Persistence of Gaussian Stationary Processes: a spectral perspective

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Joint work with

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For $T \in \{\mathbb{R}, \mathbb{Z}\}$, a random function $f : T \mapsto \mathbb{R}$ is a GSP if it is

- Gaussian: $(f(x_1), \ldots f(x_N)) \sim \mathcal{N}_{\mathbb{R}^N}(0, \Sigma_{x_1, \ldots, x_N})$,
- Stationary (shift-invariant): $(f(x_1 + s), \ldots f(x_N + s)) \overset{d}{\sim} (f(x_1), \ldots f(x_N))$,

for all $N \in \mathbb{N}$, $x_1, \ldots, x_N, s \in T$.

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Motivation: nearly any stationary noise.

- Field fluctuations
- Electromagnetic noise
- Ocean waves
- Vibrations of strings / membranes
- Data traffic
- ...

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Covariance function

$$r(s,t) = \mathbb{E}(f(s)f(t)) = r(s-t) \qquad t, s \in T.$$

Assumption: $r(\cdot)$ and $f(\cdot)$ continuous.

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Spectral measure

r continuous and positive-definite $\Rightarrow$ there exists a finite, non-negative, symmetric measure ρ over T^* ($\mathbb{Z}^* \simeq [-\pi, \pi]$ and $\mathbb{R}^* \simeq \mathbb{R}$) s.t.

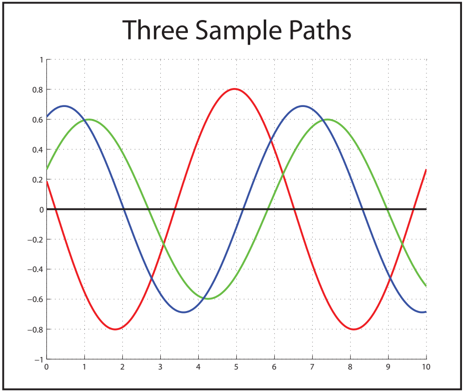
$$r(t) = \widehat{\rho}(t) = \int_{T^*} e^{-i\lambda t} d\rho(\lambda).$$

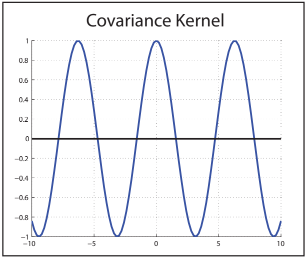
ξ_j i.i.d. $\mathcal{N}(0,1)$

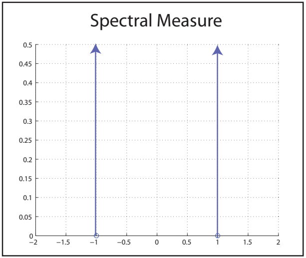
$$f(x) = \xi_0 \sin(x) + \xi_1 \cos(x)$$

$$r(x) = \cos(x)$$

$$\rho = \frac{1}{2}(\delta_1 + \delta_{-1})$$

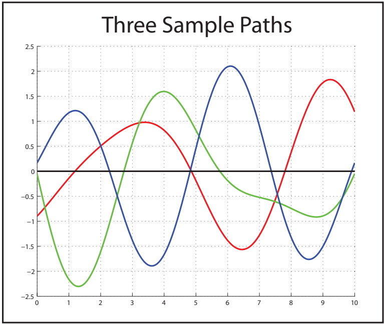


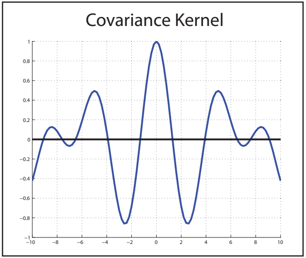


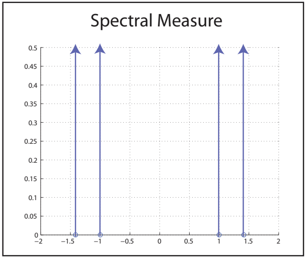


Toy-Example Ib - Almost periodic wave

$$
\begin{aligned}
f(x) = {} & \xi_0 \sin(x) + \xi_1 \cos(x) \\
& + \xi_2 \sin(\sqrt{2}x) + \xi_3 \cos(\sqrt{2}x) \\
r(x) = {} & \cos(x) + \cos(\sqrt{2}x) \\
\rho = {} & \tfrac{1}{2}\left(\delta_1 + \delta_{-1} + \delta_{\sqrt{2}} + \delta_{-\sqrt{2}}\right)
\end{aligned}
$$





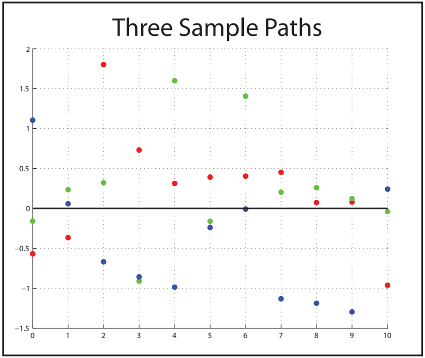


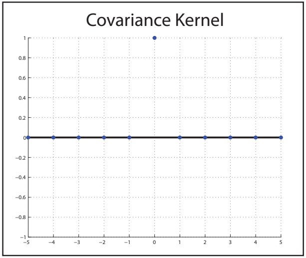
Example II - i.i.d. sequence

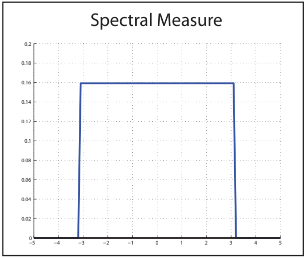
$$f(n) = \xi_n$$

$$r(n) = \delta_{n,0}$$

$$d\rho(\lambda) = \frac{1}{2\pi}\mathbb{1}_{[-\pi,\pi]}(\lambda)d\lambda$$





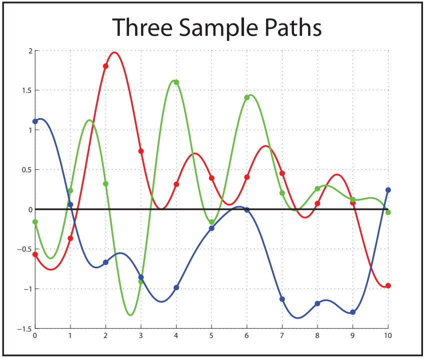


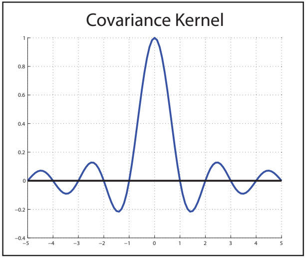
Example IIb - Sinc kernel

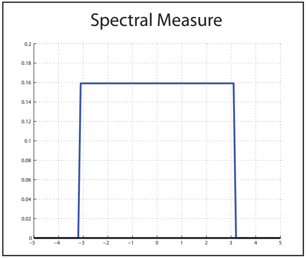
$$f(x) = \sum_{n \in \mathbb{N}} \xi_n \,\text{sinc}(x - n)$$

$$r(x) = \frac{\sin(\pi x)}{\pi x} = \text{sinc}(x)$$

$$d\rho(\lambda) = \frac{1}{2\pi} \mathbb{1}_{[-\pi,\pi]}(\lambda) d\lambda$$





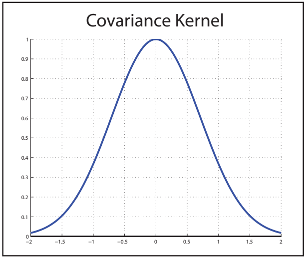


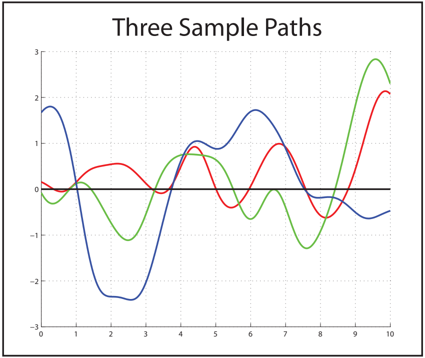
Example III - Gaussian Covariance (Fock-Bargmann)

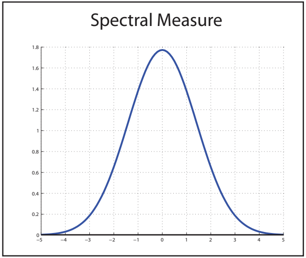
$$f(x) = \sum_{n \in \mathbb{N}} \xi_n \frac{x^n}{\sqrt{n!}} e^{-\frac{x^2}{2}}$$

$$r(x) = e^{-\frac{x^2}{2}}$$

$$d\rho(\lambda) = \sqrt{\pi} e^{-\frac{\lambda^2}{2}} d\lambda$$



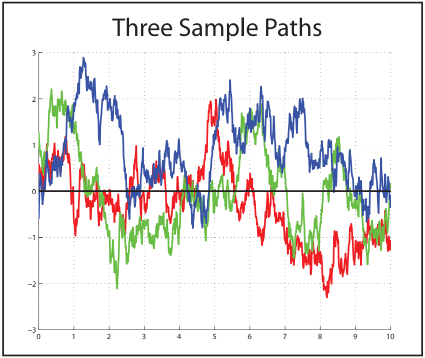


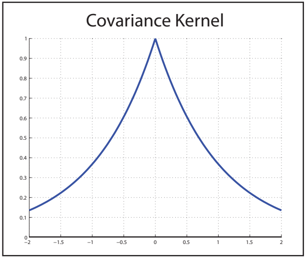


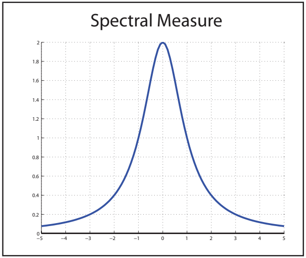
Example IV - Exponential Covariance (Ornstein-Uhlenbeck)

$$r(x) = e^{-|x|}$$

$$d\rho(\lambda) = \frac{2}{\lambda^2 + 1} d\lambda$$



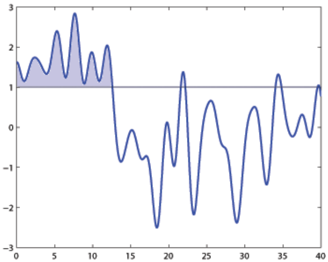




Persistence

The **persistence probability** of a stochastic process f over a level $\ell \in \mathbb{R}$ in the time interval $(0, N]$ is:F

$$P_f(N) := \mathbb{P}\Big(f(x) > \ell, \forall x \in (0, N]\Big).$$



Persistence (above the mean)

The **persistence probability** of a **centered** stochastic process f in the time interval $(0, N]$ is:F

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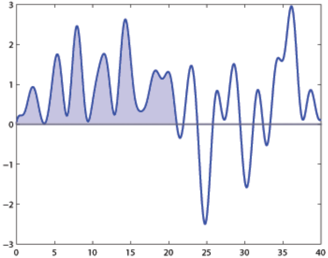

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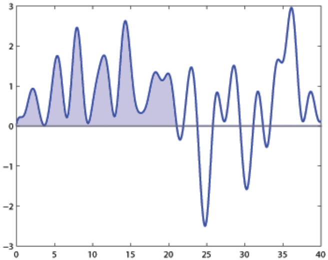
Question: For a GSP f, what is the behavior of $P_f(N)$ as $N \to \infty$?



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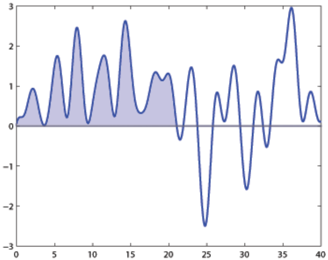

Motivation: detection theory.

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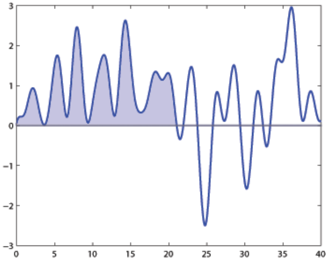

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X_n i.i.d.	$P_X(N) = 2^{-N}$
$Y_n = X_{n+1} - X_n$	$P_Y(N) = \frac{1}{(N+1)!} \asymp e^{-N \log N}$
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Engineering and Applied Mathematics (1940–1970)

- 1944 Rice - "Mathematical Analysis of Random Noise".
  - Mean number of level-crossings (Rice formula)
  - Behavior of $P(t)$ for $t \ll 1$ (short range).
- 1962 Slepian - "One-sided barrier problem".
  - Slepian's Inequality: $r_1(x) \geq r_2(x) \Rightarrow P_1(N) \geq P_2(N)$.
  - specific cases
- 1962 Newell & Rosenblatt
  - If $r(x) \to 0$ as $x \to \infty$, then $P(N) = o(N^{-\alpha})$ for any $\alpha > 0$.
  - If $|r(x)| < ax^{-\alpha}$ then $\log P(N) \leq \begin{cases} -CN & \text{if } \alpha > 1 \\ -CN/\log N & \text{if } \alpha = 1 \\ -CN^{\alpha} & \text{if } 0 < \alpha < 1 \end{cases}$
  - examples for $\log P(N) > -C\sqrt{N}\log N \gg -CN$ $(r(x) \asymp x^{-1/2})$.

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Applicable mainly when r is non-negative or absolutely summable.

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Physics (1990–2010)

- GSPs used in models for electrons in matter, diffusion, spin systems.
- Majumdar et al.: Heuristics explaining why $\log P_f(N) \asymp -\theta N$ "generically".

Probability and Anlysis (2000+)

- hole probability for Gaussian analytic functions
  - in the plane (Sodin-Tsirelson, Nishry), hyperbolic disc (Buckley et al.)
  - for sinc-kernel: $e^{-cN} < P(N) < 2^{-N}$ (Antezana-Buckley-Marzo-Olsen, '12)

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- Absence of zeroes in random polynomials,
  relations with non-stationary diffusion processes
  (Dembo-Mukherjee 2013, 2015)

Spectral perspective

Both the covariance and the spectral measure are linear under sums of independent GSPs.

Observation

If $f = f_1 \oplus f_2$ where f_1, f_2 are independent GSPs, then

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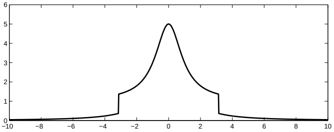

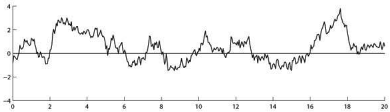

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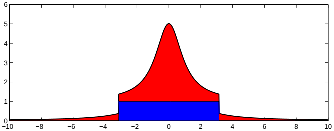

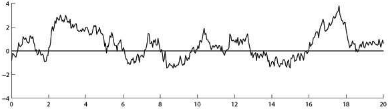

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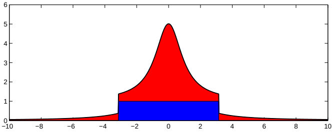

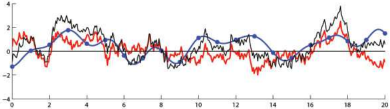

First Spectral Result

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Theorem 1 (Feldheim & F., 2013)

Suppose that ρ has density $w(\lambda)$ in $[-a, a]$ with $0 < m \leq w(x) \leq M$.
Then $\log P_f(N) \asymp -N$, that is, for some c_1, c_2,

$$-c_1 N \leq \log P_f(N) \leq -c_2 N.$$

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$\rho_{AC} \neq 0$ ("non-trivial absolutely continuous component").

Theorem 1 (Feldheim & F., 2013)

Suppose that ρ has density $w(\lambda)$ in $[-a, a]$ with $0 < m \le w(x) \le M$.
Then $\log P_f(N) \asymp -N$, that is, for some c_1, c_2,

$$-c_1 N \le \log P_f(N) \le -c_2 N.$$

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- Given in terms of ρ (not r).
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Toy Examples

$$X_n \text{ i.i.d. } \Rightarrow P_X(N) = 2^{-N} \qquad w = \mathbb{1}_{[-\pi,\pi]}$$

$$Y_n = X_{n+1} - X_n \Rightarrow P_Y(N) \asymp e^{-N \log N} \qquad w = 2(1 - \cos\lambda)\mathbb{1}_{[-\pi,\pi]}$$

$$Z_n \equiv Z_0 \Rightarrow P_Z(N) = \frac{1}{2} \qquad \rho = \delta_0$$

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New questions and conjectures

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Recent progress:

- Non-negative correlations – Dembo & Mukherjee (2013, 2015):
  - Improve upon the methods of Newell & Rosenblatt.
  - In case $r(x) \geq 0$, pinpoint the behaviour of $\log P(N)$ up to a constant.
- Lower bounds for GSP on $\mathbb{Z}$ – Krishna & Krishnapur (2016):
  - $\log P(N) \geq -CN^2$ (assuming $\rho_{AC} \neq 0$)
  - $\log P(N) \geq -CN \log N$ if in some interval around 0 the spectral measure has density $w(\lambda)$ with $w(\lambda) \geq c\lambda^{\alpha}$ $(c > 0)$.

Persistence is largely determined by the spectral behaviour near 0.

Theorem 2 (Feldheim, F., Nitzan, 2017)

Suppose that in $[-a, a]$ the spectral measure has density $w(\lambda)$ which satisfies $c_1\lambda^\alpha \le w(\lambda) \le c_2\lambda^\alpha$ for some $\alpha > -1$. Then:

$$\log P_f(N) \begin{cases} \asymp -N^{1+\alpha}\log N, & -1<\alpha<0 \quad \text{(exploding spec. at 0)} \\ \asymp -N, & \alpha = 0 \quad \text{(bounded spec. at 0)} \\ \lesssim -\alpha N \log N, & \alpha > 0 \quad \text{(vanishing spec. at 0)}. \end{cases}$$

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Implications:

- generalize Dembo-Mukherjee to sign-changing covariance.
- obtain upper bound in the spectral gap conjecture.
- the first example of $\log P(N) \le -CN \log N$.
- with Krishna-Krishnapur: pinpoints $\log P_f(N)$ up to a constant over $\mathbb{Z}$.

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Furthermore:

- $w(\lambda) \le c_2\lambda^\alpha \Rightarrow$ upper bounds, $w(\lambda) \ge c_1\lambda^\alpha \Rightarrow$ lower bounds.
- Formulated using $\rho([0,\lambda])$ for $\lambda \ll 1$.

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Question: What about lower bound over $\mathbb{R}$ when $\alpha > 0$?

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Theorem 3 (Feldheim, F., Nitzan, 2017)

Let $T = \mathbb{R}$. If the spectral measure vanishes on an interval containing 0, and on $[1, \infty)$ it has density $w(\lambda)$ such that $w(\lambda) \geq \lambda^{-100}$, then

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- heavy tail $\Rightarrow$ f is "rough" $\Rightarrow$ tiny persistence.
- light tail $\Rightarrow$ f is smooth $\Rightarrow$ matching lower bounds as over $\mathbb{Z}$ **[in progress]**

Ideas from the proof.

Main tool: spectral decomposition

Key Observation

$$\rho = \rho_1 + \rho_2 \quad \Rightarrow \quad f \stackrel{d}{=} f_1 \oplus f_2$$

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Application:

$$\rho = m\mathbb{1}\left[-\frac{\pi}{k}, \frac{\pi}{k}\right] + \mu \Rightarrow f = S \oplus g,$$

where $r_S(x) = c\ \mathrm{sinc}(\frac{x}{k})$. That is, $(S(nk))_{n\in\mathbb{Z}}$ are i.i.d.

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Proof: $\mathbb{E}[S(nk)S(mk)] = r_S((m-n)k) = 0$.

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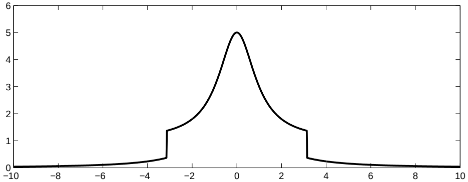

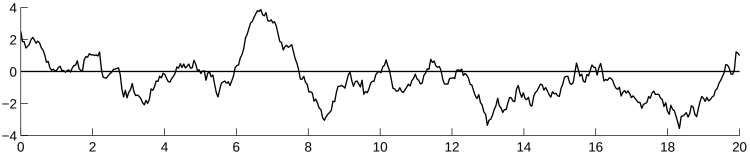

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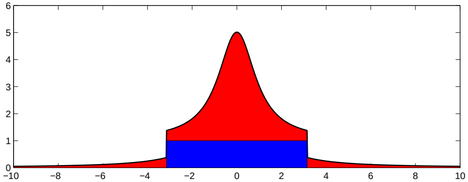

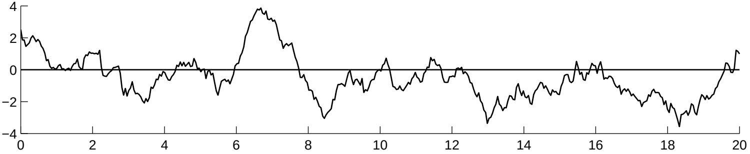

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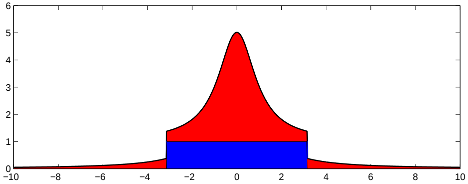

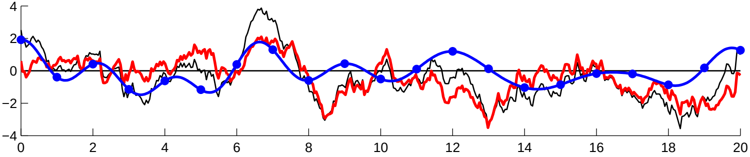

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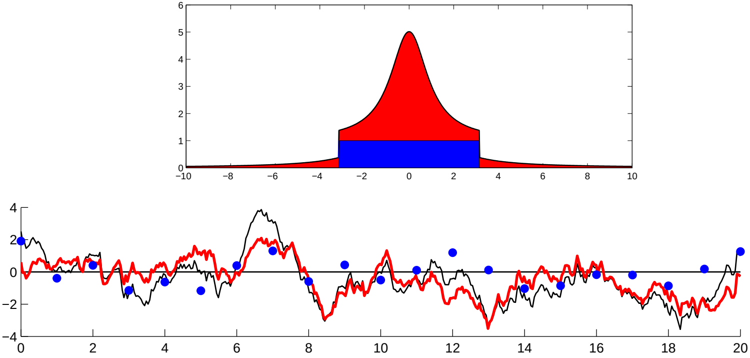

Bounded spectrum: upper bound

Lemma 1 - average of a GSP

$\frac{1}{N}\sum_{n=1}^{N} g(n) \sim \mathcal{N}_{\mathbb{R}}(0, \sigma_N^2)$, where $\sigma_N^2 \leq \frac{C}{N}$.

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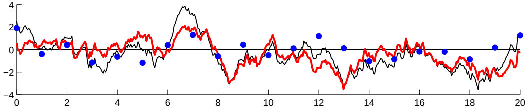

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$$\sigma_N^2 = \mathbb{E}\left(\frac{1}{N}\sum_{1}^{N} f(n)\right)^2 = \frac{1}{N^2}\sum_{1}^{N}\sum_{1}^{N} r(n-k) = \frac{1}{N}\sum_{|j|<N}\left(1-\frac{|j|}{N}\right) r(j)$$

$$= \int_{-\pi}^{\pi} \frac{\sin^2(N\frac{\lambda}{2})}{N^2\sin^2(\frac{\lambda}{2})} d\rho(\lambda) \approx \rho([0,\tfrac{1}{N}]).$$

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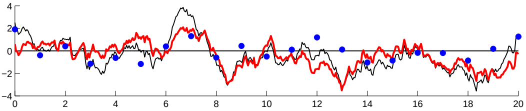

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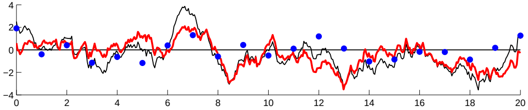

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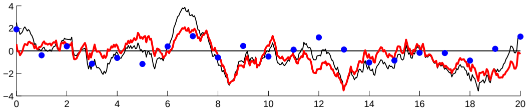

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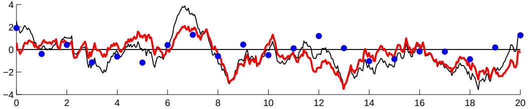

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If $Z_1, \ldots, Z_N \sim \mathcal{N}(0,1)$ i.i.d. and $\frac{1}{N}\sum_{j=1}^{N} b_j < 1$, then

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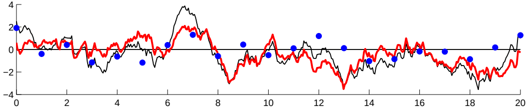

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Proof:

$$\log \mathbb{P}(Z_j > -b_j,\, 1 \le j \le N) = \log \prod_{j=1}^{N} \Phi(b_j) = \sum_{j=1}^{N} \log \Phi(b_j) \overset{\text{concav.}}{\le} N \log \Phi\left(\frac{1}{N}\sum b_j\right) \le N \log \Phi(1).$$

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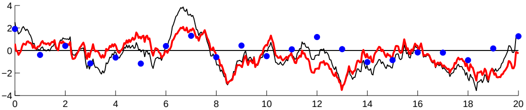

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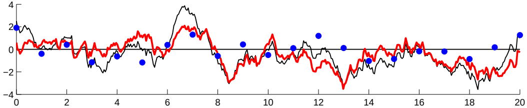

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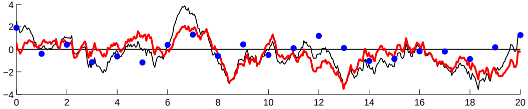

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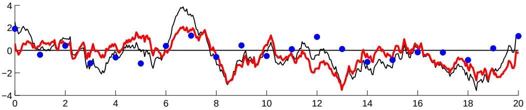

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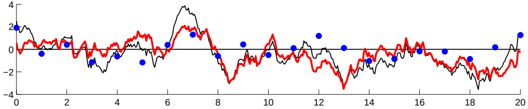

Exploding spectrum: upper bound

For any $\ell = \ell(N,\rho) > 0$,

$$P_f(N) \leq \mathbb{P}\left(S \oplus g > 0 \text{ on } (0,N] \,\middle|\, \frac{1}{N}\sum_{n=1}^{N} g(n) < \ell\right) + \mathbb{P}\left(\frac{1}{N}\sum_{n=1}^{N} g(n) \geq \ell\right)$$

Lemma 1' - average of a GSP.

$\mathrm{var}\left(\frac{1}{N}\sum_{n=1}^{N} g(n)\right) \lesssim \sigma_N^2 := \rho([0,\frac{1}{N}])$. Therefore,
$\mathbb{P}(\frac{1}{N}\sum_{n=1}^{N} g(n) \geq \ell) \leq \mathbb{P}(\sigma_N Z > \ell)$.

Lemma 2' - persistence of distorted i.i.d.

If $Z_1,\dots,Z_N \sim \mathcal{N}(0,1)$ i.i.d. and $\frac{1}{N}\sum_{j=1}^{N} b_j < \ell$, then

$$\mathbb{P}\left(Z_j + b_j > 0,\, 1 \leq j \leq N\right) \leq \mathbb{P}(Z_1 < \ell)^N.$$

Balancing equation:

$$\mathbb{P}(Z < \ell)^N \asymp \mathbb{P}(\sigma_N Z > \ell).$$

Bounded or exploding spectrum: lower bound

A new decomposition (depends on N):

$f = A \oplus h$, where A is GSP with $\rho_A = \rho|_{[-\frac{1}{N}, \frac{1}{N}]}$ **("atom-like" part)**

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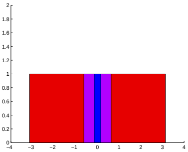

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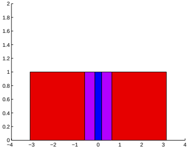

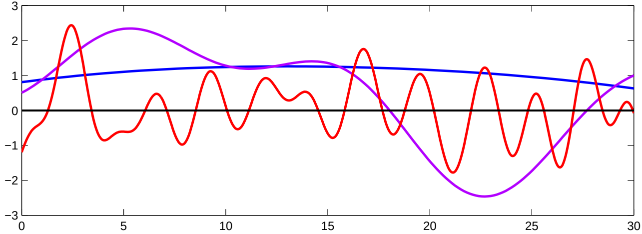

Bounded or exploding spectrum: lower bound

Lemma 3 - "atom-like" behaviour

$$\mathbb{P}(A > \ell \text{ on } (0, N]) \geq \tfrac{1}{2}\mathbb{P}(\sigma_N Z > 2\ell)$$

for every $\ell > 0$, where $Z \sim \mathcal{N}(0,1)$ and $\sigma_N^2 = \rho([-1/N, 1/N])$.

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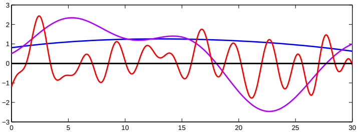

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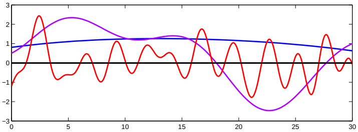

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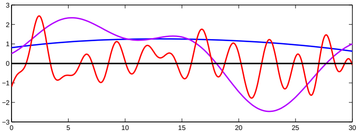

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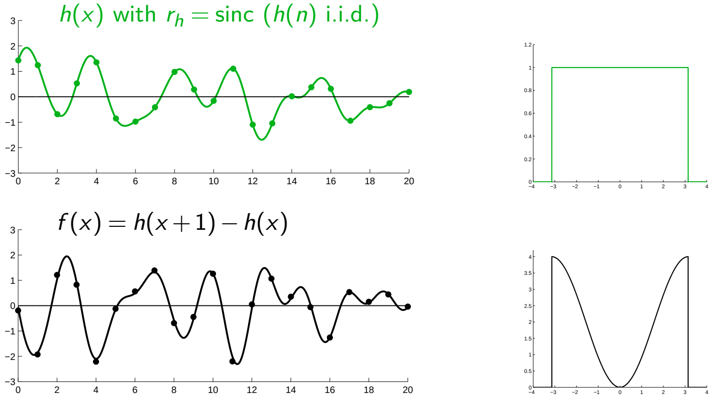
Vanishing spectrum: intuition

A GSP with vanishing spectrum is the derivative (or difference) of another GSP.

Sample path **Spectral measure**



Spectral decomposition

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Process integration

If $\int \frac{1}{\lambda^2} d\rho(\lambda) < \infty$, then there exists a GSP h such that $h' \stackrel{d}{=} f$
(and also a GSP h such that $h(x+1) - h(x) = f(x)$).

Proof: $\mathbb{E}[h(t)h(s)] = \int_{\mathbb{R}} e^{-i\lambda(t-s)} d\mu(\lambda) \quad \Rightarrow \quad \mathbb{E}[h'(t)h'(s)] = \int_{\mathbb{R}} e^{-i\lambda(t-s)} \lambda^2 d\mu(\lambda)$.

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Borell-TIS inequality

$\mathbb{P}(\sup_{[0,N]} |h| > \ell) \leq e^{-\frac{\ell^2}{2\mathrm{var}\, h(0)}}$ for a GSP h.

Anderson's lemma

$\mathbb{P}(\sup_n |X_n \oplus Y_n| \leq \ell) \leq \mathbb{P}(\sup_n |X_n| \leq \ell)$ for X_n, Y_n Gaussian centred.

Analytic lemma (FFN)

If $h : T \to \mathbb{R}$ is such that $h' > 0$ on $[0, N]$, then there exists a set $R \subseteq [0, N]$ of measure $|R| \geq \frac{N}{2}$ such that $\sup_R |h'| \leq \frac{2}{N} \cdot \sup_{[0,N]} |h|$.

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Proof uses:

- Chebyshev polynomials
  (minimizers of sup-norm on $[-1, 1]$ among monic polynomials)
- Hermite-Genocchi formula
  (for leading coefficient of an interpolation polynomial)
- Splines
  (for moving from $\mathbb{R}$ to $\mathbb{Z}$)

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$\ell = \sqrt{N \log N} \Rightarrow$ both sides are $e^{-CN \log N}$.

- high order vanishing
- $\rho_{ac} \neq 0$:
  $\rho = m\mathbb{1}_E + \mu$, $E \neq \mathbb{1}_{[-a,a]} \Rightarrow$ "almost i.i.d." on a dense subset of the lattice (Restricted Invertibility Theorem by Bourgain-Tzafriri)
- tiny persistence:
  optimize over $E = E(\rho, N)$.

Future directions

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- high dimensions
- singular measures
- existence of limiting exponent
  (e.g. $\lim_{N\to\infty} \frac{\log P_f(N)}{N}$)
- non-stationary processes

Thank you.

"Persistence can grind an iron beam down into a needle."

– – Chinese Proverb.