \quad (2.30)$$

and so $|G(x/\sqrt{m})|^m\to e^{-\alpha x^2/2}$ as $m\to\infty$, uniformly in $x\in[-R,R]$. Thus the integral in (2.29) computed on $[-R,R]$ converges to zero as $m\to\infty$.

From the expansion (2.30) we get

$$\exists\delta>0\ \forall|x|<\delta:\ |G(x)|\le e^{-\frac{\alpha x^2}{4}}.$$

Consider the integration in (2.29) for $R\le|x|\le\delta\sqrt{m}$. For such x we have $|G(x/\sqrt{m})|^m\le e^{-\frac{\alpha x^2}{4}}$, and so the integrand is less than $2e^{-\frac{\alpha x^2}{4}}$. Choosing R so that $4\int_R^\infty e^{-\frac{\alpha x^2}{4}}<\varepsilon$ will satisfy our needs.

Lastly, consider the integration on $\delta\sqrt{m}\le|x|<\infty$. By properties of Fourier transform, $\eta:=\sup_{|x|\ge\delta}|G(x)|\in(0,1)$. Thus

$$\int_{|x|\ge\delta\sqrt{m}}\left|\left|G\left(\frac{x}{\sqrt{m}}\right)\right|^m-e^{-\frac{\alpha x^2}{2}}\right|dx\le\eta^{m-\nu}\sqrt{m}\int_{\mathbb{R}}|G|^\nu+\int_{|x|\ge\delta\sqrt{m}}e^{-\frac{\alpha x^2}{2}}<\varepsilon,$$

for m large enough. Here we have used condition (b).

□

2.5 Theorem 8: Super-linear variance

In this section we prove the two items of Theorem 8, in reverse order.

2.5.1 Item (ii): Super-linear variance for particular a,b

Assume condition (2.3) holds for the particular a and b at hand. Fix a parameter $\delta>0$, and let $F=F_\delta$ be the set provided by Claim 2.4.3. The premise ensures that, if δ is small enough, at least one of the measures $(1+\lambda)\rho_{2a}|_{F_\delta}$ and $(1+\lambda)\rho_{2b}|_{F_\delta}$ does not have L^2-density. WLOG assume it is the former. At first, assume also $\rho_{2a}|_F$ is not in L^2. Repeating the arguments of the Subsection 2.4.1 we get the lower bound

$$\liminf_{T\to\infty}\frac{V_f^{a,b}(T)}{2T}\ge c_\delta\int_{\mathbb{R}}|\mathcal{F}[\mu]|^2(x)dx,$$

where $\mu=\rho_{2a}|_F$ and $c_\delta>0$. The LHS is therefore infinite, and so $L_1=\infty$.

We are left with the case that $\lambda\rho_{2a}|_{F_\delta}$ does not have L^2-density, but $\rho_{2a}|_{F_\delta}$ does (denote it by p_{2a}). The argument is similar. Continuing from (2.24) and employing Claim 2.4.3, we get

$$\begin{aligned}\liminf_{T\to\infty}\frac{V_f^{a,b}(T)}{2T} &\geq c_\delta \liminf_{\varepsilon\to 0}\int_F\int_F \frac{1}{2\varepsilon}\mathbb{1}_{(\tau-\varepsilon,\tau+\varepsilon)}(\lambda)(\lambda+\tau)^2 p_{2a}(\lambda)\ p_{2a}(\tau)d\lambda d\tau \\ &\geq c_\delta\cdot 4\int_K \lambda^2 p_{2a}(\lambda)^2 d\lambda,\end{aligned}$$

where $K\subset F$ is compact. But, by our assumption, by choosing K properly the last bound can be made arbitrarily large, so that $\lim_{T\to\infty}\frac{V_f^{a,b}(T)}{2T}=\infty$.

2.5.2 Item (i): Super-linear variance for almost all a, b

Let ρ be such that the condition in item (i) holds. If ρ has a singular component, then the condition in item (ii) holds for all possible a, b and so $L_1(a,b)=\infty$ with no exceptions. Otherwise, ρ has density $p(\lambda)$. Define the set

$$E=\{(a,b):\ a,b\in J,\ a<b,\ \text{the condition in item (ii) fails for } a,b\}.$$

If $E=\emptyset$, once again $L_1(a,b)=\infty$ for all $a,b\in J$ with no exceptions.

Assume then there is some $(a_0,b_0)\in E$. This means there exists λ_1,λ_2 such that for any pair of intervals I_1, I_2 such that $\lambda_j\in I_j$ $(j=1,2)$, both the functions $(1+\lambda^2)e^{2\pi\cdot 2a_0\lambda}p(\lambda)$ and $(1+\lambda^2)e^{2\pi\cdot 2b_0\lambda}p(\lambda)$ are in $L^2(\mathbb{R}\setminus(I_1\cup I_2))$, but at least one of them (WLOG, the former) is not in $L^2(\mathbb{R})$. Observe that the existence of such λ_1,λ_2 depends solely on $p(\lambda)$, and may therefore be regarded as independent of the point $(a_0,b_0)\in E$. Moreover, at least one among λ_1 and λ_2 (say, λ_1) is such that for any neighborhood I containing it, $p\notin L^2(I)$.

Suppose now $a,b\in E$ are such that

$$h_1^{a,b}(\lambda_1)>0, \tag{2.31}$$

where $h_1^{a,b}(\lambda)=\left(l_1^a(\lambda)e^{2\pi a\lambda}-l_1^b(\lambda)e^{2\pi b\lambda}\right)^2$ is the function appearing in the the first term of our asymptotic formula, and in the lower bound in inequality (2.24). Recall $h_1^{a,b}$ is non-negative and has only two zeroes by Claim 2.4.1.

We may choose $\delta>0$ smaller than the minimal distance between λ_1 and a zero of $h_1^{a,b}$, and then construct $F=F_\delta$ as in Claim 2.4.3. Certainly

$\lambda_1 \in F_\delta$, and so the measure $\mu = \rho_{2a}|_{F_\delta}$ is not in $L^2(\mathbb{R})$ (it is even not in $L^2(I)$ for any neighborhood I of λ_1). Just as in subsection 2.4.1 we shall get

$$\liminf_{T\to\infty} \frac{V_f^{a,b}(T)}{2T} \geq c_\delta \int_{\mathbb{R}} |\mathcal{F}[\mu]|^2(x)dx = \infty.$$

We end by showing that for a given point $\lambda_1 \in \mathbb{R}$ and a given $a \in J$, the set of $b \in J$ which do not obey (2.31) is finite. Indeed, this is the set

$$\{b \in J : \ h^{a,b}(\lambda_1) = 0\} = \{b \in J : \ \varphi(a) = \varphi(b)\}$$

where

$$\varphi(y) = e^{2\pi y \lambda_1} l_1^y(\lambda_1) = \frac{\partial}{\partial y}\left(\frac{e^{2\pi\lambda_1 y}}{r(2iy)}\right).$$

Suppose the desired set is not finite. Since φ is real-analytic, it must be constant on J. But then $r(2iy) = \frac{e^{2\pi\lambda_1 y}}{cy+d}$ for some $c, d \in \mathbb{R}$, and the corresponding spectral density would satisfy condition (2.2) for all relevant a, b. This contradiction ends the proof.

2.6 Directions of further research

2.6.1 Related fluctuation problems

Most likely, it is possible to extend the methods developed in this chapter in order to study fluctuations in other models. These include:

- the number of zeroes of a real stationary Gaussian function: it might be possible to simplify Slud's result, and to show that the variance is always at least linear under mild assumptions on the spectral measure.
- the increment of the argument of a stationary Gaussian function $f : \mathbb{R} \to \mathbb{C}$.
- smooth statistics of the zeroes of a GAF: For a compactly supported smooth test-function $\phi : D_\Delta \to \mathbb{R}$, study the number of zeroes weighted according to ϕ; that is, $\sum \phi(T^{-1}x + iy)$, where the sum is taken over all $x + iy$ which are zeroes of f.

2.6.2 A Central Limit Theorem

It is expected that the number of zeroes of a stationary GAF in long rectangles converges, after appropriate scaling, to a Gaussian distribution (i.e., a

"Central Limit Theorem", or CLT, for the zeroes). This conjecture is supported by analogous theorems applying to real functions (mentioned above). More support comes from a work by Nazarov and Sodin [34] for the "planar GAF" (a special GAF which is invariant to plane isometries), in which they prove CLT for zeroes in large balls as the radius approaches infinity. It would be desirable to prove a CLT for zeroes of a stationary GAF, beginning with the case of linear fluctuations (i.e., when $V^{a,b}(T)$ grows linearly in T).

Chapter 3

Gap probability for real stationary processes

3.1 Introduction

3.1.1 Definitions

Let T be either $\mathbb{Z}$ or $\mathbb{R}$, with the usual topology. A *Gaussian process* (GP) on T is a random function $f : T \to \mathbb{R}$ whose finite marginals, that is $(f(t_1), \dots, f(t_n))$ for any $t_1, \dots, t_n \in T$, have multi-variate centered Gaussian distribution. A GP on $\mathbb{Z}$ is called a *Gaussian sequence*, while a GP on $\mathbb{R}$ is called a *Gaussian function*. The Gaussian functions that we consider will be almost surely continuous.

A GP on T whose distribution is invariant with respect to shifts by any element of T, is called *stationary*. We abbreviate GSP, GSS and GSF for Gaussian stationary processes, sequences and functions respectively.

For a GSP f on T define the *covariance function* $r :\ T \to \mathbb{R}$ as

$$r(t) = \mathbb{E}(f(0)f(t)).$$

Observe that due to stationarity, for every $t, s \in T$ we have

$$\mathbb{E}\left[f(s)f(t)\right] = r(t-s).$$

It is not difficult to verify that $r(\cdot)$ is a positive-definite continuous function (continuity follows from almost sure continuity of f). By Bochner's theorem, there is a finite non-negative measure ρ on T^* such that

$$r(t) = \widehat{\rho}(t) := \int_{T^*} e^{-i\lambda t} d\rho(\lambda).$$

Here T^* is the dual of T, i.e. $\mathbb{Z}^* \simeq [-\pi, \pi]$ and $\mathbb{R}^* \simeq \mathbb{R}$. Notice that ρ must be symmetric, i.e., for any interval I: $\rho(-I) = \rho(I)$. We use the notation $\mathcal{M}(T^*)$ for the set of all finite non-negative symmetric measures on T^*. The measure $\rho = \rho_f \in \mathcal{M}(T^*)$ is called *the spectral measure* of the process f. Any $\rho \in \mathcal{M}(T^*)$ uniquely defines a GSP on T.

Throughout the chapter, we shall assume the following condition:

$$\exists \delta > 0 : \int_{T^*} |\lambda|^\delta d\rho(\lambda) < \infty. \tag{3.1}$$

This condition is enough to ensure that the associated process f will be a.s. continuous (see Adler and Taylor [2, Chapter 1, p. 22]). Notice that this holds trivially in case $T = \mathbb{Z}$.

3.1.2 Results

Let $f : T \to \mathbb{R}$ be a GSP. Define the "gap probability" of f to be

$$H_f(N) = \mathbb{P}\left(\forall t \in [0, N) \cap T : \ f(t) > 0\right),$$

where $N \in \mathbb{R}$ is a parameter. This describes half the probability that no sign-changes of f occurred in a time interval of length N. We study the asymptotics of this probability as $N \to \infty$. It makes no essential difference to regard N as an integer, and we usually do so.

Our main results are the following. Let f be a Gaussian stationary process on $T = \mathbb{Z}$ or $T = \mathbb{R}$, with spectral measure $\rho \in \mathcal{M}(T^*)$, satisfying (3.1).

Theorem 9 (upper bound)**.** *Suppose that there exists $a > 0$ and two numbers $M, m > 0$ such that*

$$\textit{for any interval } I \subset (-a, a), \ m|I| \le \rho(I) \le M|I|.$$

Then there exists $C > 0$ such that for all large enough N,

$$H_f(N) \le e^{-CN}.$$

Theorem 10 (lower bound)**.** *Suppose that there exists $a > 0$ and a number $m > 0$ such that*

$$\textit{for any interval } I \subset (-a, a), \ m|I| \le \rho(I).$$

Then there exists $c > 0$ such that for all large enough N,

$$H_f(N) \ge e^{-cN}.$$

Remark 3.1.1. The condition in Theorem 9 may be replaced by the following: There exist two intervals $J_1 = (-a, a)$ and J_2, and two numbers $M, m > 0$, such that

(a) for any interval $I \subset J_1$: $\rho(I) \leq M|I|$, and

(b) for any interval $I \subset J_2$: $m|I| \leq \rho(I)$.

The necessary changes in the proof are indicated in Section 3.3.1. However, the authors believe condition (a) might be enough to ensure an upper exponential bound on $H(N)$.

Remark 3.1.2. Examples for which $H(N)$ tends to zero slower than any exponential in N are known; Newell and Rosenblatt construct one in [35].

Examples for which $H(N)$ tends to zero faster than any exponential in N are also known. A simple example was pointed out to us by M. Krishnapur. Let $(Y_j)_{j\in\mathbb{Z}}$ be a GSS with i.i.d. entries, distributed $\mathcal{N}(0,1)$, and define $X_j = Y_j - Y_{j-1}$ for all $j \in \mathbb{Z}$. Then X is a GSS with $H_X(N) = \frac{1}{N!} = e^{-N\log N(1+o(1))}$. Notice that the spectral measure has density $2(1-\cos(\lambda))$, $\lambda \in [-\pi, \pi]$, which vanishes at $\lambda = 0$.

3.1.3 Overview

The rest of the chapter is organized as follows. Section 3.2 is devoted to discussion of the results. This includes an historical background, and a simple yet useful observation that we shall use (Observation 2 below). The results are then proved independently: Theorem 9 (an upper exponential bound) is proved in Section 3.3, while Theorem 10 (a lower exponential bound) is proved in Section 3.4.

3.2 Discussion

3.2.1 Background

Gap probability, sometimes referred to by the name "persistence probability" or "hole probability", was studied extensively in the 1960's, by Slepian [44], Longuet-Higgins [28], Newell-Rosenblatt [35] and others. In addition to proving some bounds and inequalities (such as the well-known "Slepian Inequality"), they developed series expansions which approximate this probability quite well for small intervals. In a few examples, exact expressions for the gap probability were calculated (see [44] and references therein).

In the last decade or two, physicists (such as Majumdar-Bray [30] and Ehrhardt-Majumdar-Bray [14]) proposed some new methods of approximation, especially for the long-range regime. These suggest that in many cases of interest the gap probability $H(N)$ behaves asymptotically like $e^{-\theta N}$, with some $\theta > 0$.

Dembo and Mukherjee [11, Theorem 1.6] observed that if the covariance function $r(t)$ is non-negative, this asymptotic behavior follows from Slepian Inequality and subadditivity. I.e., the limit

$$\theta_f = \lim_{N\to\infty} \frac{-\log H_f(N)}{N}$$

exists, and is finite and non-negative (finiteness is not mentioned explicitly in the reference, but follows easily from the proof therein using the continuity of f). To our knowledge, even in this simpler case no general bounds on θ_f were known. A computation of the limit θ_f, as well as its existence for $r(t)$ which changes sign, are open.

We note that [11], along with other works by physicists such as Schehr-Majumdar [40], draw connections between gap probabilities of GSPs, those of diffusion processes, and those of zeroes of random polynomials.

The first attempt to tackle the case where $r(t)$ is not non-negative was done by Antezana, Buckley, Marzo and Olsen [3]. They were able, using novel ideas, to obtain exponential upper and lower bounds for $H_f(N)$ for the particular case of the cardinal sine covariance $r(t) = \frac{\sin(\pi t)}{t}$, which corresponds to indicator spectral density $\mathbb{1}_{[-\pi,\pi]}$. Our results may be viewed as an extension of their result to other *stationary* Gaussian processes, using an idea of spectral decomposition. Recently Antezana, Marzo and Olsen were able to generalize this same result in the direction of Gaussian analytic functions over *de-Branges spaces* [4].

Via private communication we learned of results by Krishnapur-Maddaly regarding lower bounds for the gap probability of a GSS. It seems that our conditions for a lower exponential bound are currently stronger, but they have given very mild conditions which ensure $H_f(N) \geq e^{-cN^2}$ (where $c > 0$ is a constant, and the inequality holds for large enough N). Though the results are similar in spirit, their methods seem to be very different from ours.

Lastly we mention an analogous result for the planar Gaussian analytic function

$$\sum_{n\in\mathbb{Z}} a_n \frac{z^n}{\sqrt{n!}}, \text{ where } a_n \sim \mathcal{N}_{\mathbb{C}}(0,1) \text{ are i.i.d.}$$

Bounds concerning hole probabilities for this model were obtained by Sodin and Tsirelson [49], and later refined by Nishry [36]. Nishry showed that the probability of having no zeroes in a ball of radius R in the plane is asymptotically $e^{-(e^2/4+o(1))R^4}$, as $R \to \infty$. For discussion of such results and comparison to other point processes in the plane, see [22, Chapter 7].

3.2.2 A Key Observation

We include here the basic observation which will be used to prove both Theorems 9 and 10. We use the symbol $\oplus$ to indicate the sum of two independent processes or random variables.

Observation 2. *Let f be a GSP on T with spectral measure $\rho \in \mathcal{M}(T^*)$, and suppose $\rho = \rho_1 + \rho_2$, where $\rho_1, \rho_2 \in \mathcal{M}(T^*)$. Then the following equality holds in distribution:*

$$f \overset{d}{=} f_1 \oplus f_2,$$

where f_j is a GSP with spectral measure ρ_j ($j = 1, 2$), and f_1 is independent (as a process) from f_2.

Proof. We calculate the covariance function of $f_1 \oplus f_2$ using the independence of the processes:

$$\begin{aligned} \mathbb{E}\Big[(f_1(0) + f_2(0))\,(f_1(t) + f_2(t)) \Big] &= \mathbb{E} f_1(0) f_1(t) + \mathbb{E} f_2(0) f_2(t) \\ &= \widehat{\rho_1}(t) + \widehat{\rho_2}(t) = \widehat{\rho}(t). \end{aligned}$$

This covariance function is equal to that of f. As all processes are Gaussian, the observation follows. ☐

3.3 Theorem 9: Upper bound

This section is devoted to the proof of Theorem 9.

Let f be a GSF or GSS with spectral measure ρ, obeying the conditions of Theorem 9. Let $k \in \mathbb{N}$ be such that $\frac{\pi}{k} \le a$, and denote $J := [-\pi/k, \pi/k] \subset [-a, a]$. We decompose the spectral measure as follows:

$$d\rho(\lambda) = m 1\!\!1_J(\lambda) d\lambda + d\mu(\lambda),$$

where $\mu \in \mathcal{M}(T^*)$ is non-negative and there exists $M' > 0$ such that

$$\text{for any interval } I \subset (-a, a): \ \mu(I) \le M'|I|. \tag{3.2}$$

By Observation 2, we may represent

$$f \overset{d}{=} S \oplus g$$

where S and g are independent processes, with spectral measures $m1\!\!1_J(\lambda)$ and μ respectively.

Next, we observe that sampling S in a certain lattice results in independent random variables:

Observation 3 (indicator spectrum)**.** *The GSP $(S(t))_{t\in T}$ having spectral density $m1\!\!1_{[-\pi/k,\pi/k]}$ has the property that $(S(jk))_{j\in\mathbb{Z}}$ are i.i.d. Gaussian random variables.*

Proof. By taking the Fourier transform of the given measure, the covariance function of S is

$$\mathbb{E}\left[S(s)S(t)\right] = 2m\frac{\sin(\frac{\pi}{k}(t-s))}{t-s}.$$

Thus $S(jk)$ and $S(nk)$ are uncorrelated for any $j, n \in \mathbb{Z}$, $j \neq n$; as these are Gaussian random variables - independence follows. □

In order to apply Observation 3, we look at a certain translated lattice $\{jk + \ell : j \in \mathbb{Z}\}$ on which S is indeed independent. The translation (which we call "split") of the sampled lattice will depend on g.

More precisely, fix a number $q > 0$ (say, $q = 1$), and define the set $G \subset C(T)$ as follows:

$$G = \left\{ h \in \mathbb{Z}^{\mathbb{R}} \; : \; \frac{1}{N}\sum_{j=1}^{N} h(j) < q \right\}, \text{case of GSS}$$
$$G = \left\{ h \in C(\mathbb{R}) \; : \; \frac{1}{N}\int_0^N h(t)dt < q \right\}, \text{case of GSF.}$$

Using the law of total probability we have:

$$\begin{aligned}&\mathbb{P}\left(f(t) = S(t) + g(t) > 0,\; 0 \le t < N\right)\\ &\quad\le \mathbb{P}\left(S(t) + g(t) > 0,\; 0 \le t < N \,\middle|\, g \in G\right) + \mathbb{P}\left(g \notin G\right).\end{aligned}$$

It is enough to show that there exist $C_1, C_2 > 0$ such that for large enough N,

(i) $\mathbb{P}\left(S(t) + g(t) > 0,\; 0 \le t < N \,\middle|\, g \in G\right) \le e^{-C_1 N}$, and

(ii) $\mathbb{P}(g \notin G) \leq e^{-C_2 N}$.

We proceed the proof for the function-case, noting the sequence-case follows similar lines and is generally easier.

We begin by showing (i). It is enough to show that there is $C_1 > 0$ such that for any large enough N and any fixed $g \in G$,

$$\mathbb{P}(S(t) + g(t) > 0,\ 0 \leq t < N) \leq e^{-C_1 N}. \tag{3.3}$$

Indeed, this would imply (using the independence of g and S):

$$\begin{aligned}&\mathbb{P}(S(t) + g(t) > 0,\ 0 \leq t < N \mid g \in G)\\ &= \mathbb{E}\Big[\ \mathbb{P}\big(S(t) + g(t) > 0,\ 0 \leq t < N \,\big|\, g\big) \mid g \in G\Big] \leq e^{-C_1 N},\end{aligned}$$

as required.

To that end, we use a property which holds when the event $g \in G$ occurs, stated below.

Observation 4. *Let g be a continuous function such that $\frac{1}{N}\int_0^N g(t)dt < q$, and assume $N \in \mathbb{N}$ is divisible by k, then there exists a number $l \in [0, k)$ such that*

$$\frac{k}{N}\sum_{j=0}^{N/k-1} g(jk + \ell) < q.$$

Proof. Else, for every $l \in [0, k)$ the reverse inequality holds. Integrating it over $l \in [0, k]$ yields a contradiction. □

Now, fix a function $g \in G$. We can find a special split ℓ_g whose existence is guaranteed by Observation 4. Therefore:

$$\begin{aligned}&\mathbb{P}(S(t) + g(t) > 0,\ 0 \leq t < N)\\ &\leq \mathbb{P}(S(jk + \ell_g) + g(jk + \ell_g) > 0,\ j = 0, 1, \ldots, N/k - 1),\end{aligned}$$

where $(S(jk + \ell))_{j\in\mathbb{Z}}$ are i.i.d Gaussians (whose variance is independent of l_g), and $\frac{k}{N}\sum_{j=0}^{N/k-1} g(jk + \ell_g) < q$. The following inequality will give the desired bound.

Proposition 3.3.1. *Let $X_1, \ldots, X_N$ be i.i.d standard Gaussian random variables (distributed $\mathcal{N}(0, 1)$), and let $q \in \mathbb{R}$. There is a constant $C_q > 0$ such that for any numbers $b_1, \ldots, b_N \in \mathbb{R}$ which obey $\frac{1}{N}\sum_{j=1}^N b_j < q$, the following holds:*

$$\mathbb{P}(X_j + b_j > 0,\ 1 \leq j \leq N) \leq e^{-C_q N}.$$

Proof. Denote by $\Phi(b) = \mathbb{P}(X_1 < b)$ the cumulative distribution function of X_1. By symmetry, $\Phi(b) = \mathbb{P}(X_1 > -b)$. Using the "i.i.d." property of the variables $\{X_j\}_{j=1}^N$ we have:

$$p = \mathbb{P}\left(X_j + b_j > 0,\ 1 \le j \le N\right) = \prod_{j=1}^{N} \mathbb{P}\left(X_j > -b_j\right) = \prod_{j=1}^{N} \Phi(b_j).$$

Taking logarithm and using the concavity and monotonicity of $x \mapsto \log \Phi(x)$, we get:

$$\log p = \sum_{j=1}^{N} \log \Phi(b_j) \le N \cdot \log \Phi\left(\frac{\sum_j^N b_j}{N}\right) < N \cdot \log \Phi(q),$$

and so $C_q = -\log \Phi(q) > 0$ is the desired constant. □

In order to prove (ii), we shall use the following:

Proposition 3.3.2. $\frac{1}{N}\int_0^N g(t)dt \sim \mathcal{N}_{\mathbb{R}}(0, \sigma_N^2)$, *where* $\sigma_N^2 \le \frac{C_0}{N}$ *for all* $N \in \mathbb{N}$ *and some constant* $C_0 > 0$.

Proof. The normality of the given integral follows from general arguments of convergence of Gaussian random variables. We focus on the bound on its variance. Recall that μ denoted the spectral measure of g. We calculate the variance:

$$\begin{aligned}\sigma_N^2 &= \frac{1}{N^2}\mathbb{E}\left(\int_0^N g(t)dt\right)^2 = \frac{1}{N^2}\iint_{[0,N]^2} \mathbb{E}(g(t)g(s))dt\ ds \\ &= \frac{1}{N^2}\int_0^N\int_0^N \widehat{\mu}(t-s)dt\ ds = \frac{1}{N}\int_{|t|<N}\left(1 - \frac{|t|}{N}\right)\widehat{\mu}(t)dt.\end{aligned}$$

The change in order of integration and expectancy in the first equality is easily justified by use of Fubini's theorem.

The inverse Fourier transform of $(1 - \frac{|t|}{N})1\!\!1_{[-N,N]}(t)$ is given by

$$K_N(\lambda) = N\left(\frac{\sin(N\lambda/2)}{N\lambda/2}\right)^2 \le \min\left(N, \frac{\pi^2}{N\lambda^2}\right)$$

Using first Plancherel's Identity, and then condition (3.2) on the boundedness of μ, we get:

$$\begin{aligned}\sigma_N^2 &= \frac{1}{N}\int_{\mathbb{R}} K_N(\lambda)d\mu(\lambda) \\ &\leq \int_{|\lambda|<\frac{\pi}{N}} d\mu(\lambda) + \frac{\pi^2}{N^2}\left(\int_{\frac{\pi}{N}\leq|\lambda|<a} + \int_{|\lambda|\geq a}\right)\frac{1}{\lambda^2}d\mu(\lambda) \\ &\leq M'\cdot\frac{2\pi}{N} + \frac{\pi^2}{N^2}\left(M'\int_{\frac{\pi}{N}\leq|\lambda|<a}\frac{d\lambda}{\lambda^2} + \frac{1}{a^2}\mu(\{|\lambda|>a\})\right) \\ &\leq \frac{C_0}{N},\end{aligned}$$

where C_0 is a constant (depending on μ). □

At last, we prove (ii). Denote by γ a standard Gaussian random variable. Using the Proposition 3.3.2 together with the well-known inequality

$$\forall y>0: \ \mathbb{P}(\gamma>y) < \frac{1}{\sqrt{2\pi}y}e^{-y^2/2},$$

we get:

$$\begin{aligned}\mathbb{P}(g\notin G) = \mathbb{P}\left(\frac{1}{N}\int_0^N g(t)\geq q\right) &= \mathbb{P}(\sigma_N\cdot\gamma\geq q) = \mathbb{P}\left(\gamma\geq\frac{q}{\sigma_N}\right) \\ &\leq \frac{1}{\sqrt{2\pi}}\cdot\frac{\sigma_N}{q}e^{-\frac{1}{2}\cdot\frac{q^2}{\sigma_N^2}} \\ &\leq \frac{1}{q}\sqrt{\frac{C_0}{2\pi N}}e^{-\frac{q^2}{2C_0}N} \leq e^{-C_2N},\end{aligned}$$

for a suitable choice of $C_2>0$ (depending only on q and μ). Theorem 9 is proved. □

3.3.1 Extension: Proof of Remark 3.1.1

Remark 3.1.1 states a somewhat more general condition under which the conclusion of Theorem 9 is true. The proof is only a slight modification of the one presented. First, choose $\ell, k\in\mathbb{N}$ so that

$$J := \left[\frac{(2\ell-1)\pi}{k}, \frac{2\ell\pi}{k}\right] \subset J_2\cup(-J_2).$$

Now decompose the measure as follows:

$$d\rho(\lambda) = m1\!\!1_{J\cup -J}(\lambda)d\lambda + d\mu(\lambda).$$

By the premise, $\mu \in \mathcal{M}(T^*)$ obeys the boundedness condition (3.2) (just as before). Applying Observation 2 we get

$$f \stackrel{d}{=} S \oplus g,$$

where S has spectral measure $m\mathbb{I}_{J\cup -J}(\lambda)d\lambda$ and g has spectral measure μ. We define G as before and strive to prove items (i) and (ii). Item (ii) follows from Proposition 3.3.2 and the calculation following it with no change. The only property used in order to prove item (i) is the independence of $(S(jk))_{j\in\mathbb{Z}}$ (i.e., Observation 3). Let us show this still holds.

One way to end the argument is by calculation of the Fourier transform of $\mathbb{I}_{J\cup -J}(\lambda)d\lambda$ and observing it vanishes at kj, $j \in \mathbb{Z}$ (just as in the proof of Observation 3). We give here a more general argument, relying on two observations:

Observation 5. *Let $(f(t))_{t\in\mathbb{R}}$ be a GSF with spectral measure ρ, and $\alpha > 0$. Then the GSF $x \mapsto f(\alpha x)$ has spectral measure $\frac{1}{\alpha}\ \rho_\alpha$, where*

$$\forall I \subset \mathbb{R}: \quad \rho_\alpha(I) = \rho(\{x \in \mathbb{R}: \ \alpha x \in I\}).$$

Proof. $\mathbb{E}\left[f(\alpha t)f(\alpha s)\right] = \widehat{\rho}(\alpha(t-s)) = \frac{1}{\alpha}\ \widehat{\rho_\alpha}(t-s)$. □

Observation 6. *If $(f(t))_{t\in\mathbb{R}}$ is a GSF with spectral measure ρ, then the GSS $(f(j))_{j\in\mathbb{Z}}$ has the folded spectral measure $\rho^* \in \mathcal{M}([-\pi,\pi])$ obtained by:*

$$\rho^*(I) = \sum_{n\in\mathbb{Z}} \rho(I + 2\pi n).$$

Proof. ρ^* is the unique measure in $\mathcal{M}([-\pi,\pi])$ such that $\widehat{\rho^*}(j) = \widehat{\rho}(j)$ for any $j \in \mathbb{Z}$. □

Combining the last two observations, we get that if $(S(t))_{t\in T}$ has spectral density $m\mathbb{I}_{J\cup -J}(\cdot)$, then the spectral density of $(S(kj))_{j\in\mathbb{Z}}$ is $\frac{m}{k}\mathbb{I}_{[-\pi,\pi]}(\cdot)$. Now Observation 3 leads to the desired conclusion.

3.4 Theorem 10: Lower bound

In this section we prove Theorem 10.

3.4.1 Reducing GSS to GSF

Theorem 10 is easily reduced to the case of functions, by noticing the following:

Observation 7. *Any finite measure $\rho \in \mathcal{M}([-\pi,\pi])$ generates a GSF f and a GSS X. The distribution of $(X(j))_{j\in\mathbb{Z}}$ is the same as that of $(f(j))_{j\in\mathbb{Z}}$ (since their covariance functions coincide). Moreover, for any number N:*

$$\begin{aligned} H_f(N) &= \mathbb{P}(f(x) > 0,\ x \in [0,N) \cap \mathbb{R}) \\ &\le \mathbb{P}(f(j) > 0,\ j \in [0,N) \cap \mathbb{N}) = H_X(N). \end{aligned}$$

Therefore, in order to bound $H_X(N)$ from below where X is a GSS, it is enough to bound $H_f(N)$ from below where f is the GSF with the same spectral measure as X.

3.4.2 Proof for GSF

Let $(f(t))_{t\in\mathbb{R}}$ be a GSF with spectral measure ρ, obeying the condition of Theorem 10. By scaling f (and therefore scaling its spectral measure according to Observation 5), we may assume the condition is satisfied with $a = \pi$.

Just as in the proof of Theorem 9, we decompose the spectral measure in the following manner:

$$d\rho = m1\!\!1_{[-\pi,\pi]}(\lambda)d\lambda + d\mu.$$

Applying Observation 2 we have

$$f \stackrel{d}{=} S \oplus g$$

where S and g are independent processes, and the spectral measure of S has density $m1\!\!1_{[-\pi,\pi]}(\lambda)$.

We have:

$$\begin{aligned} H_f(N) &= \mathbb{P}\left(S(x) + g(x) > 0,\ 0 \le x < N\right) \\ &\ge \mathbb{P}\left(S(x) > d,\ 0 \le x < N\right)\ \mathbb{P}\left(|g(x)| \le \frac{d}{2},\ 0 \le x < N\right), \end{aligned} \quad (3.4)$$

where $d > 0$ is a parameter of our choice. The first probability is bounded from below by the following theorem:

Theorem 11 (Antezana, Buckley, Marzo, Olsen [3]). *Let $S(x)$ be the GSF with spectral measure $d\rho(\lambda) = 1\!\!1_{[-\pi,\pi]}(\lambda)d\lambda$. Then for any $d > 0$ there exists a constant $c_d > 0$, such that for all $N \in \mathbb{N}$,*

$$\mathbb{P}\left(S(x) > d,\ 0 \leq x < N\right) \geq e^{-c_d N}.$$

In [3], the theorem is stated for the case $d = 0$, but the above can be obtained by minor modifications to the proof given there.

We turn to bound the second probability in (3.4), i.e., the probability of the event $\{|g(x)| \leq \varepsilon,\ 0 \leq x < N\}$. This is known in literature as a "small ball probability", and is bounded from below by the following result:

Lemma 3.4.1 (Talagrand [50], Shao-Wang [42]). *Let $(f(t))_{t\in I}$ be a centered Gaussian process on a finite interval I. Suppose that for some $c > 0$ and $0 < \delta \leq 2$,*

$$d_f(s,t)^2 := \mathbb{E}|f(s) - f(t)|^2 \leq c|t-s|^\delta,\ \ s,t \in I.$$

Then, for some $K > 0$ and every $\varepsilon > 0$,

$$\mathbb{P}\left(\sup_{t\in I}|f(t)| \leq \varepsilon\right) \geq \exp\left(-\frac{K|I|}{\varepsilon^{2/\delta}}\right).$$

The proof of Lemma 3.4.1, apart from being deduced from a much more general result in Talagrand's paper, may be found in notes by Ledoux [25, Ch. 7, statement (7.13)] (in a slightly different formulation). Shao and Wang decided to omit a proof from their paper as they learned that Talagrand's result generalizes theirs; but their Theorem 1.1 is the closest formulation to the one above.

We use this lemma to prove the following:

Proposition 3.4.1. *Let f be a Gaussian stationary function on $\mathbb{R}$ with spectral measure ρ, obeying the moment condition* (3.1)*. Then for all $\varepsilon > 0$ there exists $C, K > 0$ such that for any interval I:*

$$\mathbb{P}\left(\sup_{t\in I}|f(t)| < \varepsilon\right) \geq Ce^{-K|I|}.$$

Applying the proposition to $f = g$, $I = [0,N)$ and $\varepsilon = \frac{d}{2} > 0$, will give the desired bound on the second factor in (3.4), thus ending the proof of Theorem 10.

Proof of Proposition 3.4.1. First we notice that if the moment condition (3.1) is satisfied with a certain exponent $\delta > 0$, then it is also satisfied by any smaller positive exponent. Therefore we may assume $0 < \delta < 2$.

We shall check that f obeys the condition of Lemma 3.4.1 with this same δ, i.e. that there exists a constant $c > 0$ such that

$$d_f(s,t)^2 \leq c|t-s|^\delta, \;\; s,t \in I.$$

Indeed:

$$\begin{aligned} d_f(s,t)^2 &= \mathbb{E}(f(s)-f(t))^2 = 2(r(0)-r(s-t)) \\ &= 2\int_{\mathbb{R}} \Big(1-\cos(\lambda(s-t))\Big) d\rho(\lambda) \leq 2L|t-s|^\delta \int_{\mathbb{R}} |\lambda|^\delta d\rho(\lambda), \end{aligned}$$

where $L = \sup_{x\in\mathbb{R}} \frac{1-\cos(x)}{|x|^\delta} < \infty$. The proposition follows. □

3.4.3 Directions of further research

Investigation of the gap probability

The results of this chapter show that there is a relation between spectral properties of a Gaussian stationary process, and the asymptotics of its gap probability. It would be desirable to understand this relation better, and use it to find accurate asymptotics of the gap probability; i.e., to prove existence of the limit

$$\lim_{T\to\infty} \frac{-\log H(T)}{T},$$

presumably under some spectral conditions. As mentioned earlier, this limit is known to exists if the correlation function $r(t)$ is non-negative, and is conjectured to exist in a much broader setup; but even for simple examples in which $r(t)$ changes sign, such as the sinc-kernel, the existence is open. Further, it would be interesting to compute the limit (which is unknown also when $r(t) \geq 0$), and to characterize when is it 0 or $+\infty$. This, I conjecture, has to do with the spectral measure vanishing or being unbounded near the origin, respectively.

Another direction is to extend the bounds obtained in Theorems 10 and 9 to similar probabilities for complex-valued functions. For instance, it would be desirable to bound the probability that a stationary "smooth" function $f : \mathbb{R} \to \mathbb{C}$ does not wind around zero for a long time-interval.

Exponential concentration

For stationary Gaussian functions $f : \mathbb{R} \to \mathbb{R}$, another rare event of interest is that the number of zeroes in a long interval $[0, T]$ is "far" from the mean number of zeroes (by more then αT, where $\alpha > 0$ is given). The probability of such an event is expected to be exponentially small in T (with a constant depending on α). "Exponential concentration" around the mean is natural to expect when Offord-type estimates hold (i.e., when the tail of the distribution of the number of zeroes in a compact set decays exponentially; see [22, Chapters 3.2 and 7.1]). Exponential concentration was proved for nodal lines of Gaussian spherical harmonics, by Nazarov and Sodin [31].

A significant step towards proving such a result for real Gaussian processes was done by Tsirelson in a course he gave in TAU [51]. There he proves exponential concentration for weighted counting functions of the zeroes (depending on the derivative at each zero), under the assumption of purely atomic spectral measure. As a first step, it would be natural to extend this result to arbitrary spectral measures (with similar weighted counting functions). This might shed light on the methods and conditions needed for the original problem; yet, it seems, some new ideas and techniques will still be required.

Bibliography

[1] J. Aaronson, *An introduction to infinite ergodic theory*, Mathematical Surveys and Monographs Vol. 50, American Mathematical Society (1997).

[2] R.J. Adler and J.E. Taylor, *Random Fields and Geometry*, Springer Monographs in Mathematics (2007).

[3] Antezana, J., Buckley, J., Marzo, J. and J.-F. Olsen, *Gap probabilities for the cardinal sine*, J. Math. Anal. Appl. **396** (2012), 466–472.

[4] Antezana, J., Marzo, J. and J.-F. Olsen, *Zeros of random functions generated with de Branges kernels*, to appear.

[5] J.-M. Azaïs, J.R. León, *CLT for crossings of random trigonometric polynomials*, Electron. J. Probab. **18** (2013) , 1-17.

[6] P. Bleher and D. Ridzal, $SU(1,1)$ *Random Polynomials*, Journal of Statistical Physics (2002), Vol 106, numbers 1-2, p.147-171.

[7] E. Bogomolny, O. Bohigas, and P. Leboeuf, *Quantum Chaotic Dynamics and Random Polynomials*, J. Statist. Phys. **85** (1996), 639-679.

[8] D. Chambers and E. Slud, *Central limit theorems for nonlinear functionals of stationary Gaussian processes*, Probab. Th. Rel. Fields **80** (1989), 323-346.

[9] H. Cramér and M.R. Leadbetter, *Stationary and Related Stochastic Processes*, Wiley series in Probability and Mathematical Statistics (1967).

[10] J. Cuzick, *A central limit theorem for the number of zeros of a stationary Gaussian process*, Ann. Probab. **4** (1976), 547-556.

[11] A. Dembo and S. Mukherjee, *No zero-crossings for random polynomials and the heat equation*, Annals of Probability, to appear. See arXiv:1208.2382.

[12] R. Durrett, *Probability: Theory and Examples*, Cambridge University Press, fourth edition (2010).

[13] A. Edelman and E. Kostlan, *How many zeros of a random polynomial are real?* Bull. Are. Math. Soc. (N.S), **32** (1995), 1-37.

[14] G.C. Ehrhardt, S.N. Majumdar and A.J. Bray, *Persistence exponents and the statistics of crossings and occupation times for Gaussian stationary processes*, Physical Review E **69** (2004), 016106.

[15] N. Feldheim, *Zeroes of Gaussian Analytic Functions with Translation-Invariant Distribution*, Israel Journal of Mathematics **195** (2013), 317-345.

[16] N. Feldheim, *Variance of the Number of Zeroes of Shift-Invariant Gaussian Analytic Functions*, submitted. arXiv:1309.2111.

[17] N. D. Feldheim, O. N. Feldheim, *Long gaps between sign-changes of Gaussian Stationary Processes*. To appear in IMRN, doi: 10.1093/imrn/rnu020. See also: arXiv:1307.0119.

[18] W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. 2, 2nd edition, John Wiley and Sons (1971).

[19] A. Granville, I. Wigman, *The distribution of the zeros of random trigonometric polynomials*, American Journal of Mathematics **133** (2011), 295-357.

[20] U. Grenander, *Stochastic Processes and Statistical Inference*, Arkiv for Matematik **1** (1950), 195-277.

[21] L. Hörmander, *The Analysis of Linear Partial Differential Operators*, Springer-Verlag (1983).

[22] J.B. Hough, M. Krishnapur, Y. Peres and B. Virag, *Zeroes of Gaussian Analytic Functions and Determinantal Processes*, University Lecture Series **51**, American Mathematical Society (2009).

[23] M. Kac, *On the average number fo real roots of a random algebraic equation*, Bull. Amer. Math. Soc. **18** (1943), 29-35.

[24] Y. Katznelson, *An Introduction to Harmonic Analysis* (Third Edition), Cambridge University Press (2004).

[25] Ledoux, M , *Isoperimetry and Gaussian Analysis*, Lectures on Probability Theory and Statistics, Lecture Notes in Mathematics **1648** (1996), 165-294.

[26] B.Ja. Levin, *Zeros of Entire Functions*, Amer. Math. Soc. (1964).

[27] Ju.V. Linnik and I.V. Ostrovskii, *Decomposition of Random Variables and Vectors*, Amer. Math. Soc. (1977).

[28] M. S. Longuet-Higgins, *The Distribution of Intervals between Zeros of a Stationary Random Function*, Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences **254** (1962), 557-599.

[29] B. Macdonald, *Density of Complex Zeros of a System of Real Random Polynomials*, J. Stat. Phys. **136** (2009), 807-833.

[30] Majumdar, S.N. and A.J. Bray, *Persistence with Partial Survival*, Physical Review Letters **81** (1998), 2626-2629.

[31] F. Nazarov and M. Sodin, *On the number of nodal domains of random spherical harmonics*, American Journal of Mathematics **131** (2009), 1337-1357.

[32] F. Nazarov and M. Sodin, *Random Complex Zeroes and Random Nodal Lines* Proceedings of ICM 2010 Vol. 3 (2010), 1450-1484.

[33] F. Nazarov and M. Sodin, *What is a ... Gaussian entire function?* Notices Amer. Math. Soc. **57** (2010), 375-377.

[34] F. Nazarov and M. Sodin, *Fluctuations in Random Complex Zeroes: Asymptotic Normality Revisited*, International Math. Research Notices Advance Access (2011).

[35] Newell G. F. and M. Rosenblatt , *Zero Crossing Probabilities for Gaussian Stationary Processes*, Annals Mathematical Statistics **33** (1962), 1306-1313.

[36] A. Nishry , *Asymptotics of the hole probability for zeros of random entire functions*, International Mathematics Research Notices (2010), 2925-2946.

[37] R.E.A.C. Paley and N. Wiener, *Fourier Transforms in the Complex Domain*, American Mathematical Society Colloquium Publications, Vol. XIX (1967), 163-178.

[38] V.I. Piterbarg, *Asymptotic Methods in the Theory of Gaussian Processses and Fields*, Translations of Mathematical Monographs Vol. 148, American Mathematical Society (1996).

[39] T. Prosen, *Exact Statistics of complex zeros for Gaussian random polynomials with real coefficients*, J. Phys. A **29** (1996), 4417-4423.

[40] Schehr, G. and S.N. Majumdar , *Real roots of random polynomials and zero crossing properties of diffusion equation*, Journal of Statistical Physics **132** (2008), 235–273.

[41] G. Schehr and S.N. Majumdar, *Condensation of the Roots of Real Random Polynomials on the Real Axis*, J. Stat. Phys. **135** (2009), 587-598.

[42] Q.-M. Shao and D. Wang, *Small ball probabilities of Gaussian fields*, Research Report No. 616, Department of Mathematics, National Univ. of Singapore.

[43] L.A. Shepp and R.J. Vanderbei, *The complex zeros of random polynomials*, Trans. Amer. Math. Soc. **347** (1995), 4365-4384.

[44] D. Slepian, *The one-sided barrier problem for Gaussian noise.* Bell Systems Technical Journal **41** (1962), 463-501.

[45] E. Slud, *Multiple Wiener-Ito integral expansions for level-crossing-count functionals*, Prob. Th. Rel. Fields **87** (1991) 349-364.

[46] M. Sodin, *Zeros of Gaussian analytic functions*, Math. Res. Lett. **7** (2000), 371-381.

[47] M. Sodin, *Zeroes of Gaussian analytic functions.* Proceedings European Congress of Mathematics (Stockholm, 2004), Eur. Math. Soc., Zürich (2005), 445-458,

[48] M. Sodin and B. Tsirelson, *Random complex zeroes. I. Asymptotic normality.* Israel J. Math. **144** (2004), 125-149.

[49] M. Sodin and B. Tsirelson, *Random complex zeroes. III. Decay of the hole probability.* Israel J. Math. **147** (2005), 371-379.

[50] M. Talagrand, *New Gaussian estimates for enlarged balls*, Geometric and Functional Analysis **3** (1993), 502-526.

[51] B. Tsirelson, *Gaussian Measures* (notes for gradute course in TAU), http://www.tau.ac.il/ tsirel/Courses/Gauss3/main.html.