LONG GAPS BETWEEN SIGN-CHANGES OF GAUSSIAN STATIONARY PROCESSES

NAOMI D. FELDHEIM[1,2] AND OHAD N. FELDHEIM[1,3]

Abstract. We study the probability of a real-valued stationary process to be positive on a large interval $[0, N]$. We show that if in some neighborhood of the origin the spectral measure of the process has density which is bounded away from zero and infinity, then the decay of this probability is bounded between two exponential functions in N. This generalizes similar bounds obtained for particular cases, such as a recent result by Artezana, Buckley, Marzo, Olsen.

1. Introduction

1.1. **Definitions.** Let T be either $\mathbb{Z}$ or $\mathbb{R}$, with the usual topology. A *Gaussian process* (GP) on T is a random function $f : T \to \mathbb{R}$ whose finite marginals, that is $(f(t_1), \dots, f(t_n))$ for any $t_1, \dots, t_n \in T$, have multi-variate Gaussian distribution. A GP on $\mathbb{Z}$ is called a *Gaussian sequence*, while a GP on $\mathbb{R}$ is called a *Gaussian function*. In what follows, we always assume continuity of Gaussian functions.

A GP on T whose distribution is invariant with respect to shifts by any element of T, is called *stationary*. We abbreviate GSP, GSS and GSF for Gaussian stationary processes, sequences and functions respectfully.

For a GSP f on T define the *covariance function* $r : T \to \mathbb{R}$ as

$$r(t) = \mathbb{E}(f(0)f(t)).$$

Observe that due to stationarity, for every $t, s \in T$ we have

$$\mathbb{E}\left[f(s)f(t)\right] = r(t-s).$$

It is not difficult to verify that $r(\cdot)$ is a positive-definite continuous function (see Adler and Taylor [1, Chapter 1]). By Bochner's theorem, there is a

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[1]School of Mathematical Sciences, Tel Aviv University, Tel Aviv, Israel.

[2]E-mail: trinomi@gmail.com. Research supported by the Science Foundation of the Israel Academy of Sciences and Humanities, grant 166/11.

[3]E-mail: ohad_f@netvision.net.il. Research supported by an ERC advanced grant.

finite non-negative measure ρ on T^* such that

$$r(t) = \widehat{\rho}(t) := \int_{T^*} e^{-i\lambda t} d\rho(\lambda).$$

Here T^* is the dual of T, i.e. $\mathbb{Z}^* \simeq [-\pi, \pi]$ and $\mathbb{R}^* \simeq \mathbb{R}$. We use the notation $\mathcal{M}^+(T^*)$ for the set of all finite non-negative measures on T^*. The measure $\rho = \rho_f \in \mathcal{M}^+(T^*)$ is called *the spectral measure* of the process f. Notice that ρ must be symmetric, i.e., for any interval I: $\rho(-I) = \rho(I)$. Any $\rho \in \mathcal{M}^+(T^*)$ uniquely defines a GSP f.

Throughout the paper, we shall assume the following condition:

$$\exists \delta > 0 : \int_{T^*} |\lambda|^\delta d\rho(\lambda) < \infty. \tag{1}$$

This condition is enough to ensure that the associated process f will be continuous (see once again [1, Chapter 1]). Notice that this holds trivially in case $T = \mathbb{Z}$.

1.2. **Results.** Let $f : T \to \mathbb{R}$ be a GSP. Define the "gap probability" of f to be

$$H_f(N) = \mathbb{P}\left(\forall t \in [0, N) \cap T : \; f(t) > 0\right),$$

where $N \in \mathbb{R}$ is a parameter. This describes the probability that no sign-changes of f occurred in a time interval of length N. We study the asymptotics of this probability as $N \to \infty$. It makes no essential difference to regard N as an integer, and we usually do so.

Our main results are the following. Let f be a Gaussian stationary process on $T = \mathbb{Z}$ or $T = \mathbb{R}$, with spectral measure $\rho \in \mathcal{M}^+(T^*)$, satisfying (1).

Theorem 1 (upper bound)**.** *Suppose that there exists $a > 0$ and two positive numbers $M, m > 0$ such that*

$$\text{for any interval } I \subset (-a, a), \; m|I| \le \rho(I) \le M|I|.$$

Then there exists $C = C(a, m, M) > 0$ such that for all large enough N,

$$H_f(N) \le e^{-CN}.$$

Theorem 2 (lower bound)**.** *Suppose that there exists $a > 0$ and a number $m > 0$ such that*

$$\text{for any interval } I \subset (-a, a), \; m|I| \le \rho(I).$$

Then there exists $c = c(a, m) > 0$ such that for all large enough N,

$$H_f(N) \ge e^{-cN}.$$

Remark 1.1. The condition in Theorem 1 may be replaced by the following: There exist two intervals $J_1 = (-a, a)$ and J_2, and two numbers $M, m > 0$, such that

(i) for any interval $I \subset J_1$: $\rho(I) \leq M|I|$, and
(ii) for any interval $I \subset J_2$: $m|I| \leq \rho(I)$.

The necessary changes in the proof are indicated in Section 3.1. However, the authors believe condition (i) might be enough to ensure an upper exponential bound on $H(N)$.

Remark 1.2. Examples for which $H(N)$ tends to zero slower than any exponential in N are known; Newell and Rosenblatt construct one in [10].

Examples for which $H(N)$ tends to zero faster than any exponential in N are also known. A simple example was pointed out to us by M. Krishnapur. Let $(Y_j)_{j\in\mathbb{Z}}$ be a GS with independent entries, and define $X_j = Y_j - Y_{j-1}$ for all $j \in \mathbb{Z}$. Then X is a GSS with $H_X(N) = \frac{1}{N!} \simeq e^{-CN\log N}$, for a suitable constant $C > 0$. Notice that the spectral measure has density $2(1 - \cos(\lambda))$, $\lambda \in [-\pi, \pi]$, which vanishes at $\lambda = 0$.

1.3. **Overview.** The rest of the paper is organized as follows. Section 2 is devoted to discussion of the results. This includes an historical background, and a simple yet useful observation that we shall use (Observation 1 below). The results are then proved independently: Theorem 1 (an upper exponential bound) is proved in Section 3, while Theorem 2 (a lower exponential bound) is proved in Section 4.

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2. Discussion

2.1. **Background.** Gap probability, sometimes referred to by the name "persistence probablity" or "hole probability", was studied extensively in the 1960's, by Slepian [14], Longuet-Higgins [8], Newell-Rosenblatt [10] and others. In addition to proving some bounds and inequlities (such as the well-known "Slepian inequality"), they developed series expansions which approximate this probability quite well for small intervals. In a few examples, exact expressions for the gap probability were calculated (see [14] and references therein).

In the last decade or two, physicists (such as Majumdar-Bray [9] and Ehrhardt-Majumdar-Bray [5]) proposed some new methods of approximation, especially for the long-range regime. Their predictions suggest that in many cases of interest the gap probability $H(N)$ behaves asymptotically like $e^{-\theta N}$, with some $\theta > 0$. A rigorous derivation of such a result is still lacking.

In case the covariance function $r(t)$ is non-negative, Dembo and Mukherjee [4, Theorem 1.6] proved those predictions are correct; namely, that the limit

$$\lim_{N\to\infty} \frac{-\log H_f(N)}{N}$$

exists (possibly infinite). The case when $r(t)$ changes sign, as well as computation of the limit, remain open. We note that the work last mentioned, along with other works by physicists such as Schehr-Majumdar [12], draw connections between gap probabilities of GSPs, those of diffusion processes, and those of zeros of random polynomials.

In this work we are interested in the case where $r(t)$ changes sign. A simple and interesting example is the cardinal sine covariance $r(t) = \frac{\sin(\pi t)}{t}$, which corresponds to indicator spectral density $1\!\!1_{[-\pi,\pi]}$. In an elegant recent work, Antezana, Buckley, Marzo and Olsen [2] give exponential upper and lower bounds for $H_f(N)$ (see Theorem 3 below). Our research may be viewed as a an extension of their result to other *stationary* Gaussian processes. Recently Antezana, Marzo and Olsen were able to generalize this same result in the direction of Gaussian analytic functions over *de-Branges spaces* [3].

Via private communication we learned of results by Krishnapur-Maddaly regarding lower bounds for the gap probability of a SGS. It seems that our conditions for a lower exponential bound are currently stronger, but they have given very mild conditions which ensure $H_f(N) \geq e^{-cN^2}$ (where $c > 0$ is a constant, and the inequality holds for large enough N). Though the results are similar in spirit, their methods seem to be very different from ours.

Lastly we mention an analogous result for the planar Gaussian analytic function

$$\sum_{n\in\mathbb{Z}} a_n \frac{z^n}{\sqrt{n!}}, \text{ where } a_n \sim \mathcal{N}_{\mathbb{C}}(0,1) \text{ are i.i.d.}$$

Bounds concerning hole probabilities for this model were obtained by Sodin and Tsirelson [15], and later refined by Nishry [11]. They showed that the probability of having no zeroes in a ball of radius R in the plane is asymptotically $e^{-(e^2/4+o(1))R^4}$, as $R \to \infty$. For discussion of such results and comparison to other point processes in the plane, see [6, Chapter 7].

2.2. **A Key Observation.** We include here the basic observation which will be used to prove both Theorems 1 and 2. We use the symbol $\oplus$ to indicate the sum of two independent processes or random variables.

Observation 1. *Let f be a GSP on T with spectral measure $\rho \in \mathcal{M}^+(T^*)$, and Suppose $\rho = \rho_1 + \rho_2$, where $\rho_1, \rho_2 \in \mathcal{M}^+(T^*)$. Then the following equality holds in distribution:*

$$f \overset{d}{=} f_1 \oplus f_2,$$

where f_j is a GSP with spectral measure ρ_j ($j = 1, 2$), and f_1 is independent (as a process) from f_2.

Proof. We calculate the covariance function of $f_1 \oplus f_2$ using the independence of the processes:

$$\begin{aligned} \mathbb{E}\Big[\left(f_1(0) + f_2(0)\right)\left(f_1(t) + f_2(t)\right) \Big] &= \mathbb{E} f_1(0) f_1(t) + \mathbb{E} f_2(0) f_2(t) \\ &= \widehat{\rho_1}(t) + \widehat{\rho_2}(t) = \widehat{\rho}(t). \end{aligned}$$

This covariance function is equal to that of f. As all processes are Gaussian, the observation follows. $\square$

3. Upper bound: proof of Theorem 1

This section is devoted to the proof of Theorem 1.

Let f be a GSF or GSS with spectral measure ρ, obeying the conditions of Theorem 1. Let $k \in \mathbb{N}$ be such that $\frac{\pi}{k} \le a$, and denote $J := [-\pi/k, \pi/k] \subset [-a, a]$. We decompose the spectral measure as follows:

$$d\rho(\lambda) = m 1\!\!1_J(\lambda) d\lambda + d\mu(\lambda),$$

where $\mu \in \mathcal{M}^+(T^*)$ is non-negative and there exists $M' > 0$ such that

$$\text{for any interval } I \subset (-a, a) : \ \mu(I) \le M'|I|. \tag{2}$$

By Observation 1, we may represent

$$f \overset{d}{=} S \oplus g$$

where S and g are independent processes, with spectral measures $m 1\!\!1_J(\lambda)$ and μ respectively.

Next, we observe that sampling S in a certain lattice results in independent random variables:

Observation 2 (indicator spectrum)**.** *The GSP $(S(t))_{t \in T}$ having spectral density $m 1\!\!1_{[-\pi/k, \pi/k]}$ has the property that $(S(jk))_{j \in \mathbb{Z}}$ are i.i.d. Gaussian random variables.*

Proof. By taking the Fourier transform of the given measure, the covariance function of S is

$$\mathbb{E}\left[S(s)S(t)\right] = \frac{\sin(\pi(t-s)/k)}{(t-s)}.$$

Thus $S(jk)$ and $S(mk)$ are uncorrelated for any $j, m \in \mathbb{Z}$, $j \neq m$; as these are Gaussian random variables - independence follows. $\square$

In order to apply Observation 2, we look at a certain translated lattice $\{jk + l : j \in \mathbb{Z}\}$ on which S is indeed independent. The translation (which we call "split") of the sampled lattice will depend on g.

More precisely, fix a number $q > 0$ (say, $q = 1$), and define an event E depending only on the process $(g(t))_{t \in T}$ in the following way:

$$E = \left\{ \frac{1}{N} \sum_{t=1}^{N} g(t) < q \right\}, \text{ if } g \text{ is GSS}$$

$$E = \left\{ \frac{1}{N} \int_0^N g(t)dt < q \right\}, \text{ if } g \text{ is GSF}$$

Using the law of total probability we have:

$$\begin{aligned} &\mathbb{P}\left(f(t) = S(t) + g(t) > 0,\ 0 < t \leq N\right) \\ &\quad \leq \mathbb{P}\left(S(t) + g(t) > 0,\ 0 < t \leq N \;\middle|\; E\right) + \mathbb{P}\left(E^c\right). \end{aligned}$$

It is enough to show that there exist $C_1, C_2 > 0$ such that for large enough N,

(I) $\mathbb{P}\left(S(t) + g(t) > 0,\ 0 < t \leq N \;\middle|\; E\right) \leq e^{-C_1 N}$, and

(II) $\mathbb{P}\left(E^c\right) \leq e^{-C_2 N}$.

We proceed the proof for the function-case, noting the sequence-case follows similar lines and is generally easier.

We begin by showing (I). It is enough to show that there is $C_1 > 0$ such that for any large enough N and any fixed $g \in E$,

$$\mathbb{P}\left(S(t) + g(t) > 0,\ 0 < t \leq N\right) \leq e^{-C_1 N}.$$

Indeed, this would imply (using the independence of g and S):

$$\begin{aligned} &\mathbb{P}\left(S(t) + g(t) > 0,\ 0 < t \leq N \,|\, E\right) \\ &\quad = \mathbb{E}\left(\mathbb{P}\left(S(t) + g(t) > 0,\ 0 < t \leq N\right) \;\middle|\; E\right) \leq e^{-C_1 N}, \end{aligned}$$

as required.

To that end, we use a property which holds when the event E occurs, stated below.

Observation 3. *Let g be a continuous function such that $\frac{1}{N}\int_0^N g(t)dt < q$, and assume $N \in \mathbb{N}$ is divisible by k, then there exists a number $l \in [0,k)$ such that*

$$\frac{k}{N}\sum_{j=0}^{N/k-1} g(jk+l) < q.$$

Proof. Else, for every $l \in [0,k)$ the reverse inequality holds. Integrating it over $l \in [0,k]$ yields a contradiction. □

Now, fix a function $g \in E$. We can find a special split l_g whose existence is guaranteed by Observation 3. Therefore:

$$\begin{aligned}&\mathbb{P}\left(S(t)+g(t)>0,\ 0<t\le N\right)\\ &\le \mathbb{P}\left(S(jk+l_g)+g(jk+l_g)>0,\ j=0,1,\dots,N/k-1\right),\end{aligned}$$

where $(S(jk+l))_{j\in\mathbb{Z}}$ are i.i.d Gaussians (whose variance is independent of l_g), and $\frac{k}{N}\sum_{j=0}^{N/k-1} g(jk+l_g) < q$. The following inequality will give the desired bound.

Proposition 3.1. *Let $X_1,\dots,X_N$ be i.i.d real centered Gaussian random variables, and let $q \in \mathbb{R}$. There is a constant $C_q > 0$ such that for any numbers $b_1,\dots,b_N \in \mathbb{R}$ which obey $\frac{1}{N}\sum_{j=1}^N b_j < q$, the following holds:*

$$\mathbb{P}\left(X_j+b_j>0,\ 1\le j\le N\right) \le e^{-C_q N}.$$

Proof. Without loss of generality assume $\mathrm{var}(X_1)=1$. Denote by $\Phi(b) = \mathbb{P}(X_1 < b)$ the cumulative distribution function of X_1. By symmetry, $\Phi(b) = \mathbb{P}(X_1 > -b)$. Using the "i.i.d" property of the variables $\{X_j\}_{j=1}^N$ we have:

$$p = \mathbb{P}\left(X_j+b_j>0,\ 1\le j\le N\right) = \prod_{j=1}^N \mathbb{P}\left(X_j > -b_j\right) = \prod_{j=1}^N \Phi(b_j).$$

Taking logarithm and using the concavity and monotonicity of $x \mapsto \log\Phi(x)$, we get:

$$\log p = \sum_{j=1}^N \log\Phi(b_j) \le N\cdot\log\Phi\left(\frac{\sum_j^N b_j}{N}\right) < N\cdot\log\Phi(q),$$

and so $C_q = -\log\Phi(q) > 0$ is the desired constant. □

In order to prove (II), we shall use the following:

Proposition 3.2. $\frac{1}{N}\int_0^N g(t)dt \sim \mathcal{N}_{\mathbb{R}}(0,\sigma_N^2)$, *where $\sigma_N^2 \le \frac{C_0}{N}$ for all $N \in \mathbb{N}$ and some constant $C_0 > 0$.*

Proof. The normality of the given integral follows from general arguments of convergence of Gaussian random variables. We focus on the bound on its variance. Recall that μ denoted the spectral measure of g. We calculate the variance:

$$\begin{aligned}\sigma_N^2 &= \frac{1}{N^2}\mathbb{E}\left(\int_0^N g(t)dt\right)^2 = \frac{1}{N^2}\iint_{[0,N]^2}\mathbb{E}(g(t)g(s))dt\ ds\\ &= \frac{1}{N^2}\int_0^N\int_0^N \widehat{\mu}(t-s)dt\ ds = \frac{1}{N}\int_{|t|<N}\left(1-\frac{|t|}{N}\right)\widehat{\mu}(t)dt.\end{aligned}$$

The change in order of integration and expectancy in the first equality is easily justified by use of Fubini's theorem.

The inverse Fourier transform of $(1-\frac{|t|}{N})1\!\!1_{[-N,N]}(t)$ is given by

$$K_N(\lambda) = N\left(\frac{\sin(N\lambda/2)}{N\lambda/2}\right)^2 \le \min\left(N, \frac{\pi^2}{N\lambda^2}\right)$$

Using first Plancherel's identity, and then condition (2) on the boundness of μ, we get:

$$\begin{aligned}\sigma_N^2 &= \frac{1}{N}\int_{\mathbb{R}} K_N(\lambda)d\mu(\lambda)\\ &\le \int_{|\lambda|<\frac{\pi}{N}} d\mu(\lambda) + \frac{\pi^2}{N^2}\left(\int_{\frac{\pi}{N}\le|\lambda|<a} + \int_{|\lambda|\ge a}\right)\frac{1}{\lambda^2}d\mu(\lambda)\\ &\le M'\cdot\frac{2\pi}{N} + \frac{\pi^2}{N^2}\left(M'\int_{\frac{\pi}{N}\le|\lambda|<a}\frac{d\lambda}{\lambda^2} + \frac{1}{a^2}\mu(\{|\lambda|>a\})\right)\\ &\le \frac{C_0}{N},\end{aligned}$$

where C_0 is a constant (depending on μ). □

At last, we prove (II). Denote by γ a standard Gaussian random variable (i.e., distributed $\mathcal{N}(0,1)$). Using the Proposition 3.2 together with the well-known inequality

$$\forall y>0:\ \mathbb{P}(\gamma>y) < \frac{1}{\sqrt{2\pi}y}e^{-y^2/2},$$

we get:

$$\begin{aligned}\mathbb{P}(E^c) = \mathbb{P}\left(\frac{1}{N}\int_0^N g(t)\ge q\right) &= \mathbb{P}(\sigma_N\cdot\gamma\ge q) = \mathbb{P}\left(\gamma\ge\frac{q}{\sigma_N}\right)\\ &\le \frac{1}{\sqrt{2\pi}}\cdot\frac{\sigma_N}{q}e^{-\frac{1}{2}\cdot\frac{q^2}{\sigma_N^2}}\\ &\le \frac{1}{q}\sqrt{\frac{C_0}{2\pi N}}e^{-\frac{q^2}{2C_0}N} \le e^{-C_2N},\end{aligned}$$

for a suitable choice of $C_2 > 0$ (depending only on q and μ). Theorem 1 is proved. $\square$

3.1. **Extension: Proof of Remark 1.1.** Remark 1.1 states a somewhat more general condition under which the conclusion of Theorem 1 is true. The proof is only a slight modification of the one presented. First, choose $l, k \in \mathbb{N}$ so that

$$J := \left[\frac{(2l-1)\pi}{k}, \frac{2l\pi}{k}\right] \subset J_2 \cup (-J_2).$$

Now decompose the measure as follows:

$$d\rho(\lambda) = m 1\!\!1_{J\cup -J}(\lambda) d\lambda + d\mu(\lambda).$$

By the premise, $\mu \in \mathcal{M}^+(T^*)$ obeys the boundedness condition (2) (just as before). Applying Observation 1 we get

$$f \overset{d}{=} S \oplus g,$$

where S has spectral measure $m 1\!\!1_{J\cup -J}(\lambda) d\lambda$ and g has spectral measure μ. We define E and strive to prove items (I) and (II). Item (II) follows from Proposition 3.2 and the calculation following it with no change. The only property used in order to prove item (I) is the independence of $(S(jk))_{j\in\mathbb{Z}}$ (i.e., Observation 2). Let us show this still holds.

One way to end the argument is by calculation of the Fourier transform of $1\!\!1_{J\cup -J}(\lambda)d\lambda$ and observing it vanishes at kj, $j \in \mathbb{Z}$ (just as in the proof of Observation 2). We give here a more general argument, relying on two observations:

Observation 4. *Let $(f(t))_{t\in\mathbb{R}}$ be a GSF with spectral measure ρ, and $\alpha > 0$. Then the GSF $x \mapsto f(\alpha x)$ has spectral measure ρ_α, defined by*

$$\forall I \subset \mathbb{R}: \quad \rho_\alpha(I) = \rho(\{x \in \mathbb{R}: \ \alpha x \in I\})$$

Proof. $\mathbb{E}\left[f(\alpha t) f(\alpha s)\right] = \widehat{\rho}(\alpha(t-s)) = \widehat{\rho_\alpha}(t-s)$. $\square$

Observation 5. *If $(f(t))_{t\in\mathbb{R}}$ is a GSF with spectral measure ρ, then sampling the lattice $(f(j))_{j\in\mathbb{Z}}$ has the folded spectral measure $\rho^* \in \mathcal{M}^+([-\pi,\pi])$ obtained by: $\rho^*(I) = \sum_{m\in\mathbb{Z}} \rho(I + 2\pi m)$.*

Proof. ρ^* is the unique measure in $\mathcal{M}^+([-\pi,\pi])$ such that $\widehat{\rho^*}(j) = \widehat{\rho}(j)$ for any $j \in \mathbb{Z}$. $\square$

Combining the last two observations, we get that if $(S(t))_{t\in T}$ has spectral density $m 1\!\!1_{J\cup -J}(\cdot)$, then the spectral density of $(S(kj))_{j\in\mathbb{Z}}$ is $m 1\!\!1_{[-\pi,\pi]}(\cdot)$. Now Observation 2 leads to the desired conclusion.

4. Lower bound: Proof of Theorem 2

4.1. **Reducing GSS to GSF.** Theorem 2 is easily reduced to the case of functions, by noticing the following:

Observation 6. *Any finite measure $\rho \in \mathcal{M}^+([-\pi,\pi])$ generates a GSF f and a GSS X. The distribution of $(X(j))_{j\in\mathbb{Z}}$ is the same as that of $(f(j))_{j\in\mathbb{Z}}$ (since their covariance functions coincide). Moreover, for any number N:*

$$\begin{aligned} H_f(N) &= \mathbb{P}(f(x) > 0,\ x \in [0,N)\cap\mathbb{R}) \\ &\le \mathbb{P}(f(j) > 0,\ j \in [0,N)\cap\mathbb{N}) = H_X(N). \end{aligned}$$

Therefore, in order to bound $H_X(N)$ from below where X is a GSS, it is enough to bound $H_f(N)$ from below where f is the GSF with the same spectral measure as X.

4.2. **Proof for GSF.** Let $(f(t))_{t\in\mathbb{R}}$ be a GSF with spectral measure ρ, obeying the condition of Theorem 2. By scaling f (and therefore scaling its spectral measure according to Observation 4, we may assume the condition is satisfied with $a=\pi$.

Just as in the proof of Theorem 1, we decompose the spectral measure in the following manner:

$$d\rho = m1\!\!1_{[-\pi,\pi]}(\lambda)d\lambda + d\mu.$$

Applying Observation 1 we have

$$f \overset{d}{=} S \oplus g$$

where S and g are independent processes, and the spectral measure of S has density $m1\!\!1_{[-\pi,\pi]}(\lambda)$.

We have:

$$\begin{aligned} H_f(N) &= \mathbb{P}\left(S(x)+g(x) > 0,\ 0 \le x < N\right) \\ &\ge \mathbb{P}\left(S(x) > d,\ 0\le x<N\right)\mathbb{P}\left(|g(x)| \le \frac{d}{2},\ 0 \le x < N\right), \end{aligned} \tag{3}$$

where $d>0$ is a parameter of our choice. The first probability is bounded from below by the following theorem:

Theorem 3 (Antezana, Buckley, Marzo, Olsen [2]). *Let $S(x)$ be the GSF with spectral measure $d\rho(\lambda) = 1\!\!1_{[-\pi,\pi]}(\lambda)d\lambda$. Then for any $d>0$ there exists a constant $c_d>0$, such that for all $N\in\mathbb{N}$,*

$$\mathbb{P}\left(S(x) > d,\ 0 \le x < N\right) \ge e^{-c_d N}.$$

We turn to bound the second probability in (3), i.e., the probability of the event $\{|g(x)| \le \varepsilon,\ 0 \le x < N\}$. This is known in literature as a "small ball probability", and is bounded from below by the following result:

Lemma 4.1 (Talagrand [16], Shao and Wang [13])**.** *Let $(f(t))_{t\in I}$ be a centered Gaussian process on a finite interval I. Suppose that for some $c > 0$ and $0 < \delta \le 2$,*

$$d_f(s,t)^2 := \mathbb{E}|f(s) - f(t)|^2 \le c|t-s|^\delta,\ \ s,t \in I.$$

Then, for some $K > 0$ and every $\varepsilon > 0$,

$$\mathbb{P}\left(\sup_{t\in I} |f(t)| \le \varepsilon\right) \ge \exp\left(-\frac{K|I|}{\varepsilon^{2/\delta}}\right).$$

The proof of Lemma 4.1, apart from being deduced from a much more general result in Talagrand's paper, may be found in notes by Ledoux [7, Ch. 7] (but in a slightly different version). Shao and Wang decided to omit a proof from their paper as they learned that Talagrand's result generalizes theirs; but they do include the most close formulation to the one above.

We draw the following corollary:

Corollary 4.1. *Let f be a Gaussian stationary function on $\mathbb{R}$ with spectral measure ρ, obeying the moment condition* (1)*. Then for all $\varepsilon > 0$ there exists $C, K > 0$ such that for any interval I and any $N \in \mathbb{N}$:*

$$\mathbb{P}\left(\sup_I |f| < \varepsilon\right) \ge Ce^{-K|I|}.$$

Applying the corollary to $f = g$, $I = [0, N)$ and $\varepsilon = \frac{d}{2} > 0$, will give the desired bound on the second factor in (3), thus ending the proof of Theorem 2.

Proof of Corollary 4.1. First we notice that if the moment condition (1) is satisfied with a certain exponent $\delta > 0$, then it is also satisfied by any smaller positive exponent. Therefore we may assume $0 < \delta < 2$.

We shall check that f obeys the condition of Lemma 4.1 with this same δ, i.e. that there exists a constant $c > 0$ such that

$$d_f(s,t)^2 \le c|t-s|^\delta,\ \ s,t \in I.$$

Indeed:

$$\begin{aligned} d_f(s,t)^2 &= \mathbb{E}(f(s) - f(t))^2 = 2(r(0) - r(s-t)) \\ &= 2\int_{\mathbb{R}} \Big(1 - \cos(\lambda(s-t))\Big) d\rho(\lambda) \le 2L|t-s|^\delta \int_{\mathbb{R}} |\lambda|^\delta d\rho(\lambda), \end{aligned}$$

where $L = \sup_{x\in\mathbb{R}} \frac{1-\cos(x)}{|x|^\delta} < \infty$. The Corollary follows. $\square$

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