Persistence and Ball Exponents for Gaussian Stationary Processes

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Abstract

Consider a real Gaussian stationary process f_ρ, indexed on either $\mathbb{R}$ or $\mathbb{Z}$ and admitting a spectral measure ρ. We study $\theta_\rho^\ell = -\lim_{T\to\infty} \frac{1}{T} \log \mathbb{P}\left(\inf_{t\in[0,T]} f_\rho(t) > \ell\right)$, the persistence exponent of f_ρ. We show that, if ρ has a positive density at the origin, then the persistence exponent exists; moreover, if ρ has an absolutely continuous component, then $\theta_\rho^\ell > 0$ if and only if this spectral density at the origin is finite. We further establish continuity of θ_ρ^ℓ in ℓ, in ρ (under a suitable metric) and, if ρ is compactly supported, also in dense sampling. Analogous continuity properties are shown for $\psi_\rho^\ell = -\lim_{T\to\infty} \frac{1}{T} \log \mathbb{P}\left(\inf_{t\in[0,T]} |f_\rho(t)| \le \ell\right)$, the ball exponent of f_ρ, and it is shown to be positive if and only if ρ has an absolutely continuous component.

1 Introduction

Persistence, namely the event that a real stochastic process f stays above a level ℓ over a long time interval $[0, T]$, is a well studied object since the 1950s, especially for centered *Gaussian stationary processes* (GSP) in discrete or continuous time (c.f. [7, 13, 16, 43, 45, 47, 49, 55, 57, 58] and the references therein). This has been studied in particular for the critical case $\ell = 0$ (c.f. [27, 28, 34, 51]), with applications to statistical mechanics (c.f. [14, 25, 42, 48, 62]). The persistence exponent of a GSP f over level ℓ is defined as the exponential rate of decay of the persistence probability, namely,

$$\theta_f^\ell = -\lim_{T\to\infty} \frac{1}{T} \log \mathbb{P}\left(\inf_{t\in[0,T]} f(t) > \ell\right),$$

whenever the limit exists.

Slepian, in his celebrated 1962 paper [57], conjectured that a persistence exponent should exist under mild conditions. The validity of this conjecture has often been taken for granted in the physics literature [22, 24]. Prior to this work, the only cases in which an exponent was shown to exist were non-negative correlated processes [20] (using a subadditivity argument), Markov processes (using Perron-Frobenius), and m-dependent processes (using independence properties) [6].

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Every continuous GSP $f : \mathbb{R} \to \mathbb{R}$ is characterized by a *covariance kernel* $r(t) = \operatorname{cov}(f(0), f(t))$, or, equivalently, by a *spectral measure*, that is, a finite, non-negative, symmetric measure ρ on $\mathbb{R}$ such that:

$$r(t) = \widehat{\rho}(t) = \int_{\mathbb{R}} e^{-it\lambda}\, d\rho(\lambda).$$

We denote by $f_\rho, r_\rho, \theta^\ell_\rho$ the process, covariance kernel and persistence exponents associated with ρ.

In recent years it became evident that it is possible to obtain a much more precise understanding of persistence in terms of the spectral measure of f (c.f. [26, 27, 28]). Our main result is that existence of a positive spectral density in the origin is sufficient for the existence of θ^ℓ_ρ.

1.1 Main results

Throughout, we require ρ to belong to the following class of measures with finite log–$(1+\beta)$ moment:

$$\mathcal{L} = \Big\{\rho \in \mathcal{S} \;\mid \exists \beta > 0 : \int_0^\infty \max\Big(\log^{1+\beta}\lambda, 1\Big)\, d\rho(\lambda) < \infty\Big\},$$

where $\mathcal{S}$ is the set of all spectral measures (symmetric, non-negative, finite measures on $\mathbb{R}$). The requirement $\rho \in \mathcal{L}$ is a rather mild condition which ensures continuity of the process (see [1, Sec. 1]). While we state our results for GSPs on $\mathbb{R}$, all theorems are valid for processes on $\mathbb{Z}$ as well, in which case ρ is supported on $[-\pi, \pi]$ and the condition $\rho \in \mathcal{L}$ is always satisfied (see Remark 7). Define

$$\rho'(0) := \lim_{\varepsilon \to 0} \tfrac{1}{2\varepsilon}\rho([-\varepsilon, \varepsilon])$$

whenever the limit exists, and

$$\mathcal{M} = \Big\{\rho \in \mathcal{S} \;\mid \rho'(0) \in (0, \infty]\Big\}.$$

We use the notation ρ_{ac} to denote the absolutely-continuous component of the measure ρ.

Theorem 1. *Let $\rho \in \mathcal{L} \cap \mathcal{M}$. Then, for all $\ell \in \mathbb{R}$ a persistence exponent $\theta^\ell_\rho \in [0, \infty)$ exists. Moreover, under the further assumption $\rho_{ac} \neq 0$, we have $\theta^\ell_\rho > 0$ if and only if $\rho'(0) < \infty$.*

Remark 1. Tightness of the condition $\rho \in \mathcal{M}$ is demonstrated by a counter-example, provided in Section 5.4. There we show that for the spectral density $(A + B\cos(\frac{1}{\lambda}))\mathbb{1}_{|\lambda|\le 1}$ in a certain range of $0 < B < A$, the exponent θ^0_ρ does not exist. In this example the density is bounded, compactly supported, and continuous on $[-1, 1] \setminus \{0\}$. We note that it is possible to construct examples of non-existence of θ^ℓ_ρ for any level $\ell \ge 0$, and even for all levels at once.

Remark 2. For $\rho \in \mathcal{L}$ with $\rho_{\mathrm{ac}} \neq 0$ and $\rho'(0) = 0$, we have $\theta^\ell_\rho = \infty$ for all $\ell \ge 0$. This is not covered by Theorem 1, but follows from Lemma 2.24 below. The conditions for positivity of θ^ℓ_ρ in the case $\rho_{\mathrm{ac}} = 0$ presently remain unknown.

Remark 3. For $\ell \le 0$ and $\rho \in \mathcal{M}$, it follows from our results that θ^ℓ_ρ is positive if and only if $\rho_{\mathrm{ac}} \neq 0$. For $\ell > 0$ and $\rho \in \mathcal{M}$ we conjecture that θ^ℓ_ρ is always positive.

A *ball event* is the event that a real stochastic process f stays in $[-\ell, \ell]$ over a long time interval $[0, T]$. This event is well-studied (c.f. [4, 33, 35, 39], for a complete bibliography, see [37]) with various applications (c.f. [36, 38]). Given a spectral measure ρ, denote the ℓ-ball exponent of ρ by

$$\psi_\rho^\ell = -\lim_{T\to\infty} \frac{1}{T} \log \mathbb{P}\left(\sup_{t\in[0,T]} |f_\rho(t)| < \ell \right).$$

This exponent is known to exist for all spectral measures in $\mathcal{L}$ using a subadditivity argument together with Khatri-Sidak [56] inequality (see e.g. [35]). Our main result concerning ball events is the following analogue of Theorem 1.

Theorem 2. *Let $\rho \in \mathcal{L}$. For all ℓ the ball exponent ψ_ρ^ℓ is positive if and only if $\rho_{\mathrm{ac}} \neq 0$.*

We further obtain three results concerning with the continuity and comparison of persistence and ball exponents. The first is the fact that the persistence and ball exponents are monotone in the spectral measure (in an appropriate sense), and are unaffected by the singular component.

Theorem 3. *Let $\ell \in \mathbb{R}$ and $\rho, \nu \in \mathcal{L}$. Then:*

(I) $\psi_{\rho+\nu}^\ell \geq \psi_\rho^\ell$.

(II) $\theta_{\rho+\nu}^\ell \geq \theta_\rho^\ell$, *provided that $\rho \in \mathcal{M}$ and $\nu'(0) = 0$.*

Further, equality holds in both (I) *and* (II) *if ν is purely-singular.*

Observe that Theorem 2 is an immediate corollary of Theorem 3.

Remark 4. We conjecture that equality in Theorem 3 holds if and only if ν is purely-singular.

The second result is continuity of exponents in the level ℓ and in the measure ρ, with respect to a suitable metric topology. Stating the result requires the following definitions.

For two finite measures ρ_1, ρ_2, define the total variation distance

$$d_{TV}(\rho_1, \rho_2) = \sup\{|\rho_1(E) - \rho_2(E)| : E \subseteq \mathbb{R}\}.$$

If ρ_1, ρ_2 also have a finite density at the origin, we define the metric

$$d_{\mathrm{TV}_0}(\rho_1, \rho_2) = d_{TV}(\rho_1, \rho_2) + |\rho_1'(0) - \rho_2'(0)|. \tag{1.1}$$

Fix $\alpha, \alpha', A, \beta, B > 0$ and consider the classes of measures

$$\begin{aligned} \mathcal{M}_{(\alpha,\alpha'),A} &= \left\{ \rho \in \mathcal{S} \mid \alpha \leq \frac{\rho(-x,x)}{2x} \leq \alpha', \forall x \in (0, A) \right\}, \\ \mathcal{L}_{\beta,B} &= \left\{ \rho \in \mathcal{S} \mid \int_0^\infty \max\left(\log^{1+\beta} \lambda, 1 \right) d\rho(\lambda) < B \right\}. \end{aligned} \tag{1.2}$$

Theorem 4.

(I) *The ball exponent ψ_ρ^ℓ is locally-Lipschitz continuous in $\ell \in (0, \infty)$ and uniformly d_{TV}-continuous in $\rho \in \mathcal{L}_{\beta,B}$.*

(II) *The persistence exponent θ_ρ^ℓ is locally-Lipschitz continuous in $\ell \in \mathbb{R}$ and uniformly d_{TV_0}-continuous in $\rho \in \mathcal{L}_{\beta,B} \cap \mathcal{M}_{(\alpha,\alpha'),A} \cap \mathcal{M}$.*

Remark 5. The function $\ell \mapsto \psi_\rho^\ell$ is convex (by log-concavity of centered Gaussian processes), and hence continuous. In contrast, continuity of $\ell \mapsto \theta_\rho^\ell$ is not immediate nor is it true for all spectral measures ρ. Consider, for example, $\mu = \frac{1}{2}(\delta_1 + \delta_{-1})$, which corresponds to the process $f_\mu(t) = \zeta_1 \cos(t) + \zeta_2 \sin(t)$ where $\zeta_1, \zeta_2 \sim \mathcal{N}(0,1)$ are independent. A direct analysis yields

$$\theta_\mu^\ell = \begin{cases} \infty, & \ell \geq 0, \\ 0, & \ell < 0 \end{cases}.$$

Note that the only discontinuity in this example is at the level $\ell = 0$. In fact, even when $\rho'(0) = 0$, we can establish continuity of θ_ρ^ℓ in ℓ (assuming it exists), for all $\ell < 0$. When, in addition, $\rho_{\text{ac}} \neq 0$ we have $\theta_\rho^\ell = \infty$ for all $\ell \geq 0$ (by Remark 2). Therefore whenever $\rho_{\text{ac}} \neq 0$, the map $\ell \mapsto \theta_\rho^\ell$ is always continuous away from 0. We conjecture this to hold also for purely singular measures.

The third continuity result is concerned with discrete sampling. For a spectral measure ρ define

$$\theta_{\rho;\Delta}^\ell = -\lim_{T\to\infty} \frac{1}{T} \log \mathbb{P}\left(\inf_{n\in\mathbb{Z},\, n\Delta\in[0,T]} f_\rho(n\Delta) > \ell \right),$$
$$\psi_{\rho;\Delta}^\ell = -\lim_{T\to\infty} \frac{1}{T} \log \mathbb{P}\left(\sup_{n\in\mathbb{Z},\, n\Delta\in[0,T]} |f_\rho(n\Delta)| < \ell \right),$$

whenever the limit exists. These are the persistence and ball exponents of the Gaussian stationary sequence generated by sampling f in Δ-intervals. Our next result provides conditions for convergence of $\theta_{\rho;\Delta}^\ell$ to θ_ρ^ℓ and of $\psi_{\rho;\Delta}^\ell$ to ψ_ρ^ℓ, as the sampling interval Δ approaches 0.

Theorem 5. *Let $\ell \in \mathbb{R}$ and $\rho \in \mathcal{L}$. Then:*

(I) $\lim_{\Delta\to 0} \psi_{\rho;\Delta}^\ell = \psi_\rho^\ell$.

(II) $\lim_{\Delta\to 0} \theta_{\rho;\Delta}^\ell = \theta_\rho^\ell$, *provided that $\rho \in \mathcal{M}$ has compact support.*

Remark 6. The second part of Theorem 5 may be extended to non-compactly supported spectral measures with sufficient rate of decay at infinity (for instance, ones which have density ρ' satisfying $\sup_{\lambda\in\mathbb{R}} |\lambda|^{1+\eta}\rho'(\lambda) < \infty$ for some $\eta > 0$). However, it does not extend to all $\rho \in \mathcal{L} \cap \mathcal{M}$. A counter-example is provided in Section 7.3.

Remark 7. As we have mentioned earlier, Theorems 1, 2, 3 and 4 hold true also for Gaussian processes in discrete time. The proofs remain valid without any change. For such processes the condition $\rho \in \mathcal{L}$ holds trivially.

Remark 8. A natural generalization of the problem presented here is the so called *two sided barrier problem* (considered by Shinozuka [53]), i.e. the probability that a GSP persists within a set $[a, b]$ where $a \neq -b$ and $-\infty < a < b < \infty$. Somewhat surprisingly the methods used here do not seem to generalize directly to this case as several monotonicity properties are lost. Thus the problem remains open. It is possible to show that when $a < 0 < b$, an exponential-type behavior is always demonstrated, while if $0 < a < b$ such exponential-type behavior holds only when the spectral measure is well behaved about the origin. We conjecture that, in both cases, the existence of a two-sided barrier exponent should hold whenever $\rho \in \mathcal{L} \cap \mathcal{M}$.

1.2 Background

1.2.1 Gaussian stationary processes and persistence

Gaussian processes provide good approximations for natural phenomena in which a random function is generated as a sum of nearly independent random contributions of similar scale (due to the functional CLT). When the process has a time invariant distribution, stationarity occurs and the approximating process becomes a GSP. This makes GSP an excellent model for noise, such as static interference, liquid surface fluctuations, gas density fluctuations or the shot effect fluctuation in thermionic emission. GSPs have therefore been extensively studied with motivation stemming from mathematics, physics and engineering. For an introduction to Gaussian processes see e.g. [1, 38]. In 1944, Rice [49] studied the zeroes of GSPs, introduces the notion of persistence and presented the asymptotics of the probability of persistence in short intervals $[0, T]$ as T tends to 0. In the same paper the problem of estimating persistence probability decay as T tends to ∞ was first posed. Motivated by this problem, Slepian [57] introduced in 1962 his famous inequality, and estimated the persistence probability of several examples. He conjectured that under mild decay conditions, the persistence should decay exponentially. In his words:

> "Intuition would indicate exponential falloff for a wide class of covariances."

Slepian also called for a study of continuity properties of the persistence exponent (in terms of the covariance kernel), and for numerical methods for estimating it.

Newell and Rosenblatt [45] were quick to extend Slepian's work and apply his methods to obtain rough bounds for the persistence probability for GSPs with polynomially decaying covariance. These were far from being tight, but remained the state-of-the-art for at least fifty years.

In the 1990's the interest of the physics community in persistence revived, as it turned out to be of use for analyzing rare events in spin systems and heat flows (see e.g. [22, 42] and the extensive survey [14]). Indeed, the authors of [22] were somewhat disappointed at the state of the problem:

> "To our surprise, given the correlation function of the Gaussian process the determination of this asymptotic decay turns out to be a hard unsolved problem."

In the late 2010's, Dembo and the third author [19, 20], seeking to study both the solutions of the heat equation initiated by white noise and the probability that a random polynomial has no roots, revisited Slepian's method. They observed that in the restricted case of GSPs with non-negative covariance, it is possible to use probabilistic arguments and tools from linear algebra to extend the method and obtain the exact rate of the decay of the persistence up to sub-exponential factors. Developing on this, in [5, Lem. 3.2] Aurzada and Mukherjee showed that a GSP with non-negative covariance has a positive persistence exponent if and only if its correlation is integrable. In addition, they obtain continuity results in terms of the covariance kernel for this set of processes.

The first process with sign-changing covariance kernel for which exponential-type decay was established is the sinc kernel process. This result, due to Antezana, Buckley, Marzo and Olsen [2], was a tour de force of analytic methods and direct computations. A study of this result, has led the first two authors to introduce spectral conditions which ensure exponential bounds on persistence, requiring the spectral measure to have a polynomial decay and a bounded spectral density in a small vicinity of the origin [26]. This was extended together with Nitzan [28] to conditions under which persistence decays sub- or super-exponentially, providing also new examples for extremely fast decaying persistence probabilities. All of these conditions depend on the interplay between the spectral behavior near the origin and the decay of the spectral measure near infinity. The special case of a "spectral gap" (i.e., a spectral measure which vanishes on an interval near the origin) was treated in more detail in [27].

Here, we go beyond establishing exponential-type behavior of persistence: we show the existence of a *persistence exponent* and provide several new *continuity results*. We do so by combining and expanding the spectral method of [28] and the covariance method of [20].

1.2.2 Related processes and events

Persistence is sometimes regarded more generally, as the event that a given stochastic process takes values in a specific set over a long time interval. Such events have received significant attention for a large class of examples, including random walks [11, 29] (see references there-in), Lévy processes [10, 23], Markov processes [17, 18], random polynomials [21, 41] and (spatial) processes driven by either stochastic differential equations, or by partial differential equations with random initial configuration [51, 52].

For point configurations in the plane, much attention is given to the persistence-type event of having no points at all in a large region. This event, which reflects the rigidity of the model, was studied for zeroes of random analytic functions [15, 46, 58], Coulomb gas [31] and random matrices [9], among other examples. In some cases, even very advanced questions such as the conditional behavior of the model could be handled [3, 30, 32].

The study of ball probabilities is also old and active, often referred to as "small deviations". These were studied for various models and metrics, with considerable interest in Gaussian processes and the L_∞ metric. Here the challenge is not to show existence of an exponent, but rather to compute or estimate it, especially when the width of the ball tends to zero. First results in the context of GSPs were obtained by Newell [44]. A classical bound is due to Lifshitz and Tsirelson [39], and improvements were made by Aurzada, Ibragimov, Lifshitz and van Zanten [4] as well as by Weber [63, 64].

1.3 Comparison lemmata for persistence probabilities

In this section we present three results which act as basic tools in our treatment of persistence probabilities, but are also of independent interest. These are three comparison properties of persistence probabilities: under change of level, under change of measure and under smoothing.

Denote

$$\mathcal{P}^\ell_\rho(T) := \mathbb{P}\left(\inf_{t\in[0,T]} f_\rho(t) > \ell\right), \qquad \theta^\ell_\rho(T) := -\frac{1}{T}\log \mathcal{P}^\ell_\rho(T). \tag{1.3}$$

and recall that $\theta^\ell_\rho = \lim_{T\to\infty} \theta^\ell_\rho(T)$ whenever the limit exists.

Throughout the remainder of the paper, we fix the level $\ell \in \mathbb{R}$ and parameters $\alpha, \beta, B > 0$, and consider measures in the class

$$\mathcal{M}_{\alpha,A} := \bigcup_{\alpha'>0} \mathcal{M}_{(\alpha,\alpha'),A} = \Big\{\rho \in \mathcal{S} \ \mid \alpha \leq \frac{\rho(-x,x)}{2x}, \ \forall x \in (0,A)\Big\},$$

and in $\mathcal{L}_{\beta,B}$, as introduced in (1.2). Constants may depend on (ℓ, α, β, B) implicitly. Since in some applications the parameter $A > 0$ will be varying with T, the dependence on it is made explicit.

Lemma 1.1 (continuity in levels)**.** *There exists $C > 0$ such that, for any $\mu \in \mathcal{M}_{\alpha,A} \cap \mathcal{L}_{\beta,B}$, $\delta > 0$ and $T \geq \max\{4, \frac{1}{A}\}$, we have*

$$0 \leq \theta_\mu^{\ell+\delta}(T) - \theta_\mu^{\ell-\delta}(T) \leq C\delta.$$

Lemma 1.2 (smoothing lemma)**.** *Let h be a spectral density such that $h(0) = 1$, $\widehat{h} \geq 0$ and $\widehat{h}$ is supported on $[-\frac{a}{2}, \frac{a}{2}]$. Then for any spectral measures μ and ν:*

$$\mathcal{P}^\ell_{\mu+\nu}(T+a) \leq \mathcal{P}^\ell_{\mu+h^2\nu}(T).$$

Lemma 1.3 (continuity in measure)**.** *Let $\mu \in \mathcal{M}_{\alpha,A} \cap \mathcal{L}_{\beta,B}$ and $\nu \in \mathcal{L}_{\beta,B}$ be such that $\nu(\mathbb{R}) < \varepsilon$ for some $\varepsilon > 0$.*

(I) *For all $T > \max\{4, \frac{1}{A}\}$, we have*

$$\theta^\ell_{\mu+\nu}(T) \leq \theta^\ell_\mu(T) + C_\varepsilon,$$

where C_ε obeys $\lim_{\varepsilon\to 0} C_\varepsilon = 0$.

(II) *There exist c, ε_0 and $T_0(\varepsilon)$ such that for all $\varepsilon < \varepsilon_0$ and $T > \max\{T_0, \frac{2}{A}\}$, if $\nu([-\frac{L}{T}, \frac{L}{T}]) < \varepsilon \frac{L}{T}$*

$$\theta^\ell_\mu\left(T(1-\eta)\right) \leq \theta^\ell_{\mu+\nu}(T) + C_\varepsilon,$$

where $L = \sqrt{\frac{2}{c\varepsilon}}$, $\eta = (c\varepsilon)^{1/4}$ and $\lim\limits_{\varepsilon\to 0} C_\varepsilon = 0$.

1.4 Examples

We conclude the introduction by discussing several noteworthy processes for which the existence of persistence exponents is first established by Theorem 1.

1. **Sinc kernel.** Consider the process with kernel $r(t) = \mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$. The corresponding spectral measure has density $1\!\!1_{[-\pi,\pi]}$. It is clear that $\rho \in \mathcal{M} \cap \mathcal{L}$, so by Theorem 1, for any $\ell \in \mathbb{R}$ the exponent θ^ℓ_ρ exists in $(0,\infty)$. This process received much attention [2] and has various applications to statistics and signal processing [61].

2. **Zero-order Bessel kernel.** Let $r(t) = J_0(t) = \sum_{m=0}^\infty \frac{(-t)^m}{2^m (m!)^2}$. The corresponding spectral measure ρ has a density $\frac{2}{\sqrt{1-\lambda^2}} 1\!\!1_{[-1,1]}$ (see [60]). As $\rho \in \mathcal{M} \cap \mathcal{L}$, Theorem 1 yields that, for any $\ell \in \mathbb{R}$, the exponent θ^ℓ_ρ exists in $(0,\infty)$.

3. **Moving Average over i.i.d.** For a sequence $\{U_k\}_k \in \mathbb{Z}$ such that $\sum_{k\in\mathbb{Z}} U_k^2 < \infty$, we define the moving average process by

$$X_j = \sum_{k\in\mathbb{Z}} U_k Z_{j-k} = (U * Z)(j),$$

where $\{Z_k\}_{k\in\mathbb{Z}}$ are i.i.d. $\mathcal{N}(0,1)$ random variables. The corresponding spectral measure ρ has density $|\widehat{U}(\lambda)|^2$, where $\widehat{U}(\lambda) = \sum_{k\in\mathbb{Z}} U_k e^{ik\lambda}$. Assume that $\widehat{U}$ is continuous in a neighborhood of 0 (in the wide sense) and note that $\widehat{U}(0) = \sum_{k\in\mathbb{Z}} U_k$. Then $\rho'(0) = \lim_{x\to 0} \frac{1}{x}\int_0^x |\widehat{U}(\lambda)|^2 d\lambda$ exists and equals $|\widehat{U}(0)|^2$. By Theorem 1 and Remarks 2 and 7, the persistence exponent θ_X^ℓ exists in $[0,\infty]$ for any $\ell \in \mathbb{R}$, and

$$\theta_X^\ell = \infty \iff \sum_{k\in\mathbb{Z}} U_k = 0, \quad \text{while} \quad \theta_X^\ell = 0 \iff |\sum_{k\in\mathbb{Z}} U_k| = \infty.$$

4. **Moving Average over a GSP.** More generally, let $\{Y_t\}_{t\in T}$ be a GSP over $T \in \{\mathbb{Z},\mathbb{R}\}$ with spectral measure $\rho \in \mathcal{M}\cap\mathcal{L}$. Let $U : T \to \mathbb{R}$ be such that $\widehat{U}(\lambda) = \int_T U(t)e^{it\lambda}dt$ is continuous (in the wide sense) in a neighborhood of 0, and further $\int |\widehat{U}(\lambda)|^2 d\rho(\lambda) < \infty$. Then the moving-average process defined by $X(t) = (U * Y)(t)$ is a GSP (see Obs. 2.7 below), for which θ_X^ℓ exists for all $\ell \in \mathbb{R}$. If $\widehat{U}(0)$ and $\rho'(0)$ both lie in $(0,\infty)$, then $\theta_X^\ell \in (0,\infty)$.

5. **Absolutely summable correlations.** Suppose that $\int_{\mathbb{R}} |r(t)|\, dt < \infty$ and $\int_{\mathbb{R}} r(t)dt > 0$. Then $\rho \in \mathcal{M}$ with $\rho'(0) \in (0,\infty)$, and Theorem 1 implies $\theta_\rho^\ell \in (0,\infty)$ for every ℓ. Prior to our work, such results were only known under the additional assumption that $r \geq 0$, see [5, 19].

Additional examples could be generated on noting that the class of measures $\mathcal{L}\cap\mathcal{M}$ is closed under addition, truncation, and convolution.

1.5 Outline of the paper

The paper is organized as follows. In Section 2 we present various tools needed in our proofs. In Section 3 we prove the comparison results: Lemmata 1.1, 1.2 and 1.3. The rest of the paper is dedicated to the proofs of Theorems 1–5. The order in which these were presented is not the order of their establishment. In Section 4 we prove Theorem 2 concerning ball exponents and singular measures. In Section 5 we prove the existence of a persistence exponent as stated in Theorem 1, and the continuity of ball and persistence exponents as stated in Theorem 4; we also provide a class of non-existence examples. In Section 6 we prove monotonicity and indifference of the exponents to the singular part, namely, Theorem 3. Lastly, in Section 7 we prove Theorem 5 regarding convergence of persistence exponents under sampling; an example of non-convergence is also provided.

2 Preliminaries

In this section we collect tools and observations that will serve us in the rest of the paper. Throughout the paper, we denote by f_ρ the GSP corresponding to a spectral measure ρ. The notation ρ_L is used for the restriction of a measure ρ onto the interval $[-L, L]$. We use both $\mathcal{F}[\rho]$ and $\widehat{\rho}$ to denote the Fourier transform of a finite measure ρ.

2.1 Gaussian measures on Euclidean spaces

In this section we recall several classical properties of Gaussian measures on $\mathbb{R}^d$. We start with standard estimates of the one-dimensional Gaussian distribution (see, e.g., [28, Lem. 3.13]).

Lemma 2.1. *Let $Z \sim \mathcal{N}(0,1)$. For all $x > 0$ we have:*

(a)
$$\frac{1}{\sqrt{2\pi}}\left(\frac{1}{x}-\frac{1}{x^3}\right)e^{-x^2/2} \le \mathbb{P}(Z > x) \le \frac{1}{\sqrt{2\pi}}\frac{1}{x}e^{-x^2/2},$$

In particular, for $x \ge 2$:
$$e^{-x^2} \le \mathbb{P}(Z > x) \le e^{-x^2/2}.$$

(b)
$$\sqrt{\frac{2}{\pi}}xe^{-x^2/2} \le \mathbb{P}(|Z| \le x) \le x,$$

In particular, for $0 < x \le 1$:
$$\frac{1}{4}x \le \mathbb{P}(|Z| \le x) \le x.$$

We continue with a few facts about Gaussian measures and intersections or homothety.

Observation 2.2. *Let γ_d be a centered Gaussian measure in $\mathbb{R}^d$, and let $K \subset \mathbb{R}^d$ be a convex domain containing the origin. Then for any $\alpha \ge 1$, $\gamma_d(\alpha K) \le \alpha^d \gamma_d(K)$.*

Proof. Let (Θ, r) denote the polar coordinates in $\mathbb{R}^d$ (where $\Theta = (\theta_1, \ldots, \theta_{d-1})$). We have for any convex body K:

$$\gamma_d(K) = \int_\Theta \left\{ \int_0^{R_K(\Theta)} r^{d-1} g_\Theta(r)\, dr \right\} J(\Theta) d\Theta,$$

where $g_\Theta(r)$ the density of γ_d on the ray defined by Θ, $R_K(\Theta)$ is the radius of K in direction Θ, and $r^{d-1}J(\Theta)$ is the appropriate Jacobian. Using a simple change of variable we have:

$$\begin{aligned}\gamma_d(\alpha K) &= \int_\Theta \left\{ \int_0^{\alpha R_K(\Theta)} r^{d-1} g_\Theta(r)\, dr \right\} J(\Theta) d\Theta \overset{[r=\alpha s]}{=} \alpha^d \int_\Theta \left\{ \int_0^{R_K(\Theta)} s^{d-1} g_\Theta(\alpha s) ds \right\} J(\Theta) d\Theta \\ &\le \alpha^d \int_\Theta \left\{ \int_0^{R_K(\Theta)} s^{d-1} g_\Theta(s) ds \right\} J(\Theta) d\Theta = \alpha^d \gamma_d(K),\end{aligned}$$

where the inequality is due to the fact that $g_\Theta(s)$ is decreasing in $s \in [0, \infty)$. □

We shall employ the Khatri-Sidak's inequality (see [56] or [40, Ch. 2.4]). This classical result is a particular case of the celebrated Gaussian correlation inequality for general convex sets, proved by Royen [50] in 2014.

Proposition 2.3 (Khatri-Sidak's inequality)**.** *If $(Z_1, \ldots, Z_d)$ is a centered Gaussian vector in $\mathbb{R}^d$, then for any $\{\ell_j\}_{j=1}^d \subset (0, \infty)$ one has $\mathbb{P}\left(\bigcap_{j=1}^d \{|Z_j| \le \ell_j\}\right) \ge \prod_{j=1}^d \mathbb{P}(|Z_j| \le \ell_j)$.*

An immediate consequence is the following.

Corollary 2.4. *Let f be a Gaussian process which is almost surely continuous on an interval $[0, a+b]$. Then for every $\ell > 0$,*

$$\mathbb{P}\left(\sup_{[0,a+b]} |f| < \ell\right) \geq \mathbb{P}\left(\sup_{[0,a]} |f| < \ell\right) \mathbb{P}\left(\sup_{[a,a+b]} |f| < \ell\right).$$

In the case of stationarity, Corollary 2.4 together with Fekete's subadditivity lemma yield the existence of the ball exponent. As per (1.3), denote

$$\psi_\rho^\ell(T) := -\frac{1}{T} \log \mathbb{P}\left(\sup_{t\in[0,T]} |f_\rho(t)| < \ell\right). \tag{2.1}$$

Corollary 2.5. *For any spectral measure $\rho \in \mathcal{L}$ and any $\ell > 0$, the ball exponent $\psi_\rho^\ell := \lim_{T\to\infty} \psi_\rho^\ell(T)$ exists and lies in $[0, \infty)$.*

Proposition 2.3 is also the main ingredient in the proof of the following result.

Proposition 2.6. *Let γ_d be the standard Gaussian measure in $\mathbb{R}^d$. Then there exists a constant $\kappa > 0$ such that, for any $n \geq d$ and any collection of unit vectors $u_1, \dots, u_n \in \mathbb{S}^{d-1}$, we have*

$$\gamma_d\left(\bigcap_{j=1}^n \{x \in \mathbb{R}^d : |\langle x, u_j\rangle| \leq 1\}\right) \geq \left(\frac{\kappa}{\sqrt{1+2\log\frac{n}{d}}}\right)^d.$$

The analogue of Proposition 2.6 in which γ is replaced with the Lebesgue measure is a theorem by Ball and Pajor [8]. The Gaussian case should follow from the Euclidean one (allowing for different constants), however, we give here a direct Gaussian proof suggested to us by Ori Gurel-Gurevitch.

Proof. Denote $S_j = \{x \in \mathbb{R}^d : |\langle x, u_j\rangle| \leq 1\}$. Let $\alpha > 1$ (to be chosen later). By Observation 2.2 and Proposition 2.3 we have:

$$\gamma_d\Big(\bigcap_{j=1}^n S_j\Big) = \gamma_d\Big(\frac{1}{\alpha}\bigcap_{j=1}^n (\alpha S_j)\Big) \geq \alpha^{-d}\, \gamma_d\Big(\bigcap_{j=1}^n (\alpha S_j)\Big) \geq \alpha^{-d}\gamma_d\Big(\alpha S_1\Big)^n = \alpha^{-d}\gamma_1([-\alpha,\alpha])^n,$$

where γ_1 is the standard Gaussian measure in $\mathbb{R}$. Set $\alpha = \sqrt{1+2\log\left(\frac{n}{d}\right)}$. The result will follow once we show that $\gamma_1([-\alpha,\alpha])^{n/d} \geq \kappa$ for all $n \geq d$ and some constant $\kappa > 0$ which is independent of n and d. If $\frac{n}{d} < e^2$, we bound by

$$\gamma_1([-\alpha,\alpha])^{n/d} \geq \gamma_1([-1,1])^{e^2},$$

while if $\frac{n}{d} \geq e^2$ we apply the first part of Lemma 2.1 to get

$$\gamma_1([-\alpha,\alpha])^{n/d} \geq (1 - 2\,\mathbb{P}(\mathcal{N}(0,1)\geq\alpha))^{n/d} \geq \left(1-2e^{-\alpha^2/2}\right)^{n/d} \geq \left(1-2\frac{d}{n}\right)^{n/d} \geq e^{-2}.$$

Taking $\kappa = 0.05 < \min\left\{\gamma_1([-1,1])^{e^2}, e^{-2}\right\}$, the result follows. ∎

2.2 Operations on GSPs

This section collects observations about the spectral measure of a GSP which was generated from another GSP by a single operation, such as: convolution with a fixed kernel, scaling, lattice sampling and discrete derivation. In what follows, ρ is a spectral measure and f_ρ is the corresponding GSP.

Observation 2.7. *Assume that $h \in L^2_\rho(\mathbb{R})$. Then $f_\rho * \widehat{h}$ is a GSP with covariance kernel $\widehat{\rho} * \widehat{h} * (\widehat{h}(-\cdot))$ and spectral measure $|h(\lambda)|^2 d\rho(\lambda)$.*

Proof. Denote $H = \widehat{h}$. The random process $W(x) = (f_\rho * H)(x) = \int f_\rho(t) H(x-t) dt$ is Gaussian, with covariance kernel given by:

$$\begin{aligned}
\mathbb{E}[W(x)W(y)] &= \mathbb{E}\left[\int f_\rho(x-t)H(t)\ dt \int f_\rho(y-s)H(s)\ ds\right] \\
&= \iint \mathbb{E}[f_\rho(x-t)f_\rho(y-s)]H(t)H(s)\ dt\ ds \\
&= \iint \widehat{\rho}(x-y+s-t)H(t)H(s)\ dt\ ds \\
&= \int (\widehat{\rho} * H)(x-y+s)H(s)\ ds \\
&= (\widehat{\rho} * H * (H(-\cdot))\,(x-y) = \mathcal{F}\left[|h|^2\rho\right](x-y).
\end{aligned}$$

□

Observation 2.8. *For any $\varepsilon > 0$, the sequence $\{f_\rho(j\varepsilon)\}_{j\in\mathbb{Z}}$ has the folded spectral measure ρ^*_ε, which is supported on $[-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}]$ and given by:*

$$\rho^*_\varepsilon(I) = \rho\left(\bigcup_{n\in\mathbb{Z}} (I + \tfrac{2\pi}{\varepsilon}n)\right).$$

Proof. ρ^*_ε is the unique spectral measure supported on $[-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}]$ such that $\mathcal{F}[\rho^*_\varepsilon](j\varepsilon) = \mathcal{F}[\rho](j\varepsilon)$ for any $j \in \mathbb{Z}$. □

2.3 Decompositions of GSPs

The following claims provide two types of decompositions of a GSP into independent components. The first is a series representation, which may be found in [28, Claim 3.8].

Lemma 2.9 (Hilbert decomposition)**.** *Let $\rho \in \mathcal{L}$ and let φ_n be an orthonormal basis in $\mathcal{L}^2_\rho(T^*)$ which satisfies, for every $n \in \mathbb{N}$, $\varphi_n(-\lambda) = \overline{\varphi_n(\lambda)}$. Denote $\Phi_n(t) = \int_\mathbb{R} e^{-i\lambda t}\varphi_n(\lambda)d\rho(\lambda)$. Then*

$$f(t) = \sum_n \zeta_n \Phi_n(t), \quad \zeta_n \sim \mathcal{N}(0,1) \text{ i.i.d.}$$

is a continuous GSP over T with spectral measure ρ.

Lemma 2.9 has the following useful consequence.

Corollary 2.10 (one component)**.** *Let $\rho \in \mathcal{L}$ and let $\varphi \in \mathcal{L}^2_\rho$ be a real symmetric function such that $\|\varphi\|_{\mathcal{L}^2_\rho} = 1$. Write $\Phi(t) = \int_\mathbb{R} e^{-i\lambda t}\varphi(\lambda)d\rho(\lambda)$, then we have the decomposition*

$$f_\rho(t) \overset{d}{=} \zeta \cdot \Phi(t) \oplus g,$$

where $\zeta \sim \mathcal{N}(0,1)$ and g is a Gaussian process which is independent of ζ.

The second is the spectral decomposition which appeared in [26, Obs. 1].

Lemma 2.11 (spectral decomposition)**.** *If ρ_j is a spectral measure for $j \in \{0,1,2\}$ and $\rho_0 = \rho_1 + \rho_2$, then $f_{\rho_0} \overset{d}{=} f_{\rho_1} \oplus f_{\rho_2}$.*

One application of the spectral decomposition is the following simple yet useful lemma. Recall the notation (1.3) and (2.1),

Lemma 2.12. *For any spectral measure ρ, ν and any $\ell \in \mathbb{R}$, $\delta > 0$ and $T > 0$, we have the following.*

(a) $\psi^{\ell}_{\rho+\nu}(T) \leq \psi^{\ell-\delta}_{\rho}(T) + \psi^{\delta}_{\nu}(T)$.

(b) $\theta^{\ell}_{\rho+\nu}(T) \leq \theta^{\ell+\delta}_{\rho}(T) + \psi^{\delta}_{\nu}(T)$.

Proof. **Part (a).** Using Lemma 2.11, we have

$$\mathbb{P}\left(\inf_{t\in[0,T]} f_{\rho+\nu}(t) > \ell\right) = \mathbb{P}\left(\inf_{t\in[0,T]} (f_\rho(t) \oplus f_\nu(t)) > \ell\right) \geq \mathbb{P}\left(\inf_{t\in[0,T]} f_\rho(t) > \ell + \delta\right) \mathbb{P}\left(\sup_{t\in[0,T]} |f_\nu(t)| < \delta\right),$$

which upon taking log and dividing by T yields the desired inequality.

Part (b). Again using Lemma 2.11, we have

$$\mathbb{P}\left(\sup_{t\in[0,T]} |f_{\rho+\nu}(t)| < \ell\right) = \mathbb{P}\left(\sup_{t\in[0,T]} |f_\rho(t) \oplus f_\nu(t)| < \ell\right) \geq \mathbb{P}\left(\sup_{t\in[0,T]} |f_\rho(t)| < \ell - \delta\right) \mathbb{P}\left(\sup_{t\in[0,T]} |f_\nu(t)| < \delta\right),$$

which upon taking log and dividing by T yields the desired inequality.

□

2.4 Classical Gaussian tools

In this section we recall classical tools from the theory of Gaussian processes. We start with the celebrated Slepian's lemma, see [1, Thm. 2.1.2] or [57].

Proposition 2.13 (Slepian)**.** *Let X and Y be centered Gaussian processes on $I \subset \mathbb{R}$. Suppose that*

$$\mathbb{E}[X_t X_s] \leq \mathbb{E}[Y_t Y_s], \quad \mathbb{E}[X_t^2] = \mathbb{E}[Y_t^2], \quad \forall t, s \in I.$$

Then for any $\ell \in \mathbb{R}$ one has

$$\mathbb{P}\left(\sup_I X > \ell\right) \leq \mathbb{P}\left(\sup_I Y > \ell\right).$$

The following famous concentration bound is due to Borell and Tsirelson-Ibragimov-Sudakov, see [1, Thm. 2.1.1].

Proposition 2.14 (Borell-TIS)**.** *Let X be a centered Gaussian process on I which is almost surely bounded. Then for all $u > 0$ we have:*

$$\mathbb{P}\left(\sup_I X - \mathbb{E}\sup_I X > u\right) \le \exp\left(-\frac{u^2}{2\sigma}\right),$$

where $\sigma = \sup_{t\in I} \operatorname{var} X(t)$.

The expected supremum of a Gaussian process is often bounded using Dudley's metric-entropy method [1, Thm. 1.3.3]. For a Gaussian process H on an interval I, we let

$$d_H(a,b) := \sqrt{\mathbb{E}(H(a)-H(b))^2}, \quad a,b \in I \tag{2.2}$$

be the canonical semi-metric induced by H, and denote $\operatorname{diam}_H(I) = \sup_{a,b\in I} d_H(a,b)$. For any $x > 0$, the covering number $N_H(x)$ is the minimal number of d_H-balls of radius x which cover I.

Proposition 2.15 (Dudley's bound)**.** *There exists a universal constant $K > 0$ such that for any Gaussian process H on I we have*

$$\mathbb{E}\sup_I H \le K \int_0^{diam_H(I)} \sqrt{\log N_H(x)}\, dx.$$

Lastly we recall a comparison between ball probabilities due to Anderson [40, Ch. 2.3].

Proposition 2.16 (Anderson)**.** *Let X, Y be two independent, centered Gaussian processes on I. Then for any $\ell > 0$,*

$$\mathbb{P}\Big(\sup_I |X \oplus Y| \le \ell\Big) \le \mathbb{P}\Big(\sup_I |X| \le \ell\Big).$$

2.5 Supremum

In this section we apply tools from Section 2.4 in order to estimate events concerning the supremum a GSP whose spectral measure is in the class $\mathcal{L}_{\beta,B}$.

Lemma 2.17. *Suppose that $\beta < e-1$ and $B > 0$. Then for every $\rho \in \mathcal{L}_{\beta,B}$ we have*

$$0 \le r(0) - r(t) \le \frac{3B}{\log^{1+\beta}(1/t)}.$$

Proof. Recall that $t < 1$ and observe that:

$$\begin{aligned} r(0)-r(t) &= \int_{\mathbb{R}} \Big(1-\cos(\lambda t)\Big) d\rho(\lambda) = \left(\int_{|\lambda|<\frac{1}{\sqrt{t}}} + \int_{|\lambda|\ge\frac{1}{\sqrt{t}}}\right)\Big(1-\cos(\lambda t)\Big) d\rho(\lambda) \\ &\le \int_{|\lambda|<\frac{1}{\sqrt{t}}} \frac{\lambda^2 t^2 d\rho(\lambda)}{2} + \frac{2}{\log^{1+\beta}(1/t)} \int_{|\lambda|\ge\frac{1}{\sqrt{t}}} \log^{1+\beta}\lambda\; d\rho(\lambda) \\ &\le Bt + \frac{2B}{\log^{1+\beta}(1/t)} \le \frac{3B}{\log^{1+\beta}(1/t)}. \end{aligned}$$

In the last step we used that $1+\beta < e$.

□

If ρ is compactly supported, then Lemma 2.17 may be replaced by the following observation.

Observation 2.18. *Suppose that ρ is supported on $[-D, D]$. Then*

$$0 \le r(0) - r(t) \le \tfrac{1}{2} D^2 r(0) t^2.$$

Proof. $r(0) - r(t) = \int_{\mathbb{R}} (1 - \cos(\lambda t)) d\rho(\lambda) \le \int_{|\lambda| \le D} \frac{\lambda^2 t^2}{2} d\rho(\lambda) \le \frac{D^2}{2} r(0) t^2$. □

Lemma 2.19. *There exists $C = C(\beta, B)$ such that for any $\rho \in \mathcal{L}_{\beta,B}$ and any $h \le 1$,*

$$\mathbb{E} \sup_{[0,h]} |f_\rho| < C \sup_{t \in [0,h]} |r(0) - r(t)|^{\frac{\beta}{2(1+\beta)}}.$$

Proof. Let d_f be the canonical semi-metric induced by f (defined via (2.2)). Using stationarity and Lemma 2.17 we have, for any $s \in \mathbb{R}$ and $t > 0$,

$$d_f(s, s+t) = \sqrt{2|r(0) - r(t)|} \le \frac{\sqrt{6B}}{\log^{(1+\beta)/2}(1/t)} =: \psi(t).$$

This yields that for an interval $I = [0, h]$ we have

$$N_f(x) \le \max\left(1, \frac{|I|}{\psi^{-1}(x)}\right) = \max\left(1, h \exp\left[\left(\frac{6B}{x^2}\right)^{\frac{1}{1+\beta}}\right]\right),$$

and

$$\mathrm{diam}_f(I) = \sup_{x,y \in I} d_f(x,y) \le \sup_{t \in [0,h]} \sqrt{2(r(0) - r(t))}.$$

By Dudley's bound (Proposition 2.15) there exists a universal constant $K > 0$ such that

$$\mathbb{E} \sup_{[0,h]} |f_\rho| < K \int_0^{\mathrm{diam}(I)} \sqrt{\left(\left(\frac{6B}{x^2}\right)^{\frac{1}{1+\beta}} + \log h\right)_+} \, dx \le K (6B)^{\frac{1}{2(1+\beta)}} \int_0^{\mathrm{diam(I)}} x^{-\frac{1}{1+\beta}} \, dx$$
$$\le C(\beta, B) \sup_{t \in [0,h]} |r(0) - r(t)|^{\frac{\beta}{2(1+\beta)}}.$$
□

Lemma 2.20. *There exist $C_1, C_2 > 0$ (depending only on β, B) such that for any $\nu \in \mathcal{L}_{\beta,B}$ and any $m > 0$,*

$$\mathbb{P}\left(\sup_{[0,1]} |f_\nu| > (C_1 m + C_2)\, \nu(\mathbb{R})^{\frac{\beta}{2(1+\beta)}}\right) < 2e^{-m^2/2}.$$

Proof. Denote $I = [0, 1]$. Note that

$$\sqrt{\sup_{t \in I} \mathrm{var}\, f_\nu(t)} = \sqrt{r(0)} = \sqrt{\nu(\mathbb{R})} \le C_1 \nu(\mathbb{R})^{\frac{\beta}{2(1+\beta)}},$$

and by Lemma 2.19 we have

$$\mathbb{E} \sup_I |f_\nu| < C_2 \nu(\mathbb{R})^{\frac{\beta}{2(1+\beta)}},$$

where C_1 and C_2 depend only on β, B. By Proposition 2.14 (Borell-TIS theorem), we have for any $m > 0$:

$$\mathbb{P}\left(\sup_I f_\nu > (C_1 m + C_2)\nu(\mathbb{R})^{\frac{\beta}{2(1+\beta)}}\right) \leq \mathbb{P}\left(\sup_I f_\nu > \mathbb{E}\sup_I f_\nu + m\sqrt{\sup_{t\in I} \operatorname{var} f_\nu(t)}\right) < e^{-m^2/2}.$$

The statement then follows using symmetry of f_ν and $-f_\nu$ and a union bound. □

Lemma 2.21. *Let W be a centered Gaussian process on $[0,T]$, satisfying*

$$\sup_{t\in[0,T]} \operatorname{var} W(t) \leq \frac{\varepsilon}{T}, \tag{2.3}$$

$$\sup_{t\in[0,T]} \operatorname{var}\,(W(t+h) - W(t)) \leq c^2 h^2, \tag{2.4}$$

for some $\varepsilon \in (0,1)$ and $c > 0$ and all $h > 0$. Then for any $\delta > 0$ there exists $T_0 = T_0(\delta, c)$ such that if $T > T_0(\delta, c)$ we have:

$$\mathbb{P}\left(\sup_{[0,T]} |W| > \delta\right) \leq 2e^{-\frac{\delta^2}{8\varepsilon}T}.$$

Proof. Let d_W be the canonical semi-metric induced by W, as in (2.2). Using (2.4) we have

$$d_W(t, t+h) = \sqrt{\operatorname{var}\,(W(t+h) - W(t))} \leq ch,$$

for any $t, h > 0$. This yields that the covering number $N_W(x)$ of the interval $[0,T]$ in the d_W-metric obeys $N_W(x) \leq \frac{cT}{x}$. Moreover using (2.3) we have $\operatorname{diam}_W([0,T]) = \max_{x,y\in[0,T]} d_W(x,y) \leq \sqrt{2\frac{\varepsilon}{T}} \leq \sqrt{\frac{2}{T}}$. Consequently by Dudley's bound (Proposition 2.15):

$$\mathbb{E}\sup_{[0,T]} |W| < K\int_0^{\sqrt{\frac{2}{T}}} \sqrt{\log\frac{cT}{x}}dx \leq K\left(\sqrt{\frac{2}{T}}\cdot\sqrt{\log(cT)} + \int_0^{\sqrt{\frac{2}{T}}}\sqrt{\log\left(\tfrac{1}{x}\right)}dx\right).$$

Hence there exists $T_0 = T_0(\delta, c)$ such that if $T > T_0(\delta, c)$ we obtain

$$\mathbb{E}\sup_{[0,T]} |W| \leq \frac{\delta}{2}. \tag{2.5}$$

Using Borell-TIS (Proposition 2.14) with (2.3) and (2.5) we obtain that, for $T > T_0$,

$$\mathbb{P}\left(\sup_{[0,T]} |W| > \delta\right) \leq 2\mathbb{P}\left(\sup_{[0,T]} W > \delta\right) \leq 2\mathbb{P}\left(\sup_{[0,T]} W - \mathbb{E}\sup_{[0,T]} W > \frac{\delta}{2}\right) \leq 2e^{-\frac{\delta^2}{8\varepsilon}T}.$$ □

2.6 Bounds on ball and persistence exponents

In this section we present apriori bounds on ball and persistence probabilities, which hold uniformly for spectral measures in the class $\mathcal{L}_{\beta,B}$ or $\mathcal{M}_{\alpha,A} \cap \mathcal{L}_{\beta,B}$ and a given level ℓ. The first such bound is a slightly stronger version of [28, Lemma 3.12] or [59], as we assume a finite log-moment instead of a finite polynomial moment.

Lemma 2.22. *There exists $C = C(\beta, B, \ell) \in (0,\infty)$ such that for all $\rho \in \mathcal{L}_{\beta,B}$ and $T \geq 1$:*

$$\mathbb{P}\left(\sup_{[0,T]} |f_\rho| < \ell\right) \geq e^{-CT}, \ \textit{or equivalently } \psi_\rho^\ell(T) \leq C.$$

Proof. By Khatri-Sidak's inequality (Cor. 2.4) we have, for any $h > 0$,

$$\mathbb{P}\left(\sup_{[0,T]} |f_\rho| < \ell\right) \geq \mathbb{P}\left(\sup_{[0,h]} |f_\rho| < \ell\right)^{T/h}. \tag{2.6}$$

Combining Lemma 2.17 and Lemma 2.19 we have

$$\mathbb{E}\sup_{[0,h]} |f_\rho| \leq C \log^{-\beta/2}\left(\tfrac{1}{h}\right).$$

Consequently there exists h, depending on ℓ, β, B, such that $\mathbb{E}\sup_{[0,h]} |f_\rho| < \frac{\ell}{2}$. An application of Markov's inequality gives $\mathbb{P}\left(\sup_{[0,T]} |f_\rho| > \ell\right) \leq \frac{1}{2}$. For this h, inequality (2.6) gives

$$\mathbb{P}\left(\sup_{[0,T]} |f_\rho| < \ell\right) \geq 2^{-T/h}.$$

□

The next result is a somewhat more general version of [26, Theorem 2].

Lemma 2.23. *There exists $C \in (0,\infty)$ such that for all $\rho \in \mathcal{M}_{\alpha,A} \cap \mathcal{L}_{\beta,B}$ and all $T \geq \frac{1}{A}$:*

$$\mathcal{P}_\rho^\ell(T) \geq e^{-CT}, \quad \textit{or equivalently,} \quad \theta_\rho^\ell(T) \leq C.$$

Proof. Recall the notation $\rho_L = \rho|_{[-L,L]}$ for the restriction of a measure ρ to the interval $[-L, L]$. For a fixed (arbitrary) $m > 0$ we have

$$\theta_\rho^\ell(T) \leq \theta_{\rho_{1/T}}^{\ell+m}(T) + \psi_{\rho-\rho_{1/T}}^m(T) \leq \theta_{\rho_{1/T}}^{\ell+m}(T) + \psi_\rho^m(1),$$

where the first inequality holds by Lemma 2.12(b), and the second one follows from the inequalities by Anderson (Proposition 2.16) and Khatri-Sidak (Proposition 2.3). The covariance function corresponding to $\rho_{1/T}$ is

$$\mathcal{F}[\rho_{1/T}](t) = \int_{-1/T}^{1/T} \cos(\lambda t) d\rho(\lambda) \geq \rho([-\tfrac{1}{T}, \tfrac{1}{T}]) \cos(\tfrac{1}{T} t).$$

Notice that the RHS of the last inequality is the covariance function of the process

$$a_T \cos(\tfrac{1}{T} t) + b_T \sin(\tfrac{1}{T} t), \qquad \text{where } a_T, b_T \sim \mathcal{N}\left(0, \rho\left([-\tfrac{1}{T}, \tfrac{1}{T}]\right)\right) \text{ are i.i.d.}$$

By Slepian's inequality (Proposition 2.13) we have

$$\begin{aligned}\theta^{\ell+m}_{\rho_{1/T}}(T) &\le -\frac{1}{T}\log\mathbb{P}\left(\forall t\in[0,T]:\ a_T\cos(\tfrac{1}{T}t)+b_T\sin(\tfrac{1}{T}t)>\ell+m\right)\\ &\le -\frac{1}{T}\log\mathbb{P}(a_T>\tfrac{\ell+m}{\cos 1})-\frac{1}{T}\log\mathbb{P}\left(b_T>0\right)\\ &\le -\frac{1}{T}\log\mathbb{P}\left(\sqrt{\rho([-\tfrac{1}{T},\tfrac{1}{T}])}\,Z>\tfrac{\ell+m}{\cos 1}\right)+\frac{\log 2}{T},\end{aligned}$$

where $Z\sim\mathcal{N}(0,1)$. We have thus proved that for any $\rho\in\mathcal{L}_{\beta,B}$,

$$\limsup_{T\to\infty}\theta^{\ell}_{\rho}(T)\le\psi^m_\rho(1)-\limsup_{T\to\infty}\frac{1}{T}\log\mathbb{P}\left(\sqrt{\rho([-\tfrac{1}{T},\tfrac{1}{T}])}\,Z>\tfrac{\ell+m}{\cos 1}\right).\tag{2.7}$$

By Lemma 2.22, it holds that $\psi^m_\rho(1)\le C(\beta,B)<\infty$. To see that the second term is bounded, recall that $\rho([-\frac{1}{T},\frac{1}{T}])\ge\frac{2\alpha}{T}$ whenever $T\ge\frac{1}{A}$, and thus

$$\limsup_{T\to\infty}\theta^{\ell+m}_{\rho_{1/T}}(T)\le-\frac{1}{T}\log\mathbb{P}\left(Z>\frac{\ell+m}{\sqrt{2\alpha}\cos 1}\sqrt{T}\right)\le\frac{1}{2\alpha}\left(\frac{\ell+m}{\cos 1}\right)^2.$$

The last step uses Lemma 2.1 and assumes that m is such that $\frac{\ell+m}{\sqrt{2\alpha}\cos 1}>2$. ☐

The reverse direction of Lemma 2.23 requires some extra assumptions.

Lemma 2.24. *Let $\rho\in\mathcal{L}_{\beta,B}$ and $\ell\in\mathbb{R}$. Suppose ρ is such that $\rho((-x,x))\le 2\alpha' x$ for all $x\in(0,A')$. Suppose further that $d\rho\ge m\boldsymbol{1}_E(\lambda)d\lambda$ where $d\lambda$ is the Lebesgue measure, $m>0$ and E is a Lebesgue-measurable set of positive measure. Then there exist $C=C(\alpha',A',\beta,B,\ell,m,|E|)>0$ such that for all $T>1$:*

$$\mathcal{P}^{\ell}_{\rho}(T)\le e^{-CT},\quad \textit{or equivalently,}\quad \theta^{\ell}_{\rho}(T)\ge C.$$

If $\ell\ge 0$, then the constant C satisfies $\lim_{\alpha'\to 0}C(\alpha')=\infty$.

Proof. This inequality was proved in [28, Prop. 3], which is a corollary of Theorem 5.1 there. The assumption throughout that paper is that $\int|\lambda|^{\delta}d\rho(\lambda)<\infty$ for some $\delta>0$ and that $\ell=0$; however, the proof in our case (corresponding to $\gamma=1$ and $b=\alpha'$ there) applies as soon as $\rho\in\mathcal{L}$ and for any level $\ell\in\mathbb{R}$. The dependence of C on the parameter α' follows from [28, Remark 2]. ☐

Proposition 2.25. *Let $\rho\in\mathcal{L}$ and $\ell\in\mathbb{R}$. If $\lim\limits_{\varepsilon\to 0}\frac{\rho(-\varepsilon,\varepsilon)}{2\varepsilon}=\infty$ then $\theta^{\ell}_{\rho}=0$.*

Proof of Proposition 2.25. Fix a large parameter $m>0$. Using Lemma 2.1 and the fact that $\lim_{T\to\infty}T\rho([-\frac{1}{T},\frac{1}{T}])=\infty$, we deduce that

$$\limsup_{T\to\infty}\frac{1}{T}\log\mathbb{P}\left(\sqrt{\rho([-\tfrac{1}{T},\tfrac{1}{T}])}\,Z>\tfrac{\ell+m}{\cos 1}\right)=0.$$

Plugging this into (2.7) we obtain

$$0\le\limsup_{T\to\infty}\theta^{\ell}_{\rho}(T)\le\psi^m_\rho(1).$$

The desired conclusion follows on letting $m\to\infty$. ☐

3 Proofs of the comparison lemmata

3.1 Proof of Lemma 1.1: continuity in levels

Fix $T > 0$, $\delta > 0$. The inequality $\mathcal{P}_\mu^{\ell+\delta}(T) \le \mathcal{P}_\mu^{\ell-\delta}(T)$, or equivalently $\theta_\mu^{\ell-\delta}(T) \le \theta_\mu^{\ell+\delta}(T)$, follows from inclusion of events. It remains to bound the difference $\theta_\mu^{\ell+\delta}(T) - \theta_\mu^{\ell-\delta}(T)$. Let

$$\varphi_T := \frac{1}{\sigma_T} 1\!\!1_{[-\frac{1}{T}, \frac{1}{T}]}, \quad \text{where } \sigma_T^2 := \int_{-1/T}^{1/T} d\mu(\lambda),$$

so that $\varphi_T \in \mathcal{L}_\mu^2$ and $\|\varphi_T\|_{\mathcal{L}_\mu^2} = 1$. Denote $\psi_T(x) = \int_{\mathbb{R}} e^{-i\lambda x} \varphi_T(\lambda) d\mu(\lambda)$ and invoke Corollary 2.10 to obtain the decomposition

$$f_\mu(t) \overset{d}{=} \zeta \psi_T(t) \oplus R_T(t),$$

where $\zeta \sim N(0,1)$ and R_T is a centered Gaussian process on $[0,T]$. Note that on the interval $[0,T]$ the function ψ_T satisfies

$$\psi_T(x) = \frac{1}{\sigma_T} \int_{-\frac{1}{T}}^{\frac{1}{T}} \cos(x\lambda) d\mu(\lambda) \ge \frac{1}{\sigma_T} \cos\left(\tfrac{x}{T}\right) \sigma_T^2 \ge \frac{1}{2}\sigma_T \ge \frac{1}{c\sqrt{T}},$$

for $T \ge \frac{1}{A}$ and $c = \sqrt{\frac{2}{\alpha}}$.

Denote $\widetilde{f}(x) = (\zeta - c\delta\sqrt{T})\psi_T(x) \oplus R_T(x)$ and observe that

$$f(x) = \widetilde{f}(x) + c\delta\sqrt{T}\psi_T(x),$$

and $\widetilde{f}(x) \overset{d}{=} \widetilde{\zeta}\psi_T(x) \oplus R_T(x)$ with $\widetilde{\zeta} \sim \mathcal{N}(-c\delta\sqrt{T}, 1)$. Since $2\delta \cdot c\sqrt{T}\psi_T(x) \ge 2\delta$, we have

$$\mathcal{P}_\mu^{\ell+\delta}(T) = \mathbb{P}(f > \ell + \delta \text{ on } [0,T]) \ge \mathbb{P}(\widetilde{f} > \ell - \delta \text{ on } [0,T]). \tag{3.1}$$

By Lemma 2.23, there exists $M \in (0,\infty)$ such that $\mathcal{P}_\mu^{\ell}(T) \ge 4e^{-MT/2}$. Taking $M \ge 1$, invoke Part (a) of Lemma 2.1 for $T \ge 4$, to obtain $2e^{-MT/2} \ge \mathbb{P}(|\zeta| \ge \sqrt{MT})$, getting

$$\mathcal{P}_\mu^{\ell-\delta}(T) \ge \mathcal{P}_\mu^{\ell}(T) \ge 2\mathbb{P}(|\zeta| \ge \sqrt{MT}). \tag{3.2}$$

Starting from (3.1) and using Radon-Nikodim derivative estimate we obtain

$$\begin{aligned}
\mathcal{P}_\mu^{\ell+\delta}(T) &\ge \mathbb{P}\Big(\widetilde{f} \ge \ell - \delta \text{ on } [0,T], |\zeta| < \sqrt{MT}\Big) \\
&\ge \inf_{|x| \le \sqrt{MT}} \left|\frac{d\zeta}{d\widetilde{\zeta}}\right| \mathbb{P}\Big(f \ge \ell - \delta \text{ on } [0,T], |\zeta| < \sqrt{MT}\Big) \\
&\ge \inf_{|x| \le \sqrt{MT}} e^{c\delta\sqrt{T}\cdot x} \Big(\mathbb{P}\Big(f \ge \ell - \delta \text{ on } [0,T]\Big) - \mathbb{P}(|\zeta| \ge \sqrt{MT})\Big) \\
&\ge \tfrac{1}{2} e^{-c\delta\sqrt{MT}} \mathbb{P}\Big(f \ge \ell - \delta \text{ on } [0,T]\Big) = \tfrac{1}{2} e^{-c\delta\sqrt{MT}} \mathcal{P}_\mu^{\ell-\delta}(T),
\end{aligned}$$

where the last inequality is due to (3.2). Taking logarithm, this yields

$$\log \mathcal{P}_\mu^{\ell-\delta}(T) - \log \mathcal{P}_\mu^{\ell+\delta}(T) \leq C\delta T,$$

for some $C > 0$ and all $T > \max(4, \frac{1}{A})$, as required.

3.2 Proof of Lemma 1.2: smoothing increases persistence

Denoting $H = \widehat{h}$, the assumptions of the lemma are $H \geq 0$, $\int_{\mathbb{R}} H = 1$ and $\operatorname{sprt} H \subseteq [-\frac{a}{2}, \frac{a}{2}]$. By Observation 2.7, the measure $h^2\nu$ is the spectral measure of the GSP $f_\nu * H$. Therefore our objective is to show that

$$\mathbb{P}\Big(f_\mu \oplus f_\nu > \ell \text{ on } [0, T+a]\Big) \leq \mathbb{P}\Big(f_\mu \oplus (f_\nu * H) > \ell \text{ on } [0, T]\Big). \tag{3.3}$$

Gaussian measures are well-known to be log-concave (see [12, Example 2.3]). In particular,

$$\int H(s) \log \mathbb{P}\Big(X(t) + v(s,t) > \ell,\ \forall t \in [0,T]\Big)\, ds \leq \log \mathbb{P}\Big(X(t) + \int H(s) v(s,t) ds > \ell,\ \forall t \in [0,T]\Big),$$

where $v(s,t) : \mathbb{R} \times [0,T] \to \mathbb{R}$ is continuous in t for every fixed $s \in \mathbb{R}$. Thus, given a continuous function $v : \mathbb{R} \to \mathbb{R}$, we may apply this with $v(s,t) = v(t-s)$ to obtain

$$(\star) := \int H(s) \log \mathbb{P}\Big(X(t) + v(t-s) > \ell,\ \forall t \in [0,T]\Big) \leq \log \mathbb{P}\Big(X(t) + (H * v)(t) > \ell,\ \forall t \in [0,T]\Big).$$

When X is stationary, we have

$$\begin{aligned}
(\star) &= \int_{\mathbb{R}} H(s) \log \mathbb{P}\Big(X(t+s) + v(t) > \ell,\ \forall t \in [-s, T-s]\Big)\ ds \\
&\geq \int_{\mathbb{R}} H(s) \log \mathbb{P}\Big(X(t+s) + v(t) > \ell,\ \forall t \in [-\tfrac{a}{2}, T + \tfrac{a}{2}]\Big)\ ds \\
&= \int_{\mathbb{R}} H(s) \log \mathbb{P}\Big(X(t) + v(t) > \ell,\ \forall t \in [-\tfrac{a}{2}, T + \tfrac{a}{2}]\Big)\ ds \\
&= \log \mathbb{P}\Big(X(t) + v(t) > \ell,\ \forall t \in [-\tfrac{a}{2}, T + \tfrac{a}{2}]\Big),
\end{aligned}$$

where the last line uses $h(0) = \int_{\mathbb{R}} H(s) ds = 1$. Putting these together, we obtain

$$\mathbb{P}\Big(X(t) + v(t) > \ell,\ \forall t \in [-\tfrac{a}{2}, T + \tfrac{a}{2}]\Big) \leq \mathbb{P}\Big(X(t) + (H * v)(t) > \ell,\ \forall t \in [0,T]\Big).$$

Given a real valued stochastic process Y, independent of X, with almost-surely continuous path, we may apply this to deduce to deduce that

$$\mathbb{P}\Big(X(t) \oplus Y(t) > \ell,\ \forall t \in [-\tfrac{a}{2}, T + \tfrac{a}{2}]\Big) \leq \mathbb{P}\Big(X(t) \oplus (H * Y)(t) > \ell,\ \forall t \in [0,T]\Big).$$

When Y is also stationary, we may replace the interval $[-\frac{a}{2}, T + \frac{a}{2}]$ in the last inequality with $[0, T+a]$. Applying this to $X = f_\mu$ and $Y = f_\nu$ we obtain (3.3) as required.

3.3 Proof of Lemma 1.3: continuity in measure

3.3.1 Proof of Part I

Let $\delta > 0$ (to be chosen later). By Lemma 2.12(b) we have:

$$\theta^{\ell}_{\mu+\nu}(T) - \theta^{\ell+\delta}_{\mu}(T) \leq \psi^{\delta}_{\nu}(T). \tag{3.4}$$

By Corollary 2.4, we have for $T > 1$

$$\psi^{\delta}_{\nu}(T) = -\frac{1}{T}\log \mathbb{P}\Big(|f_\nu| < \delta \text{ on } [0,T]\Big) \leq -2\log \mathbb{P}\Big(\sup_{[0,1]}|f_\nu| < \delta\Big). \tag{3.5}$$

Next apply Lemma 2.20 with $u = \sqrt{2\log(1/\varepsilon)}$, using the fact that $\nu(\mathbb{R}) < \varepsilon$, to get

$$-\log \mathbb{P}\Big(\sup_{[0,1]}|f_\nu| < (C_1\sqrt{2\log(1/\varepsilon)} + C_2)\varepsilon^{\frac{\beta}{2(1+\beta)}}\Big) \leq -\log(1-2\varepsilon) \leq 2\varepsilon,$$

where $C_i = C_i(\beta, B)$ for $i \in \{1,2\}$. Thus choosing $\delta(\varepsilon) = (C_1\sqrt{2\log(1/\varepsilon)} + C_2)\varepsilon^{\frac{\beta}{2(1+\beta)}}$ we have $\lim_{\varepsilon\to 0}\delta(\varepsilon) = 0$ and

$$-\log \mathbb{P}\Big(\sup_{[0,1]}|f_\nu| < \delta(\varepsilon)\Big) \leq 4\varepsilon. \tag{3.6}$$

Combining (3.4), (3.5) and (3.6) we obtain:

$$\theta^{\ell}_{\mu+\nu}(T) - \theta^{\ell+\delta(\varepsilon)}_{\mu}(T) \leq 4\varepsilon.$$

By Lemma 1.1,

$$\theta^{\ell+\delta(\varepsilon)}_{\mu}(T) - \theta^{\ell}_{\mu}(T) \leq C\delta(\varepsilon),$$

where $C > 0$ and $T > \max(4, \frac{1}{A})$. The last two inequalities together yield the desired conclusion.

3.4 An auxiliary result

For proving Lemma 1.3(II) we shall need the following proposition.

Proposition 3.1. *For any $\eta \in (0, \frac{1}{2})$ there exists $T_0(\eta)$ with the following property. Let $m \geq 0$, $T > \max\{T_0, \frac{1}{A}\}$ and $L \in \left(\max\left(\frac{4m}{\eta^3}, \frac{1}{\eta^2}\right), T\right)$. Let $\mu \in \mathcal{M}_{\alpha,A} \cap \mathcal{L}_{\beta,B}$ and $\nu \in \mathcal{L}_{\beta,B}$ be such that $\nu((-\frac{L}{T}, \frac{L}{T})) \leq m\frac{L}{T}$. Then*

$$\theta^{\ell}_{\mu+h^2\nu}(T(1-\eta)) \leq \theta^{\ell}_{\mu+\nu}(T) + C_\eta,$$

where $h(x) = h_{L/T}(x) = \max\left(0, 1 - \frac{T}{L}|x|\right)$ and $\lim_{\eta\to 0} C_\eta = 0$.

Proof. Fix η, m, L as in the proposition, and let $T > 0$. Denote $H(x) = \widehat{h}(x) = \frac{L}{T}\mathrm{sinc}^2\left(\frac{L}{T}x\right)$ (where $\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$). Denote

$$U(x) := H(x)\mathbb{1}_{\{|x|<\eta T\}}, \qquad V(x) := H(x)\mathbb{1}_{\{|x|\geq\eta T\}},$$

$$u := \mathcal{F}^{-1}[U], \qquad v := \mathcal{F}^{-1}[V].$$

By Observation 2.7,

$$f_\nu * H \overset{\mathrm{d}}{=} f_{h^2\nu}, \qquad f_\nu * U \overset{\mathrm{d}}{=} f_{u^2\nu}, \qquad f_\nu * V \overset{\mathrm{d}}{=} f_{v^2\nu},$$

and by spectral decomposition (Lemma 2.11),

$$f_{h^2\nu} \overset{\mathrm{d}}{=} f_{u^2\nu} \oplus f_{v^2\nu}. \tag{3.7}$$

Equipped with Lemma 1.1, in order to establish the proposition it would suffice to show that there exists some $\delta = \delta(\eta)$, satisfying $\lim_{\eta\to 0}\delta(\eta) = 0$, for which

$$\theta^{\ell-\delta}_{\mu+h^2\nu}(T(1-\eta)) \le \theta^{\ell}_{\mu+\nu}(T) + C_\eta,$$

where $\lim_{\eta\to 0} C_\eta = 0$. Denoting $I_T = [0,(1-\eta)T]$, we have

$$\begin{aligned}
\mathbb{P}\Big(f_\mu \oplus f_{h^2\nu} > \ell - \delta \text{ on } I_T\Big) &= \mathbb{P}\Big(f_\mu \oplus f_{u^2\nu} \oplus f_{v^2\nu} > \ell - \delta \text{ on } I_T\Big) \\
&\ge \mathbb{P}\Big(f_\mu \oplus f_{u^2\nu} > \ell,\ \sup|f_{v^2\nu}| \le \delta \text{ on } I_T\Big), \\
&\ge \mathbb{P}\Big(f_\mu \oplus f_{u^2\nu} > \ell \text{ on } I_T\Big) - \mathbb{P}\Big(\sup_{I_T}|f_{v^2\nu}| > \delta\Big).
\end{aligned} \tag{3.8}$$

To establish the proposition it would therefore suffice to show

$$\mathbb{P}\Big(f_\mu \oplus f_{u^2\nu} > \ell \text{ on } I_T\Big) \ge \mathbb{P}\Big(f_\mu \oplus f_\nu > \ell \text{ on } [0,T]\Big)e^{\widetilde{C}_\eta T}, \tag{3.9}$$

$$\mathbb{P}\Big(\sup_{I_T}|f_{v^2\nu}| > \delta\Big) \le \tfrac{1}{2}\mathbb{P}\Big(f_\mu \oplus f_{u^2\nu} > \ell \text{ on } I_T\Big), \tag{3.10}$$

where $\lim_{\eta\to 0}\widetilde{C}_\eta = 0$. To see (3.9) we first note that U is compactly supported on $[-\eta T, \eta T]$. Hence, by Lemma 1.2, we have

$$\mathbb{P}\Big(f_\mu \oplus f_{u^2\nu} > \ell \text{ on } I_T\Big) \ge \mathbb{P}\Big(f_\mu \oplus f_{a\nu} > \ell \text{ on } I_T\Big), \tag{3.11}$$

where $a = a(L,\eta) = \int_{\mathbb{R}} U = \int_{-L\eta}^{L\eta}\operatorname{sinc}^2(x)dx$. Since $L > \frac{1}{\eta^2}$ we have

$$a(\eta) = 1 - 2\int_{1/\eta}^{\infty}\operatorname{sinc}^2(x)dx \ge 1 - \frac{2}{\pi^2}\eta,$$

so that

$$\begin{aligned}
\log\mathbb{P}\Big(f_\mu \oplus f_{a\nu} > \ell \text{ on } [0,T]\Big) &= \log\mathbb{P}\Big(f_{a^{-1}\mu+\nu} > \frac{\ell}{\sqrt{a}} \text{ on } [0,T]\Big) \\
&\ge \log\mathbb{P}\Big(f_{\mu+\nu} > \tfrac{\ell}{\sqrt{a}} \text{ on } [0,T]\Big) - C^{(1)}_\eta T \qquad \text{by Lemma 1.3(I)} \\
&\ge \log\mathbb{P}\Big(f_{\mu+\nu} > \ell \text{ on } [0,T]\Big) - C^{(2)}_\eta T, \qquad \text{by Lemma 1.1}
\end{aligned}$$

where $\lim_{\eta\to 0} C^{(i)}_\eta = 0$ for $i = 1,2$. This establishes (3.9).

To see (3.10), it is our purpose to apply Lemma 2.21. Since h is supported on $[-\frac{L}{T}, \frac{L}{T}]$, we deduce that the same is true for u and v, by (3.7). We compute

$$\begin{aligned}
\mathrm{var}\left((f_{v^2\nu})(t)\right) &= \int_{-L/T}^{L/T} v^2 d\nu \\
&\leq (\sup_{\mathbb{R}} |v|)^2 \int_{-L/T}^{L/T} d\nu \leq \left(\int |V|\right)^2 m\frac{L}{T} \\
&= m\frac{L}{T}\left(2\frac{L}{T}\int_{\eta T}^{\infty} \mathrm{sinc}^2\left(\frac{L}{T}x\right) dx\right)^2 = 4m\frac{L}{T}\left(\int_{\eta L}^{\infty} \mathrm{sinc}^2(y)\, dy\right)^2 \\
&\leq 4m\frac{L}{T}\left(\frac{1}{\eta L}\right)^2 = \frac{4m}{\eta^2 L}\cdot\frac{1}{T} \leq \frac{\eta}{T},
\end{aligned} \tag{3.12}$$

where the last step uses $L > \frac{4m}{\eta^3}$. Applying Observation 2.18, we have

$$\mathrm{var}\left((f_{v^2\nu})(t) - (f_{v^2\nu}(0)\right) \leq \left(\frac{L}{T}\right)^2 \cdot \frac{\eta}{T}\cdot t^2 \leq t^2. \tag{3.13}$$

Plugging (3.12) and (3.13) into Lemma 2.21 we obtain

$$\mathbb{P}\left(\sup_{I_T} |f_{v^2\nu}| > \delta\right) \leq 2e^{-\frac{\delta^2(1-\eta)}{8\eta}T} \leq 2e^{-\frac{\delta^2}{16\eta}T},$$

provided that $L > \frac{4m}{\eta^3}$ and $T > T_0(\eta)$. Next, we use the existence of an a priori bound,

$$\mathbb{P}\Big(f_\mu \oplus (f_\nu * H_T) > \ell \text{ on } I_T\Big) \geq e^{-MT},$$

which is guaranteed to hold for all $T \geq \frac{2}{A}$ and some $M \in (0,\infty)$, by Lemma 2.23. Setting $\delta = \sqrt{32\eta M}$ we have

$$\mathbb{P}\Big(\sup_{I_T} |f_\nu * V_T| > \delta\Big) \leq e^{-MT} \leq \mathbb{P}\Big(f_\mu \oplus (f_{h^2\nu}) > \ell - \delta \text{ on } I_T\Big), \tag{3.14}$$

which establishes (3.10) and thus the proposition. □

For the next result, recall that $\mu_L = \mu|_{[-L,L]}$ is the restriction of the measure μ onto $[-L, L]$.

Corollary 3.2 (truncation)**.** *For any $\eta \in (0, \frac{1}{2})$ there exists $T_0(\eta)$ with the following property. For any $T > \max\{T_0, \frac{1}{A}\}$, $L \in \left(\frac{1}{\eta^2}, T\right)$, $\mu \in \mathcal{M}_{\alpha,A} \cap \mathcal{L}_{\beta,B}$ and $\nu \in \mathcal{L}_{\beta,B}$ it holds that:*

$$\theta^\ell_{\mu+\nu_{\frac{L}{T}}}(T(1-\eta)) \leq \theta^\ell_{\mu+\nu}(T) + C_\eta,$$

where $\lim_{\eta\to 0} C_\eta = 0$.

Proof. Note that $(\nu - \nu|_{L/T})$ vanishes on the interval $(-\frac{L}{T}, \frac{L}{T})$. By Proposition 3.1 with $m = 0$, we have

$$\theta^{\ell}_{\mu+\nu_{\frac{L}{T}}}(T(1-\eta)) = \theta^{\ell}_{\mu+\nu_{\frac{L}{T}}+h^2(\nu-\nu_{\frac{L}{T}})}(T(1-\eta)) \leq \theta^{\ell}_{\mu+\nu}(T) + C_\eta$$

for all $L > \eta^{-2}$ and $T > \max\{T_0(\eta), \frac{1}{A}\}$. The first equality in the last display uses the fact that $h^2(\nu - \nu_{\frac{L}{T}})$ is the identically 0 measure. □

3.5 Proof of Part II

By Lemma 2.23, there exists $M \in (0, \infty)$ such that

$$\forall T > \tfrac{1}{A}: \quad \mathbb{P}\left(\inf_{[0,T]} f_\mu > \ell - 1\right) \geq e^{-MT}. \tag{3.15}$$

Choose $\eta = (\varepsilon M)^{1/4}$ and $L = \sqrt{\frac{2}{\varepsilon M}}$, and note that $L > \frac{1}{\eta^2}$. By Corollary 3.2, we have

$$\theta^{\ell}_{\mu+\nu_{\frac{L}{T}}}(T(1-\eta)) \leq \theta^{\ell}_{\mu+\nu}(T) + C^{(1)}_\varepsilon, \tag{3.16}$$

for all $T > \max\{T^{(1)}_0(\varepsilon), \frac{1}{A}\}$, where $\lim_{\varepsilon\to 0} C^{(1)}_\varepsilon = 0$.

Denoting $m = \nu([-\frac{L}{T}, \frac{L}{T}])$ we observe that $\mathcal{F}[\mu + \nu_{\frac{L}{T}}] \leq \mathcal{F}[\mu + m\delta_0]$, so that Slepian's inequality (Proposition 2.13) yields

$$\theta^{\ell}_{\mu+m\delta_0}(T(1-\eta)) \leq \theta^{\ell}_{\mu+\nu_{\frac{L}{T}}}(T(1-\eta)). \tag{3.17}$$

Proceeding to estimate the LHS of (3.17), with $Z \sim N(0,1)$ we have

$$\begin{aligned}
\mathbb{P}\left(\inf_{[0,T(1-\eta)]} f_{\mu+m\delta_0} > \ell\right) &= \mathbb{P}\left(\inf_{[0,T(1-\eta)]} f_\mu \oplus \sqrt{m}Z > \ell\right) \\
&\leq \mathbb{P}\left(Z > \left(\frac{T}{m}\right)^{1/4}\right) + \mathbb{P}\left(\inf_{[0,T(1-\eta)]} f_\mu > \ell - (mT)^{1/4}\right) \\
&\leq \mathbb{P}\left(Z > \sqrt{\frac{T}{\sqrt{\varepsilon L}}}\right) + \mathbb{P}\left(\inf_{[0,T(1-\eta)]} f_\mu > \ell - \sqrt{\varepsilon L}\right),
\end{aligned} \tag{3.18}$$

where the last inequality uses the given assumption that $m \leq \varepsilon\frac{L}{T}$. Since $\varepsilon L < 1$ for all $\varepsilon < \varepsilon_0$, we have, by (3.15), that the second term in the RHS of (3.18) is bounded below by e^{-MT} for $T \geq 2\max\{4, \frac{1}{A}\}$. By Lemma 2.1, the first term in the RHS of (3.18) is bounded above by $e^{-\frac{T}{2\sqrt{\varepsilon L}}}$, for all $T > 4\sqrt{\varepsilon L}$. Thus for $\varepsilon < \varepsilon_0$ and $T > 2\max\{4, \frac{1}{A}\}$ the second term is larger than the first, and we get

$$\mathbb{P}\left(\inf_{[0,T(1-\eta)]} f_{\mu+m\delta_0} > \ell\right) \leq 2\mathbb{P}\left(\inf_{[0,T(1-\eta)]} f_\mu > \ell - (\varepsilon L)^{1/4}\right).$$

By taking log and dividing by T we obtain, for $T \geq \max\{T_0(\varepsilon), \frac{2}{A}\}$, that

$$\theta^{\ell}_{\mu+m\delta_0}(T(1-\eta)) \geq \theta^{\ell-\sqrt{\varepsilon L}}_{\mu}(T(1-\eta)) + \tfrac{\log 2}{T}. \tag{3.19}$$

Finally, using Lemma 1.1, we have

$$\theta_\mu^\ell\left(T(1-\eta)\right) \le \theta_\mu^{\ell-\sqrt{\varepsilon L}}\left(T(1-\eta)\right) + C_\varepsilon^{(3)} \tag{3.20}$$

for some $C_\varepsilon^{(3)}$ such that $\lim_{\varepsilon\to 0} C_\varepsilon^{(3)} = 0$. Combining (3.16), (3.17), (3.19) and (3.20) we get

$$\theta_\mu^\ell\left(T(1-\eta)\right) \le \theta_{\mu+\nu}^\ell\left(T\right) + C_\varepsilon$$

for $T > \max\{T_0(\varepsilon), \frac{2}{A}\}$, which is the desired conclusion.

4 Ball exponent of singular measures

In this section we establish Theorem 2. We rely on the following proposition.

Proposition 4.1. *Let ρ be a purely-singular measure supported on $[-\pi,\pi]$. Then any $\delta > 0$ satisfies*

$$\psi_{\rho;1}^\delta = \lim_{T\to\infty} \frac{1}{T} \log \mathbb{P}\left(\sup_{j\in[0,T]\cap\mathbb{Z}} |f_\rho(j)| < \delta\right) = 0.$$

Using this, we prove Theorem 2 in Section 4.1. We discuss a useful approximation method of a GSP with compactly supported spectral measure in Section 4.2. This method is used to prove Proposition 4.1 in Section 4.3.

4.1 Proof of Theorem 2: characterization of vanishing ball exponent

Let $\rho \in \mathcal{L}$ and $\ell > 0$. Assume first that ρ is purely singular. For any given $\varepsilon > 0$, we define:

$$f_{\rho;\varepsilon}(t) = f_\rho\left(\varepsilon \left\lfloor \tfrac{t}{\varepsilon} \right\rfloor\right).$$

Note that $f_{\rho;\varepsilon}$ is a centered Gaussian process, though it is not stationary. By Khatri-Sidak's inequality (Proposition 2.3):

$$\psi_\rho^\ell(T) \le \psi_{\rho;\varepsilon}^{\ell/2}(T) - \log \mathbb{P}\left(\sup_{[0,1]} |f_\rho - f_{\rho;\varepsilon}| < \tfrac{\ell}{2}\right)$$

By letting $T \to \infty$ and using Proposition 4.1 we get

$$0 \le \psi_\rho^\ell \le \psi_{\rho;\varepsilon}^{\ell/2} - \log \mathbb{P}\left(\sup_{[0,1]} |f_\rho - f_{\rho;\varepsilon}| < \tfrac{\ell}{2}\right) = -\log \mathbb{P}\left(\sup_{[0,1]} |f_\rho - f_{\rho;\varepsilon}| < \tfrac{\ell}{2}\right).$$

The desired conclusion follows by letting $\varepsilon \to 0$, and noting that $\sup |f_\rho - f_{\rho;\varepsilon}|$ converges to 0 almost surely (by continuity of sample paths).

Assume now that ρ is not purely singular, that is, $\rho_{ac} \ne 0$. By [28, Claim 3.4], there exist $\Lambda = \{\lambda_n\}$ of positive density a, and a constant $b > 0$, such that $f(\lambda_n) \overset{d}{=} bZ_n \oplus g_n$, where Z_n are i.i.d. standard normal random variables and g_n is a Gaussian process on $\mathbb{Z}$. By Anderson's inequality (Proposition 2.16), this implies

$$\mathbb{P}\left(\sup_{[0,T]}|f_\rho| < \ell\right) \le \mathbb{P}\left(\max_{\lambda\in\Lambda\cap[0,T]}|f_\rho(\lambda)| < \ell\right)$$
$$\le \mathbb{P}\left(\max_{n\in\mathbb{N}\cap[0,\frac{a}{2}T]}|bZ_n| \le \ell\right) = \mathbb{P}\left(|Z| \le \frac{\ell}{b}\right)^{\lfloor\frac{a}{2}T\rfloor} \le e^{-CT},$$

where $C > 0$ depends on ℓ and the spectral measure ρ. Thus $\psi_\rho^\ell \ge C > 0$, as required.

4.2 A spectral approximation method

The following lemma presents an approximation method for GSPs with compactly supported spectral measure.

Lemma 4.2. *Let ρ be a spectral measure supported on $[-D, D]$. For $n \in \mathbb{N}$ we denote the intervals $I_1^n = [0, \frac{D}{n}]$, $I_{-1}^n = [-\frac{D}{n}, 0)$ and $I_{\pm j}^n = \pm((j-1)\frac{D}{n}, j\frac{D}{n}]$ for $j \in \{2, \ldots, n\}$, as well as*

$$C_j(t) = \frac{1}{\rho(I_j)}\int_{I_j}\cos(\lambda t)d\rho(\lambda), \quad S_j(t) = \frac{1}{\rho(I_j)}\int_{I_j}\sin(\lambda t)d\rho(\lambda). \tag{4.1}$$

Then,

$$f_\rho(t) \overset{d}{=} \sum_{j=1}^{n}\sqrt{\rho(I_j \cup I_{-j})}\Big(\zeta_j C_j(t) \oplus \eta_j S_j(t)\Big) \oplus R_n(t), \tag{4.2}$$

where $\{\zeta_j\}_{j=1}^n \cup \{\eta_j\}_{j=1}^n$ are i.i.d. $\mathcal{N}(0,1)$-distributed random variables, and $R_n(t)$ is a Gaussian process independent of them for which

$$\sup_{t\in[0,T]}\operatorname{var}(R_n(t)) \le \frac{1}{2}\left(\frac{DT}{n}\right)^2\rho([-D,D]) \tag{4.3}$$

and for any $h \in \mathbb{R}$,

$$\operatorname{var}(R_n(t) - R_n(t+h)) \le D^2\rho([-D,D])h^2. \tag{4.4}$$

Proof. Notice that

$$C_j(t) = \mathcal{F}\left[\tfrac{1}{2\rho(I_j)}\left(1\!\!1_{I_j} + 1\!\!1_{I_{-j}}\right)\, d\rho\right](t), \quad S_j(t) = \mathcal{F}\left[\tfrac{1}{2\rho(I_j)}\left(1\!\!1_{I_j} - 1\!\!1_{I_{-j}}\right)\, d\rho\right](t),$$

and

$$\left\|\frac{1}{2\rho(I_j)}(1\!\!1_{I_j} \pm 1\!\!1_{I_{-j}})\right\|_{L^2_\rho} = \frac{1}{2\rho(I_j)}\sqrt{\int(1\!\!1^2_{I_j} + 1\!\!1^2_{-I_j})d\rho} = \frac{1}{\sqrt{2\rho(I_j)}}.$$

Since $\{1\!\!1_{I_j} \pm 1\!\!1_{-I_j}\}_{j=0}^n$ is an orthogonal system in $\mathcal{L}^2_\rho$, by the Hilbert decomposition (Lemma 2.9) we have the representation (4.2). Let $t \in [0,T]$. By (4.2),

$$\begin{aligned}\operatorname{var}\big(R_n(t)) &= \rho([-D,D]) - \operatorname{var}\left(\sum_{j=0}^{n}\sqrt{\rho(I_j \cup I_{-j})}\Big(\zeta_j C_j(t) \oplus \eta_j S_j(t)\Big)\right)\\ &= \sum_{j=1}^{n}\rho(I_j \cup I_{-j})\Big(1 - (C_j^2(t) + S_j^2(t))\Big). \end{aligned} \tag{4.5}$$

Using (4.1) we compute, for each $j \in [n]$, that

$$\begin{aligned} C_j^2(t) + S_j^2(t) &= \frac{1}{\rho(I_j)^2}\left[\left(\int_{I_j}\cos(\lambda t)d\rho(\lambda)\right)^2 + \left(\int_{I_j}\sin(\lambda t)d\rho(\lambda)\right)^2\right]\\ &= \frac{1}{\rho(I_j)^2}\int_{I_j}\int_{I_j}\Big(\cos(\lambda_1 t)\cos(\lambda_2 t) + \sin(\lambda_1 t)\sin(\lambda_2 t)\Big)d\rho(\lambda_1)d\rho(\lambda_2)\\ &= \frac{1}{\rho(I_j)^2}\int_{I_j}\int_{I_j}\cos((\lambda_1-\lambda_2)t)d\rho(\lambda_1)d\rho(\lambda_2)\\ &\geq \frac{1}{\rho(I_j)^2}\int_{I_j}\int_{I_j}\left(1 - \tfrac{1}{2}|(\lambda_1-\lambda_2)t|^2\right)d\rho(\lambda_1)d\rho(\lambda_2) \geq 1 - \frac{D^2T^2}{2n^2}, \end{aligned}$$

where in the last step we used that $|(\lambda_1-\lambda_2)t| \leq \frac{D}{n}T$ for any $\lambda_1,\lambda_2 \in I_j^n$ and $t \in [0,T]$. From (4.5) we now obtain:

$$\operatorname{var}\big(R_n)(t) \leq \frac{1}{2}\left(\frac{DT}{n}\right)^2\sum_{j=1}^{n}\rho(I_j \cup I_{-j}) = \frac{1}{2}\left(\frac{DT}{n}\right)^2\rho([-D,D]), \quad \forall t \in [0,T],$$

thus verifying (4.3). Moreover, we have by Anderson's lemma and Observation 2.18:

$$\operatorname{var}\left(R_n(t) - R_n(t+h)\right) \leq \operatorname{var}\left(f_\rho(t) - f_\rho(t+h)\right) \leq D^2\rho([-D,D])h^2,$$

which establishes (4.4). □

4.3 Proof of Proposition 4.1

Assume without loss of generality that $\rho([-\pi,\pi]) = 1$. Let $n \in \mathbb{N}$ and $D = \pi$, and for $|j| \in [n] := \{1,2,\ldots,n\}$ define C_j and S_j as in (4.1). Then by Jensen's inequality we have

$$\sup_{t\in\mathbb{R}}\left\{|C_j(t)|^2 + |S_j(t)|^2\right\} \leq 1. \tag{4.6}$$

By Lemma 4.2, the decomposition (4.2) holds, with bounds as in (4.3) and (4.4).

Fix $\varepsilon > 0$ and partition the indices in $[n]$ into

$$\mathcal{A}_{n,\varepsilon} = \left|\left\{j \in [n]:\ \rho(I_j \cup I_{-j}) \geq \frac{\pi\varepsilon}{n}\right\}\right|, \quad \mathcal{B}_{n,\varepsilon} = [n] \setminus \mathcal{A}_{n,\varepsilon}, \tag{4.7}$$

defining accordingly the functions

$$A_{n,\varepsilon}(t) = \sum_{j\in\mathcal{A}_{n,\varepsilon}} \sqrt{\rho(I_j \cup I_{-j})}\Big(\zeta_j C_j(t) \oplus \eta_j S_j(t)\Big),$$

$$B_{n,\varepsilon}(t) = \sum_{j\in\mathcal{B}_{n,\varepsilon}} \sqrt{\rho(I_j \cup I_{-j})}\Big(\zeta_j C_j(t) \oplus \eta_j S_j(t)\Big),$$

so that (4.2) becomes

$$f_\rho \overset{d}{=} A_{n,\varepsilon} \oplus B_{n,\varepsilon} \oplus R_n. \tag{4.8}$$

Observe that

$$\mathbb{P}\Big(|f_\rho| < \delta \ \text{ on } [0,T]\cap\mathbb{Z}\Big) \geq \mathbb{P}\Big(\sup_{[0,T]\cap\mathbb{Z}} \big|A_{n,\varepsilon}\big| < \tfrac{\delta}{3}\Big)\mathbb{P}\Big(\sup_{[0,T]\cap\mathbb{Z}} \big|B_{n,\varepsilon}\big| < \tfrac{\delta}{3}\Big)\mathbb{P}\Big(\sup_{[0,T]\cap\mathbb{Z}} \big|R_n\big| < \tfrac{\delta}{3}\Big)$$

Given $T \in \mathbb{N}$, we carry out this decomposition with $n = mT$ (the parameter $m \in \mathbb{N}$ will be chosen later). Proposition 4.1 reduces to the following claims.

Claim 4.3. *For any fixed $m \in \mathbb{N}$ and $\varepsilon > 0$,*

$$\lim_{T\to\infty} \frac{1}{T} \log \mathbb{P}\left(\sup_{[0,T]} \big|A_{mT,\varepsilon}\big| < \tfrac{\delta}{3}\right) = 0.$$

Claim 4.4. *For any $m \in \mathbb{N}$,*

$$\lim_{\varepsilon\to 0}\lim_{T\to\infty} \frac{1}{T} \log \mathbb{P}\left(\sup_{[0,T]} \big|B_{mT,\varepsilon}\big| < \tfrac{\delta}{3}\right) = 0.$$

Claim 4.5.

$$\lim_{m\to\infty}\lim_{T\to\infty} \frac{1}{T} \log \mathbb{P}\left(\sup_{[0,T]} \big|R_{mT}\big| < \tfrac{\delta}{3}\right) = 0.$$

We turn to verify the claims.

Proof of Claim 4.3. Fix $m \in \mathbb{N}$ and $\varepsilon > 0$. We begin by showing that

$$\lim_{T\to\infty} \frac{|\mathcal{A}_{mT,\varepsilon}|}{T} = 0. \tag{4.9}$$

To this end we use the following classical fact (see [54, Chapter VII.6, Thm 2]): given a filtration $\mathcal{F}_T \nearrow \mathcal{F}$, a measure ρ is purely singular with respect to another measure μ if and only if

$$\lim_{T\to\infty} \frac{d\rho|_{\mathcal{F}_T}}{d\mu|_{\mathcal{F}_T}} \overset{\mu\text{-a.s.}}{=} 0.$$

Since almost sure convergence implies convergence in probability, this implies that if ρ is singular w.r.t. μ then for any $\varepsilon > 0$ we have

$$\lim_{T\to\infty} \mu\left(\frac{d\rho|_{\mathcal{F}_T}}{d\mu|_{\mathcal{F}_T}} > \varepsilon\right) = 0. \tag{4.10}$$

We apply this, taking ρ to be the given spectral measure, μ – the Lebesgue measure on $[-\pi,\pi]$ and $\mathcal{F}_T = \sigma(\{I_j^{mT}\}_{1\le|j|\le mT})$. Observing that

$$\frac{d\rho|_{\mathcal{F}_T}}{d\mu|_{\mathcal{F}_T}} = \sum_{1\le|j|\le mT} \frac{\rho(I_j^{mT})}{\mu(I_j^{mT})} 1\!\!1_{I_j} = \sum_{1\le|j|\le mT} \frac{\rho(I_j^{mT})}{\pi/(mT)} 1\!\!1_{I_j},$$

we obtain from (4.10):

$$\mu\left(\sum_j \frac{\rho(I_j^{mT})}{\pi/(mT)} 1\!\!1_{I_j} > \varepsilon\right) = \frac{\pi\left|\left\{j\in\{\pm1,\dots,\pm mT\}\ :\ \rho(I_j)\ge \frac{\pi}{mT}\varepsilon\right\}\right|}{mT} \overset{T\to\infty}{\longrightarrow} 0. \tag{4.11}$$

Combining this with (4.7), and recalling that $\rho(I_j)\ge\rho(I_{-j})$ for $j\ge1$ (equality holds for $j\ne1$) we obtain that

$$\lim_{T\to\infty}\frac{|\mathcal{A}_{mT,\varepsilon}|}{T} = \lim_{T\to\infty}\frac{\left|\left\{j\in[n]\ :\ \rho(I_j)\ge\frac{\pi}{mT}\varepsilon\right\}\right|}{T} = 0,$$

thus (4.9) is established.

Denoting $d = 2|\mathcal{A}_{mT,\varepsilon}|$, we recall that $A_{mT,\varepsilon}(t) = \langle u(t),\zeta\rangle$ is an inner product in $\mathbb{R}^d$ between

$$u(t) = \left(\sqrt{\rho(I_j\cup I_{-j})}C_j(t), \sqrt{\rho(I_j\cup I_{-j})}S_j(t)\right)_{j\in\mathcal{A}_{mT,\varepsilon}}$$

and a standard d-dimensional multi-normal random vector $\zeta\sim\gamma_d$. Note that, by (4.6),

$$\|u(t)\|^2 = \sum_{j\in\mathcal{A}_{mT,\varepsilon}} \rho(I_j\cup I_{-j})(C_j^2(t)+S_j^2(t)) \le \sum_{0\le j<mT}\rho(I_j\cup I_{-j}) = \rho([-\pi,\pi]) = 1.$$

Hence,

$$\begin{aligned}
\mathbb{P}\left(\bigcap_{k=1}^T\left\{|A_{mT,\varepsilon}(k)|<\frac{\delta}{3}\right\}\right) &= \gamma_d\left(\bigcap_{k=1}^T\left\{\zeta\in\mathbb{R}^d:\ |\langle\zeta,u(k)\rangle|\le\tfrac{\delta}{3}\right\}\right) \\
&\ge \gamma_d\left(\bigcap_{k=1}^T\left\{\zeta\in\mathbb{R}^d:\ \left|\left\langle\zeta,\tfrac{u(k)}{\|u(k)\|}\right\rangle\right|\le\tfrac{\delta}{3}\right\}\right) && \|u(k)\|\le1 \\
&\ge \left(\tfrac{\delta}{3}\right)^d\gamma_d\left(\bigcap_{k=1}^T\left\{\zeta\in\mathbb{R}^d:\ \left|\left\langle\zeta,\tfrac{u(k)}{\|u(k)\|}\right\rangle\right|\le1\right\}\right) && \text{by Obs. 2.2} \\
&\ge \left(\frac{\delta\kappa}{3\sqrt{1+2\log\frac{T}{d}}}\right)^d, && \text{by Prop. 2.6}
\end{aligned}$$

where κ is a universal constant. Thus we obtain

$$\lim_{T\to\infty}\frac{1}{T}\log\mathbb{P}\left(\left\{\bigcap_{k=1}^T\left|A_{mT,\varepsilon}(k)\right|<\tfrac{\delta}{3}\right\}\right) \ge \lim_{T\to\infty}\frac{2|\mathcal{A}_{mT,\varepsilon}|}{T}\log\left(\frac{c\delta}{\sqrt{\log\left(\frac{T}{2|\mathcal{A}_{mT,\varepsilon}|}\right)}}\right) = 0,$$

where $c > 0$ is a universal constant, and the last equality follows from (4.9). The Claim follows. $\square$

Proof of Claim 4.4. Fix $m \in \mathbb{N}$. Using (4.6) and (4.7), we have for any $0 \le t \le T$ and $\varepsilon > 0$,

$$\begin{aligned}\operatorname{var} B_{mT,\varepsilon}(t) &= \sum_{j \in \mathcal{B}_{mT,\varepsilon}} \rho(I_j \cup I_{-j}) \left(C_j^2(t) + S_j^2(t)\right) \\ &\le |\mathcal{B}_{mT,\varepsilon}| \cdot 2 \max_{j \in \mathcal{B}_{mT,\varepsilon}} \rho(I_j) \sup_{t \in [0,T]} \left(C_j^2(t) + S_j^2(t)\right) \\ &\le mT \cdot \frac{2\pi\varepsilon}{mT} \cdot 1 = 2\pi\varepsilon.\end{aligned}$$

Using Khatri-Sidak's inequality (Proposition 2.3) and Lemma 2.1, we have:

$$\begin{aligned}-\frac{1}{T} \log \mathbb{P}\left(\sup_{t \in (0,T] \cap \mathbb{N}} \left|B_{mT,\varepsilon}(t)\right| < \tfrac{\delta}{3} \right) &\le -\frac{1}{T} \sum_{t=1}^{T} \log \mathbb{P}\left(\left|B_{mT,\varepsilon}(t)\right| < \tfrac{\delta}{3} \right) \\ &\le -\log \mathbb{P}\left(\sqrt{2\pi\varepsilon} |\mathcal{N}(0,1)| < \tfrac{\delta}{3} \right) \\ &\le -\log\left(1 - 2e^{-\frac{\delta^2}{36\pi\varepsilon}}\right) \xrightarrow{\varepsilon \to 0} 0.\end{aligned}$$

$\square$

Proof of Claim 4.5. By (4.3),

$$\operatorname{var}\left(R_{mT}\right)(t) \le \frac{\pi^2}{2m^2}, \quad \forall t \in [0,T].$$

Now using Khatri-Sidak's inequality (Proposition 2.3) and a tail estimate (Lemma 2.1), we have:

$$\begin{aligned}-\frac{1}{T} \log \mathbb{P}\left(\sup_{t \in (0,T] \cap \mathbb{N}} \left|R_{mT}(t)\right| < \tfrac{\delta}{3} \right) &\le -\frac{1}{T} \sum_{t=1}^{T} \log \mathbb{P}\left(\left|R_{mT}(t)\right| < \tfrac{\delta}{3} \right) \\ &\le -\log \mathbb{P}\left(\tfrac{\pi}{\sqrt{2}m} |\mathcal{N}(0,1)| < \tfrac{\delta}{3} \right) \\ &\le -\log\left(1 - 2e^{-\frac{\delta^2}{9\pi^2} m^2}\right) \xrightarrow{m \to \infty} 0.\end{aligned}$$

$\square$

5 Existence and continuity of the persistence exponent

In this section we prove Theorems 1 and 4. Our method is to approximate the spectral measure by smooth spectral measures, for which it is easier to prove the existence of the exponent, and then use the comparison lemmata (proved in Section 3) in order to retrieve existence for the original measure. To make this idea concrete, we formulate three auxiliary results. The first provides existence of the persistence exponent for smooth compactly supported spectral densities.

Proposition 5.1. *Let ρ be an absolutely continuous spectral measure with compactly supported density which is twice differentiable on $\mathbb{R}$. Then $\theta_\rho^\ell := \lim_{T \to \infty} \theta_\rho^\ell(T)$ exists in $(0, \infty]$.*

The second result states that persistence exponents are close if the spectral measures are equal near the origin and close in total variation.

Proposition 5.2. *There exist c and $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ there exist $T_0(\varepsilon)$ with the following property: denoting $L = \sqrt{\frac{2}{c\varepsilon}}$ and $\eta = (c\varepsilon)^{1/4}$, if $T > \max\{T_0, \frac{2}{A}\}$ and $\mu, \nu \in \mathcal{M}_{\alpha,A} \cap \mathcal{L}_{\beta,B}$ are such that $d_{TV}(\mu,\nu) < \varepsilon$ and $\mu|_{[-\frac{L}{T},\frac{L}{T}]} = \nu|_{[-\frac{L}{T},\frac{L}{T}]}$, then*

$$\theta^{\ell}_{\nu}(T(1-\eta)) \leq \theta^{\ell}_{\mu}(T) + C_\varepsilon$$

where $\lim_{\varepsilon\to 0} C_\varepsilon = 0$.

By the last result, persistence exponents are close also when the spectral measures are equal away from the origin and close near the origin.

Proposition 5.3. *Fix $L, \varepsilon, m > 0$. There exists $T_0 = T_0(L,\varepsilon,m)$ such that, for all $T > T_0$, the following holds: Suppose that $\mu, \nu \in \mathcal{M}_{\alpha,1/T} \cap \mathcal{L}_{\beta,B}$ are such that $\mu|_{\mathbb{R}\setminus[-\frac{L}{T},\frac{L}{T}]} = \nu|_{\mathbb{R}\setminus[-\frac{L}{T},\frac{L}{T}]}$ and $\forall \lambda \in (0, L/T): \ |\mu(-\lambda,\lambda) - \nu(-\lambda,\lambda)| \leq 2\varepsilon\lambda$, while also $\nu\left(\left(-\frac{L}{T},\frac{L}{T}\right)\right) \leq m\frac{L}{T}$ and $\mu\left(\left(-\frac{L}{T},\frac{L}{T}\right)\right) \leq m\frac{L}{T}$. Then*

$$|\theta^{\ell}_{\mu}(T) - \theta^{\ell}_{\nu}(T)| < C_\varepsilon,$$

where $C_\varepsilon = C_\varepsilon(L,m)$ and $\lim_{\varepsilon\to 0} C_\varepsilon = 0$.

We proceed as follows. First we prove Theorems 1 and 4 in Sections 5.1 and 5.2, respectively. Then we present the proofs of the auxiliary results (Propositions 5.1, 5.2 and 5.3) in Section 5.3. To show the tightness of our conditions, we provide in Section 5.4 a counter-example to the existence of a persistence exponent.

5.1 Proof of Theorem 1: existence of the persistence exponent

Let $\rho \in \mathcal{M} \cap \mathcal{L}$. If $\rho'(0) = \infty$, then it follows from Proposition 2.25 that $\theta^{\ell}_{\rho} = 0$ for all $\ell \in \mathbb{R}$. Thus we assume $\rho'(0) < \infty$. Consequently, $\rho \in \mathcal{M}_{(\alpha,\alpha'),A} \cap \mathcal{L}_{\beta,B}$ for some $\alpha, \alpha', A, \beta, B > 0$ which are fixed throughout the proof.

Let $c, \varepsilon_0 > 0$ be the constants whose existence is guaranteed by Proposition 5.2. Given $\varepsilon < \varepsilon_0$, denote

$$L(\varepsilon) = \sqrt{\frac{2}{c\varepsilon}}, \quad \eta(\varepsilon) = (c\varepsilon)^{1/4}.$$

For a given $T > 0$, we approximate the spectral measure in several steps by smoother and smoother measures, without altering the persistence exponent $\theta^{\ell}_{\rho}(T)$ significantly. We treat separately the measure near the origin, i.e., in the interval $[-\frac{L}{T},\frac{L}{T}]$, and the measure away from the origin, on the remainder of $\mathbb{R}$.

Step 1: discarding the singular part away from the origin. We write $\rho = \rho_{\text{ac}} + \rho_{\text{sing}}$ where ρ_{ac} is absolutely continuous and ρ_{sing} is purely singular. For a given $T > 0$, define

$$\mu_T := \rho - \left(\rho_{\text{sing}}|_{\mathbb{R}\setminus[-\frac{L}{T},\frac{L}{T}]}\right) = \rho|_{[-\frac{L}{T},\frac{L}{T}]} + \rho_{ac}|_{\mathbb{R}\setminus[-\frac{L}{T},\frac{L}{T}]}.$$

Let us show that upper and lower limits of $\theta^{\ell}_{\rho}(T)$ and $\theta^{\ell}_{\mu_T}(T)$ are close. Noting that $L > \frac{1}{\eta^2}$, we may apply Corollary 3.2 to obatin the existence of $T_1(\varepsilon, A)$ such that for all $T > T_1$ we have

$$\theta^{\ell}_{\mu_T}((1-\eta)T) \leq \theta^{\ell}_{\rho}(T) + C^{(1)}_{\varepsilon}, \tag{5.1}$$

where $\lim_{\varepsilon\to 0} C_\varepsilon^{(1)} = 0$. On the other hand, for $T > T_2(A)$ we have

$$\begin{aligned}
\theta_\rho^\ell(T) &\le \theta_\rho^{\ell-\varepsilon}(T) + C_\varepsilon^{(2)} && \text{by Lemma 1.1}\\
&\le \theta_{\mu_T}^\ell(T) + \psi^\varepsilon_{\rho_{\text{sing}}|_{\mathbb{R}\setminus[-\frac{L}{T},\frac{L}{T}]}}(T) + C_\varepsilon^{(2)} && \text{by Lemma 2.12(b)}\\
&\le \theta_{\mu_T}^\ell(T) + \psi^\varepsilon_{\rho_{\text{sing}}}(T) + C_\varepsilon^{(2)}, && \text{by Proposition 2.16}
\end{aligned}$$

where $\lim_{\varepsilon\to 0} C_\varepsilon^{(2)} = 0$. By Theorem 2 we have $\lim\limits_{T\to\infty} \psi^\varepsilon_{\rho_{\text{sing}}}(T) = 0$, so that

$$\limsup_{T\to\infty} \theta_\rho^\ell(T) \le \limsup_{T\to\infty} \theta_{\mu_T}^\ell(T) + C_\varepsilon^{(2)}.$$

Together with (5.1) we conclude that

$$\liminf_{T\to\infty} \theta_{\mu_T}^\ell(T) - C_\varepsilon^{(1)} \le \liminf_{T\to\infty} \theta_\rho^\ell(T) \le \limsup_{T\to\infty} \theta_\rho^\ell(T) \le \limsup_{T\to\infty} \theta_{\mu_T}^\ell(T) + C_\varepsilon^{(2)}. \tag{5.2}$$

Step 2: smooth approximation away from the origin. We shall employ the following approximation claim, which could be proved by standard analysis arguments.

Claim 5.4. *Let $\rho \in \mathcal{M} \cap \mathcal{M}_{(\alpha,\alpha'),A} \cap \mathcal{L}_{\beta,B}$, and let $g \in L^1(\mathbb{R})$ be the density of ρ_{ac}. Let $\varepsilon > 0$ be given. Then there exists an absolutely continuous measure $\nu \in \mathcal{M} \cap \mathcal{M}_{(\alpha,\alpha'),\varepsilon} \cap \mathcal{L}_{\beta,B}$ with smooth and compactly supported density $h \in C_0^\infty(\mathbb{R})$ satisfying*

$$\int_{\mathbb{R}} |h-g| < \varepsilon, \quad \textit{and} \quad \forall \lambda \in (-\varepsilon,\varepsilon):\ h(\lambda) = \rho'(0).$$

Let ν be the measure obtained by applying Claim 5.4 with our given ρ and ε. For any given $T > 0$, define the measure σ_T which approximates the measure ρ away from the origin by the smooth measure ν:

$$\sigma_T := \rho|_{[-\frac{L}{T},\frac{L}{T}]} + \nu|_{\mathbb{R}\setminus[-\frac{L}{T},\frac{L}{T}]}.$$

Note that $d_{\text{TV}}(\sigma_T,\mu_T) = \frac{1}{2}\int_{\mathbb{R}\setminus[-\frac{L}{T},\frac{L}{T}]} |h-g| < \frac{\varepsilon}{2}$. By Proposition 5.2, if $\varepsilon < \varepsilon_0$ and $T > T_3(\varepsilon)$, then

$$\theta_{\sigma_T}^\ell\left(T(1-\eta)\right) \le \theta_{\mu_T}^\ell(T) + C_\varepsilon^{(3)}, \quad \theta_{\mu_T}^\ell(T) \le \theta_{\sigma_T}^\ell\left(\tfrac{T}{1-\eta}\right) + C_\varepsilon^{(3)},$$

where $\lim_{\varepsilon\to 0} C_\varepsilon^{(3)} = 0$. We conclude that

$$\liminf_{T\to\infty} \theta_{\sigma_T}^\ell((1-\eta)T) - C_\varepsilon^{(3)} \le \liminf_{T\to\infty} \theta_{\mu_T}^\ell(T) \le \limsup_{T\to\infty} \theta_{\mu_T}^\ell(T) \le \limsup_{T\to\infty} \theta_{\sigma_T}^\ell\left(\tfrac{T}{1-\eta}\right) + C_\varepsilon^{(3)}. \tag{5.3}$$

Step 3: smooth approximation near the origin. Next we verify the conditions of Proposition 5.3, which will allow us to compare $\theta_{\sigma_T}^\ell(T)$ and $\theta_\nu^\ell(T)$. First we note that $\sigma_T|_{\mathbb{R}\setminus[-\frac{L}{T},\frac{L}{T}]} = \nu|_{\mathbb{R}\setminus[-\frac{L}{T},\frac{L}{T}]}$. Recalling that on the interval $[-\frac{L}{T},\frac{L}{T}]$ we have $\sigma_T = \rho$ and $d\nu = \rho'(0)d\lambda$, we obtain that $\sigma_T, \nu \in \mathcal{M}_{(\alpha,\alpha'),\frac{L}{T}} \cap \mathcal{L}_{\beta,B}$ Moreover, for any $\lambda \in \left(0, \frac{L}{T}\right)$ we have

$$\sup_{\lambda\in\left(0,\frac{L}{T}\right)} \frac{|\sigma_T(-\lambda,\lambda)-\nu(-\lambda,\lambda)|}{2\lambda} = \sup_{\lambda\in\left(0,\frac{L}{T}\right)} \left|\frac{\rho(-\lambda,\lambda)}{2\lambda} - \rho'(0)\right| \le \sup_{\lambda\in(0,\varepsilon)} \left|\frac{\rho(-\lambda,\lambda)}{2\lambda} - \rho'(0)\right| =: \delta(\varepsilon),$$

where $\lim_{\varepsilon\to 0}\delta(\varepsilon) = 0$. We may thus apply Proposition 5.3 to get that, for $T > T_4(\varepsilon, \alpha', A)$,

$$\left|\theta^\ell_{\sigma_T}\left((1-\eta)T\right) - \theta^\ell_\nu\left((1-\eta)T\right)\right| \le C^{(4)}_\varepsilon, \quad \left|\theta^\ell_{\sigma_T}\left(\tfrac{T}{1-\eta}\right) - \theta^\ell_\nu\left(\tfrac{T}{1-\eta}\right)\right| \le C^{(4)}_\varepsilon, \tag{5.4}$$

where $\lim_{\varepsilon\to 0} C^{(4)}_\varepsilon = 0$ (here $C^{(4)}_\varepsilon$ depends on ρ).

Step 4: smooth measures have an exponent. Since ν has a smooth and compactly supported density, by Proposition 5.1, $\lim_{T\to\infty}\theta^\ell_\nu(T) = \theta^\ell_\nu$ exists. Thus, for $T > T_0(\varepsilon,\alpha', A)$, we have

$$\begin{aligned}
\limsup_{T\to\infty}\theta^\ell_\rho(T) - \liminf_{T\to\infty}\theta^\ell_\rho(T) &\le \limsup_{T\to\infty}\theta^\ell_{\mu_T}(T) - \liminf_{T\to\infty}\theta^\ell_{\mu_T}(T) + C^{(1)}_\varepsilon + C^{(2)}_\varepsilon && \text{by (5.2)}\\
&\le \limsup_{T\to\infty}\theta^\ell_{\sigma_T}\left(\tfrac{T}{1-\eta}\right) - \liminf_{T\to\infty}\theta^\ell_{\sigma_T}\left((1-\eta)T\right) + \sum_{j=1}^{3} C^{(j)}_\varepsilon && \text{by (5.3)}\\
&\le \limsup_{T\to\infty}\theta^\ell_{\nu}\left(\tfrac{T}{1-\eta}\right) - \liminf_{T\to\infty}\theta^\ell_{\nu}\left((1-\eta)T\right) + \sum_{j=1}^{4} C^{(j)}_\varepsilon && \text{by (5.4)}\\
&= \sum_{j=1}^{4} C^{(j)}_\varepsilon. && \text{by Prop. 5.1}
\end{aligned}$$

We conclude that $\limsup_{T\to\infty}\theta^\ell_\rho(T) = \liminf_{T\to\infty}\theta^\ell_\rho(T)$, as required.

As a by-product of our proof, we have shown the following.

Proposition 5.5. *Let $\rho \in \mathcal{M}\cap\mathcal{L}$ and $\varepsilon > 0$, and let ν be the corresponding smooth measure from Claim 5.4. Then*

$$\left|\theta^\ell_\rho - \theta^\ell_\nu\right| \le C_\varepsilon,$$

where $\lim_{\varepsilon\to 0} C_\varepsilon = 0$ and C_ε depends on ρ.

5.2 Proof of Theorem 4: Continuity

5.2.1 Part I: continuity of the ball exponent

Continuity in ℓ: The function $\ell \mapsto \psi^\ell_\rho$ on $(0,\infty)$ is finite-valued (by Lemma 2.22) and convex (by log-concavity of Gaussian measures). Hence $\ell \mapsto \psi^\ell_\rho$ is continuous and differentiable almost everywhere. Furthetmore,

$$0 \ge \frac{\partial}{\partial\ell}\psi^\ell_\rho \ge \psi^{\ell+1}_\rho - \psi^\ell_\rho \ge -\psi^\ell_\rho \ge -C,$$

where $C = C(\beta, B)$ is the constant from Lemma 2.22. This shows that $\ell \mapsto \psi^\ell_\rho$ is locally Lipshitz and uniformly continuous in the class $\rho \in \mathcal{L}_{\beta,B}$.

Continuity in TV: We start with a general claim.

Claim 5.6. *Suppose μ and ν are two spectral measures. Then there exists a spectral measure γ such that $\gamma \geq \max\{\mu,\nu\}$ and $\max\{d_{\mathrm{TV}}(\gamma,\mu), d_{\mathrm{TV}}(\gamma,\nu)\} \leq 2d_{\mathrm{TV}}(\mu,\nu)$. Moreover, if $\mu = \nu$ on an interval I, then $\gamma = \nu$ on I.*

Proof. Setting $\sigma = \mu - \nu$, we note that σ is a finite signed measure. Let $\sigma = \sigma_+ - \sigma_-$ be the Hahn-Jordan decomposition. Define $\gamma := \nu + \sigma_+$. Clearly, if $\mu = \nu$ on an interval I then $\gamma = \nu$ on I. Next observe that $\gamma \geq \nu$ and $\gamma = \nu + \sigma_+ \geq \nu + \sigma = \mu$. Moreover, by the Hahn-Jordan theorem,

$$
\begin{aligned}
&d_{\mathrm{TV}}(\gamma,\nu) = d_{\mathrm{TV}}(\sigma_+,0) \leq d_{\mathrm{TV}}(\mu,\nu),\\
&d_{\mathrm{TV}}(\gamma,\mu) \leq d_{\mathrm{TV}}(\gamma,\nu) + d_{\mathrm{TV}}(\nu,\mu) \leq 2d_{\mathrm{TV}}(\mu,\nu).
\end{aligned}
$$

□

Let $\mu,\nu \in \mathcal{L}$ such that $d_{\mathrm{TV}}(\mu,\nu) < \varepsilon$. Let γ be the spectral measure constructed in Claim 5.6. Since $\gamma \geq \max\{\mu,\nu\}$ we have for arbitrary $\delta \in (0,\ell)$ that

$$\psi^{\ell}_{\mu} \leq \psi^{\ell}_{\gamma} \leq \psi^{\ell-\delta}_{\nu} + \psi^{\delta}_{\gamma-\nu},$$

where the first inequality is due to Anderson (Proposition 2.16), and the second is Lemma 2.12(a). Applying Khatri-Sidak (Proposition 2.3) and Lemma 2.20, there is a choice of $\delta = \delta(\varepsilon)$ with $\lim_{\varepsilon\to 0}\delta(\varepsilon) = 0$ and

$$\psi^{\delta(\varepsilon)}_{\gamma-\nu} \leq \psi^{\delta(\varepsilon)}_{\gamma-\nu}(1) \leq \varepsilon.$$

By the uniform continuity of $\ell \mapsto \psi^{\ell}_{\rho}$ in the class $\mathcal{L}_{\beta,B}$ (proven above), it holds that

$$\psi^{\ell-\delta(\varepsilon)}_{\nu} \leq \psi^{\ell}_{\nu} + C_{\varepsilon},$$

where $\lim\limits_{\varepsilon\to 0} C_{\varepsilon} = 0$. We deduce that $\psi^{\ell}_{\mu} \leq \psi^{\ell}_{\nu} + \widetilde{C}_{\varepsilon}$, where $\lim_{\varepsilon\to 0}\widetilde{C}_{\varepsilon} = 0$. Since the roles of (μ,ν) are symmetric, we conclude our proof.

5.2.2 Part II: continuity of the persistence exponent

Continuity in ℓ: is an immediate consequence of Lemma 1.1.
Continuity in TV_0**:** Recall the definitions in (1.2), and let $\mu,\nu \in \mathcal{M}\cap\mathcal{M}_{(\alpha,\alpha'),A}\cap\mathcal{L}_{\beta,B}$ be such that $d_{\mathrm{TV}_0}(\mu,\nu) < \varepsilon$. Let c,ε_0 be the constants whose existence is guaranteed by Proposition 5.2. Let $\varepsilon < \varepsilon_0$ and set $L = \sqrt{\frac{2}{c\varepsilon}}$, $\eta = (c\varepsilon)^{1/4}$. For a given $T > 0$ define

$$\sigma = \mu|_{[-\frac{L}{T},\frac{L}{T}]} + \nu|_{[-\frac{L}{T},\frac{L}{T}]^c},$$

and note that $\sigma \in \mathcal{M}_{\alpha,\frac{L}{T}}\cap\mathcal{L}_{\beta,B}$. Using Proposition 5.2 with μ and σ we conclude that, for $T > T_1(\varepsilon)$, we have

$$\theta^{\ell}_{\mu}\left(T(1-\eta)\right) \leq \theta^{\ell}_{\sigma}(T) + C^{(1)}_{\varepsilon},$$

where $\lim\limits_{\varepsilon\to 0} C^{(1)}_{\varepsilon} = 0$. Since $\mu \in \mathcal{M}_{(\alpha,\alpha'),A}$, we have $\mu((-\frac{L}{T},\frac{L}{T})) < 2\alpha'\frac{L}{T}$ for all $T > \frac{L}{A}$, which also implies $\sigma((-\frac{L}{T},\frac{L}{T})) < 2\alpha'\frac{L}{T}$. A similar statement holds for $\nu \in \mathcal{M}_{(\alpha,\alpha'),A}$. Applying Proposition 5.3

with σ and ν we get that, for $T > T_3(\varepsilon, \alpha')$,

$$\theta_\sigma^\ell(T) \le \theta_\nu^\ell(T) + C_\varepsilon^{(2)}$$

where $C_\varepsilon^{(2)} = C_\varepsilon^{(2)}(\alpha')$ satisfies $\lim_{\varepsilon \to 0} C_\varepsilon^{(2)} = 0$. Combining the last two displayed formulas, we get:

$$\theta_\mu^\ell \left(T(1-\eta)\right) \le \theta_\nu^\ell(T) + C_\varepsilon^{(1)} + C_\varepsilon^{(2)}.$$

By Theorem 1 we may take the limit as $T \to \infty$ to obtain

$$\theta_\mu^\ell \le \theta_\nu^\ell + C_\varepsilon^{(1)} + C_\varepsilon^{(2)}.$$

Since the roles of (μ, ν) are symmetric, the uniform continuity of $\rho \mapsto \theta_\rho^\ell$ in $\mathcal{M} \cap \mathcal{M}_{(\alpha,\alpha'),A} \cap \mathcal{L}_{\beta,B}$ follows.

5.3 Proofs of the auxiliary propositions

5.3.1 Proof of Proposition 5.1

We begin by stating a comparison lemma for persistence probabilities of "approximately stationary" Gaussian processes, which implies Proposition 5.1 and is useful in the proof of Theorem 5.

Lemma 5.7. *Let $\eta > 1$ and $c > 0$. Suppose that $\{X(t)\}_{t\ge 0}$ is a centered Gaussian process with $\mathbb{E}X(t)^2 = 1$, such that*

$$\mathbb{E}X(s)X(s+t) \le c|t|^{-\eta}, \quad \textit{for all } s, t \ge 0.$$

Fix $\mu \in \mathcal{M}_{\alpha,A} \cap \mathcal{L}_{\beta,B}$ and set

$$\xi_{M,\ell} := \sup_{u \in [\ell-1,\ell+1]} \sup_{s \ge 0} \left| \frac{\mathbb{P}\left(\inf_{t\in[s,s+M]} X(t) > u\right)}{\mathbb{P}\left(\inf_{t\in[0,M]} f_\mu(t) > u\right)} - 1 \right|.$$

Then there exist $C = C(\eta)$, such that, for all $T \ge M$,

$$\theta_\mu^\ell(M) - \widetilde{\theta}^\ell(T) \le C M^{\frac{1-\eta}{2+\eta}} + \frac{\xi_{M,\ell}}{M},$$

where $\widetilde{\theta}^\ell(T) := -\frac{1}{T}\log \mathbb{P}(\inf_{t\in[0,T]} X(t) > \ell)$.

Lemma 5.7 implies Proposition 5.1. Given ρ as Proposition 5.1, there exists $c = c(\rho)$ such that

$$\widehat{\rho}(t) \le \frac{c}{|t|^2}, \forall t \in \mathbb{R}.$$

Then, setting $X(\cdot) = f_\rho(\cdot)$, we satisfy the conditions of Lemma 5.7 with $\xi_{M,\ell} = 0$ and $\eta = 2$. The lemma thus yields the existence of $C < \infty$ such that, for all $T \ge M$ we have

$$\theta_\rho^\ell(M) \le \theta_\rho^\ell(T) + CM^{-1/4}.$$

Upon taking $\limsup_{M\to\infty} \liminf_{T\to\infty}$ we arrive at

$$\limsup_{M\to\infty} \theta_\rho^\ell(M) \leq \liminf_{T\to\infty} \theta_\rho^\ell(T),$$

and the existence of $\lim_{T\to\infty} \theta_\rho^\ell(T)$ follows. The limit must be positive by Lemma 2.24. $\square$

Proof of Lemma 5.7. We extend the methods of Dembo-Mukherjee appearing in [19, Theorem 1.6] (see also [20, Lemma 3.1]) to the case where correlations are not necessarily non-negative.

Let $\delta = C' M^{\frac{1-\eta}{2+\eta}}$ where $C' = C'(\eta) > 0$ will be chosen later. Let $T \geq M$ be given.

For $i \geq 1$, set $s_i := (1+\delta)Mi$, $I_i := [s_i - M, s_i]$ and $N := \lfloor \frac{T}{M(1+\delta)} \rfloor$. Note that $\{I_i\}$ are disjoint and $\cup_{i=1}^N I_i \subset [0,T]$. Consequently,

$$\mathbb{P}\left(\inf_{[0,T]} X(t) > \ell \right) \leq \mathbb{P}\left(\inf_{t \in \cup_{i=1}^N I_i} X(t) > \ell \right).$$

Define an $N \times N$ matrix $\mathbf{B}_N$ by setting $\mathbf{B}_N(i,i) := 1$ and, for $i \neq j$,

$$\mathbf{B}_N(i,j) := \frac{c}{\gamma} \sup_{s \in I_i, t \in I_j} |s-t|^{-\eta} \leq \frac{c|i-j|^{-\eta}}{\gamma M^\eta \delta^\eta},$$

where $\gamma := \gamma_{M,\delta} = 4c(\delta M)^{-\eta} \sum_{i=1}^\infty i^{-\eta}$, so that

$$\max_{1\leq i \leq N} \sum_{j \neq i} \mathbf{B}_N(i,j) \leq \frac{2c}{\gamma \delta^\eta M^\eta} \sum_{i=1}^\infty i^{-\eta} \leq \frac{1}{2}.$$

Thus, by the Gershgorin circle theorem, all the eigenvalues of $\mathbf{B}_N$ lie within the interval $\left[\frac{1}{2}, \frac{3}{2}\right]$, and hence $\mathbf{B}_N$ is positive definite. Setting $r(s,t) := \mathbb{E}X(s)X(t)$, we claim that for any $s \in I_i, t \in I_j$ we have

$$r(s,t) \leq (1-\gamma) r(s,t) \mathbb{1}_{i=j} + \gamma \mathbf{B}_N(i,j). \tag{5.5}$$

To see this, observe that (5.5) is equivalent to

$$r(s,t) \leq \begin{cases} (1-\gamma) r(s,t) + \gamma, & i = j, \\ c \sup_{s \in I_i, t \in I_j} |s-t|^{-\eta}, & i \neq j, \end{cases}$$

both of which are immediate from our assumptions. The RHS of (5.5) is the correlation function of the centered non-stationary Gaussian process on $\cup_{i=1}^N I_i$ defined by $t \mapsto \sqrt{1-\gamma} X^{(i)}(t) + \sqrt{\gamma} Z_i$ for $t \in I_i$, where

- $\mathbf{Z} := (Z_1, \cdots, Z_N)$ is a centered Gaussian vector with covariance $\mathbf{B}_N$,
- $\{X^{(i)}\}$ are i.i.d. copies of $X(\cdot)$, independent of $\mathbf{Z}$.

Using Slepian's inequality (Proposition 2.13) together with (5.5) yields

$$\begin{aligned}\mathbb{P}\left(\inf_{t\in\cup_{i=1}^N I_i} X(t) > \ell\right) &\le \mathbb{P}\left(\inf_{t\in I_i} \sqrt{1-\gamma}X^{(i)}(t) + \sqrt{\gamma}Z_i > \ell, 1\le i\le N\right)\\ &= \mathbb{E}\left[\prod_{i=1}^N \mathbb{P}\left(\inf_{t\in I_i} X(t) > \ell - \frac{\sqrt{\gamma}}{\sqrt{1-\gamma}}Z_i \,\Big|\, \mathbf{Z}\right)\right]\\ &\le \mathbb{E}\prod_{i=1}^N \left[\mathbb{P}\left(\inf_{t\in I_i} X(t) > \ell - 2\varepsilon\right) + \mathbb{1}\left\{Z_i > \varepsilon\gamma^{-1/2}\right\}\right]\\ &\le \mathbb{E}\prod_{i=1}^N \left[\mathbb{P}\left(\inf_{t\in[0,M]} f_\mu(t) > \ell - 2\varepsilon\right)(1+\xi_{M,\ell}) + \mathbb{1}\left\{Z_i > \varepsilon\gamma^{-1/2}\right\}\right], \quad (5.6)\end{aligned}$$

for any $\varepsilon\in(0,\frac{1}{2})$, where the one before last line uses the fact that $\sqrt{1-\gamma}\ge\frac{1}{2}$ for δM large enough (depending on η). Next we note that for any collection of distinct indices $i_1,\cdots,i_m\in[N]$, the covariance matrix Σ of $(Z_{i_1},\ldots,Z_{i_m})$ has eignevalues within $[\frac{1}{2},\frac{3}{2}]$. Consequently,

$$\begin{aligned}\mathbb{P}\left(Z_{i_\ell} > \varepsilon\gamma^{-1/2},\, 1\le\ell\le m\right) &= \det(\Sigma)^{-1/2}(2\pi)^{m/2}\int_{(\varepsilon\gamma^{-1/2},\infty)^m} e^{-\frac{1}{2}\mathbf{z}\Sigma^{-1}\mathbf{z}}d\mathbf{z}\\ &\le \frac{2^{m/2}}{(2\pi)^{m/2}}\int_{(\varepsilon\gamma^{-1/2},\infty)^m} e^{-\frac{1}{3}\|\mathbf{z}\|_2^2}d\mathbf{z}\\ &= 3^{m/2}\mathbb{P}\left(Z > \sqrt{\tfrac{2}{3}}\varepsilon\gamma^{-1/2}\right)^m,\end{aligned}$$

where $Z\sim\mathcal{N}(0,1)$. Along with (5.6), this gives

$$\begin{aligned}\mathbb{P}\left(\inf_{t\in[0,T]} X(t) > \ell\right) &\le \sum_{m=0}^N \binom{N}{m}\left[\mathbb{P}\left(\inf_{t\in[0,M]} f_\mu(t) > \ell-2\varepsilon\right)(1+\xi_{M,\ell})\right]^{N-m} 3^{m/2}\mathbb{P}\left(Z > \sqrt{\tfrac{2}{3}}\varepsilon\gamma^{-1/2}\right)^m\\ &= \left[\mathbb{P}\left(\inf_{t\in[0,M]} f_\mu(t) > \ell-2\varepsilon\right)(1+\xi_{M,\ell}) + \sqrt{3}\mathbb{P}\left(Z > \sqrt{\tfrac{2}{3}}\varepsilon\gamma^{-1/2}\right)\right]^N \quad (5.7)\end{aligned}$$

Standard bounds on Gaussian tails (Lemma 2.1), and the fact that $\gamma = c(\eta)^{-1}(\delta M)^{-\eta}$, give

$$\mathbb{P}\left(Z > \sqrt{\tfrac{2}{3}}\varepsilon\gamma^{-1/2}\right) \le \exp\left(-\frac{\varepsilon^2}{3\gamma}\right) = \exp\left(-c(\eta)\varepsilon^2\delta^\eta M^\eta\right)$$

for some $c(\eta)>0$. By Lemma 2.23 there exists $K\in(0,\infty)$ such that, for all $M>1$,

$$\mathbb{P}\left(\inf_{t\in[0,M]} f_\mu(t) > \ell-2\varepsilon\right) \ge \mathbb{P}\left(\inf_{t\in[0,M]} f_\mu(t) > \ell-1\right) \ge e^{-KM}. \quad (5.8)$$

Thus, there exist $C'=C'(\eta)$ such that for the choice $\varepsilon=\delta=C'M^{\frac{1-\eta}{2+\eta}}$ and all $M>0$ we have

$$\sqrt{3}\mathbb{P}\left(Z \ge \sqrt{\tfrac{2}{3}}\varepsilon\gamma^{-1/2}\right) \le \mathbb{P}\left(\inf_{t\in[0,M]} f_\mu(t) > \ell-2\varepsilon\right).$$

Plugging this into (5.7) yields

$$\mathbb{P}\left(\inf_{[0,T]} X(t) > \ell\right) \leq \left[2\left(1+\xi_{M,\ell}\right)\mathbb{P}\left(\inf_{t\in[0,M]} f_\mu(t) > \ell - 2\varepsilon\right)\right]^N .$$

Taking log on both sides and dividing by T gives

$$\begin{aligned}\frac{1}{T}\log\mathbb{P}\left(\inf_{[0,T]} X(t) > \ell\right) &\leq \frac{N}{T}\left(\log 2 + \log\left(1+\xi_{M,\ell}\right) + \log \mathcal{P}_\mu^{\ell-2\varepsilon}(M)\right)\\ &\leq \frac{\log 2}{M} + \frac{\xi_{M,\ell}}{M} + \frac{1}{M}\log\mathcal{P}_\mu^{\ell-2\varepsilon}(M) + \left|\left(\frac{N}{T}-\frac{1}{M}\right)\log\mathcal{P}_\mu^{\ell-2\varepsilon}(M)\right|\\ &\leq \frac{\log 2 + K + \xi_{M,\ell}}{M} + \frac{1}{M}\log\mathcal{P}_\mu^{\ell-2\varepsilon}(M),\end{aligned}$$

where in the last step we used (5.8) and the fact that $MN \leq T \leq M(N+1)$. Applying Lemma 1.1, and noting that $\varepsilon = \delta \geq \frac{1}{M}$, we obtain

$$\theta_\mu^\ell(M) - \widetilde{\theta}^\ell(T) \leq C''\delta + \frac{\xi_{M,\ell}}{M}.$$

The proof is concluded on recalling the definition of δ. □

5.3.2 Proof of Proposition 5.2

Let $\varepsilon > 0$. Let $c > 0$ be the constant whose existence is given by Part II of Lemma 1.3, and define $L = \sqrt{\frac{2}{c\varepsilon}}$, $\eta = (c\varepsilon)^{1/4}$. For a given $T > 0$, let $\mu, \nu \in \mathcal{M}_{\alpha,A} \cap \mathcal{L}_{\beta,B}$ be such that $d_{TV}(\mu,\nu) < \varepsilon$ and $\mu|_{[-\frac{L}{T},\frac{L}{T}]} = \nu|_{[-\frac{L}{T},\frac{L}{T}]}$. By Claim 5.6, there exists a measure γ such that

- $\gamma \geq \max\{\mu,\nu\}$,
- $\max\{d_{\mathrm{TV}}(\gamma,\mu), d_{\mathrm{TV}}(\gamma,\nu)\} < 2\varepsilon$, and
- $\gamma = \mu = \nu$ on $[-\frac{L}{T},\frac{L}{T}]$.

Using Part I of Lemma 1.3, if $T > \max\{4, A^{-1}\}$ then

$$\theta_\gamma^\ell(T) \leq \theta_\mu^\ell(T) + C_\varepsilon^{(1)}, \tag{5.9}$$

where $\lim\limits_{\varepsilon\to 0} C_\varepsilon^{(1)} = 0$. By Part II of Lemma 1.3, there exist ε_0 and $T_0 = T_0(\varepsilon)$ such that, if $\varepsilon < \varepsilon_0$ and $T > \max\{T_0, \frac{2}{A}\}$, we have

$$\theta_\nu^\ell\left(T(1-\eta)\right) \leq \theta_\gamma^\ell(T) + C_\varepsilon^{(2)}, \tag{5.10}$$

where $\lim\limits_{\varepsilon\to 0} C_\varepsilon^{(2)} = 0$. Combining (5.9) and (5.10) yields

$$\theta_\nu^\ell\left(T(1-\eta)\right) \leq \theta_\mu^\ell(T) + C_\varepsilon^{(1)} + C_\varepsilon^{(2)},$$

provided that $T > \max\{T_0, \frac{2}{A}\}$ and $\varepsilon < \varepsilon_0$, as desired.

5.3.3 Proof of Proposition 5.3

Proof. Let $\rho = \mu|_{\mathbb{R}\setminus[-\frac{L}{T},\frac{L}{T}]} = \nu|_{\mathbb{R}\setminus[-\frac{L}{T},\frac{L}{T}]}$ and fix $\tau = \tau(L,\varepsilon,m) > 0$ to be chosen later. Write $n = \lceil \frac{L}{\tau} \rceil$ and let $\{I_j\}_{j\in\pm[n]}$ be the decomposition of $[-\frac{L}{T},\frac{L}{T}]$ used in Lemma 4.2. For $\chi \in \{\mu,\nu\}$, we write $\chi^j := \chi(I_j \cup I_{-j})$ and

$$U_n^\chi(t) = \sum_{j=1}^n \sqrt{\chi^j}\Big(\zeta_j^\chi C_j^\chi(t) \oplus \eta_j^\chi S_j^\chi(t)\Big).$$

Using Lemma 4.2, we obtain the decomposition

$$f_\chi(t) = U_n^\chi(t) \oplus R_n^\chi(t) \oplus f_\rho(t),$$

where R_n^χ is a Gaussian process, which satisfies for all $T > T_0(L)$,

$$\begin{aligned} &\mathrm{var}\left(R_n^\chi(t)\right) \le \tfrac{1}{2}\tau^2\chi\big([-\tfrac{L}{T},\tfrac{L}{T}]\big) \le \tfrac{1}{2}\tau^2 m\tfrac{L}{T}, \\ &\mathrm{var}\left(R_n^\chi(t) - R_n^\chi(t+h)\right) \le (\tfrac{L}{T})^2\chi\big([-\tfrac{L}{T},\tfrac{L}{T}]\big)h^2 \le m(\tfrac{L}{T})^3h^2. \end{aligned} \tag{5.11}$$

Next, we couple f_μ and f_ν by taking $\zeta_j^\mu = \zeta_j^\nu$ and $\eta_j^\mu = \eta_j^\nu$ with $(U_n^\mu(t), R_n^\mu(t))$ and $(U_n^\nu(t), R_n^\nu(t))$ independent. Observe that by containment of events, we have for all $\delta > 0$,

$$\mathcal{P}_\mu^\ell(T) \le \mathcal{P}_\nu^{\ell-3\delta}(T) + \mathbb{P}\left(\sup_{[0,T]} |U_n^\mu - U_n^\nu| > \delta\right) + \mathbb{P}\left(\sup_{[0,T]} |R_n^\mu| > \delta\right) + \mathbb{P}\left(\sup_{[0,T]} |R_n^\nu| > \delta\right).$$

In the remainder of the proof, we show that for a particular choice of $\delta = \delta(\varepsilon, L, m)$ tending to 0 as $\varepsilon \to 0$, we have

$$\frac{\mathcal{P}_\mu^\ell(T)}{2} > \mathbb{P}\left(\sup_{[0,T]} |U_n^\mu - U_n^\nu| > \delta\right) + \mathbb{P}\left(\sup_{[0,T]} |R_n^\mu| > \delta\right) + \mathbb{P}\left(\sup_{[0,T]} |R_n^\nu| > \delta\right). \tag{5.12}$$

Indeed, this will imply that $\theta_\mu^\ell(T) \le \theta_\nu^{\ell-3\delta}(T) + \frac{\log 2}{T}$, from which, using continuity of $\theta_\mu^\ell(T)$ as a function of ℓ (Lemma 1.1), we will obtain $\theta_\mu^\ell(T) \le \theta_\nu^\ell(T) + C_\varepsilon$. As μ and ν are interchangeable, the proposition will readily follow.

We first bound $\mathbb{P}\left(\sup_{[0,T]} |R_n^\chi| > \delta\right)$ for $\chi \in \{\mu,\nu\}$. By (5.11) we may apply Lemma 2.21 to obtain that, for $T > T_0(L,\varepsilon,m)$,

$$\mathbb{P}\left(\sup_{[0,T]} |R_n^\chi| > \delta\right) \le 2e^{-\frac{\delta^2}{4\tau^2 mL}T}. \tag{5.13}$$

Next we bound $\mathbb{P}\left(\sup_{[0,T]} |U_n^\mu - U_n^\nu| > \delta\right)$. Once again we wish to apply Lemma 2.21; let us verify its conditions. We first compute

$$\mathrm{var}\,\left(U_n^\mu(t) - U_n^\nu(t)\right) = \sum_{j=1}^n \left(\sqrt{\mu^j}C_j^\mu(t) - \sqrt{\nu^j}C_j^\nu(t)\right)^2 + \sum_{j=1}^n \left(\sqrt{\mu^j}S_j^\mu(t) - \sqrt{\nu^j}S_j^\nu(t)\right)^2. \tag{5.14}$$

We bound the first term by

$$\sum_{j=1}^{n}\left(\sqrt{\mu^j}C_j^{\mu}(t)-\sqrt{\nu^j}C_j^{\nu}(t)\right)^2\le 2\sum_{j=1}^{n}\mu^j\left(C_j^{\mu}(t)-C_j^{\nu}(t)\right)^2+2\sum_{j=1}^{n}\left(\sqrt{\mu^j}-\sqrt{\nu^j}\right)^2C_j^{\nu}(t)^2. \quad (5.15)$$

We note that

$$|C_j^{\mu}(t)-\cos(j\tfrac{\tau}{T}t)|=\int_{I_j}\left(\cos(\lambda t)-\cos(j\tfrac{\tau}{T}t)\right)\frac{d\mu(\lambda)}{\mu(I_j)}\le\tau,$$

so that

$$\sum_{j=1}^{n}\mu^j\left(C_j^{\mu}(t)-C_j^{\nu}(t)\right)^2\le 4\tau^2\sum_{j=1}^{n}\mu^j\le 4\tau^2m\frac{L}{T}. \quad (5.16)$$

Next we notice that, if $\max\{\mu^j,\nu^j\}\le\frac{\tau\varepsilon n}{T}$, then

$$\left(\sqrt{\mu^j}-\sqrt{\nu^j}\right)^2\le\frac{\tau\varepsilon n}{T};$$

while if $\max\{\mu^j,\nu^j\}>\frac{\tau\varepsilon n}{T}$ then

$$\left(\sqrt{\mu^j}-\sqrt{\nu^j}\right)^2=\frac{\left(\mu^j-\nu^j\right)^2}{\left(\sqrt{\mu^j}+\sqrt{\nu^j}\right)^2}\le\frac{T}{\tau\varepsilon n}\left(\mu^j-\nu^j\right)^2.$$

Using the assumption that $|\mu(-\lambda,\lambda)-\nu(-\lambda,\lambda)|\le 2\varepsilon\lambda$ for all $\lambda\in(0,\alpha)$, we have

$$\left|\mu^j-\nu^j\right|\le\left|\mu\left(-\tfrac{j\tau}{T},\tfrac{j\tau}{T}\right)-\nu\left(-\tfrac{j\tau}{T},\tfrac{j\tau}{T}\right)\right|+\left|\mu\left(-\tfrac{(j-1)\tau}{T},\tfrac{(j-1)\tau}{T}\right)-\nu\left(-\tfrac{(j-1)\tau}{T},\tfrac{(j-1)\tau}{T}\right)\right|\le 4\frac{j\tau\varepsilon}{T}.$$

Recalling that $C_j(t)\le 1$, we obtain

$$\sum_{j=1}^{n}\left(\sqrt{\mu^j}-\sqrt{\nu^j}\right)^2C_j^{\nu}(t)^2\le n\left(\frac{\tau\varepsilon n}{T}+\frac{T}{\tau\varepsilon n}\cdot\left(4\frac{n\tau\varepsilon}{T}\right)^2\right)=17n\cdot\frac{\tau\varepsilon n}{T}\le 2^7\cdot\frac{\varepsilon L^2}{\tau T},$$

where the last step uses $n=\lceil\frac{L}{\tau}\rceil\le 2\frac{L}{\tau}$. Putting this together with (5.16) into (5.15), we get

$$\sum_{j=1}^{n}\left(\sqrt{\mu^j}C_j^{\mu}(t)-\sqrt{\nu^j}C_j^{\nu}(t)\right)^2\le 2^8\frac{L}{T}\left(\tau^2m+\varepsilon\frac{L}{\tau}\right).$$

Applying the same chain of arguments yields $\sum_{j=1}^{n}\left(\sqrt{\mu^j}S_j^{\mu}(t)-\sqrt{\nu^j}S_j^{\nu}(t)\right)^2\le 2^8\frac{L}{T}\left(\tau^2m+\varepsilon\frac{L}{\tau}\right)$. Plugging these bounds into (5.14), we conclude that,

$$\mathrm{var}\left(U_n^{\mu}(t)-U_n^{\nu}(t)\right)\le 2^8\frac{L}{T}\left(\tau^2m+\varepsilon\frac{L}{\tau}\right).$$

Next, writing $\Delta U_n^{\chi}(t)=U_n^{\chi}(t+h)-U_n^{\chi}(t)$, we compute $\mathrm{var}\left(\Delta U_n^{\mu}(t)-\Delta U_n^{\nu}(t)\right)$.

$$\begin{aligned}\operatorname{var}\left(\Delta U_n^{\mu}(t)-\Delta U_n^{\nu}(t)\right) &\le 2\operatorname{var}\left(\Delta U_n^{\mu}(t)\right)+2\operatorname{var}\left(\Delta U_n^{\nu}(t)\right)\\ &\le 2\operatorname{var}\left(f_{\mu_1}(t+h)-f_{\mu_1}(t)\right)+2\operatorname{var}\left(f_{\nu_1}(t+h)-f_{\nu_1}(t)\right)\\ &\le \left(\mu([-1,1])+\nu([-1,1])\right)h^2\end{aligned}$$

where the last inequality uses Obs. 2.18. (Note that we assume $T > L$, so that $[-\frac{L}{T}, \frac{L}{T}] \subseteq [-1, 1]$.) Since $\mu, \nu \in \mathcal{L}_{\beta,B}$, the quantities $\mu([-1, 1])$ and $\nu([-1, 1])$ have an upper bound depending only on β and B. We have thus shown the existance of $c = c(\beta, B)$ for which

$$\operatorname{var}\left(\Delta U_n^{\mu}(t)-\Delta U_n^{\nu}(t)\right) \le c^2h^2.$$

Applying Lemma 2.21, we have, for $T > T_0(L, \varepsilon, m)$,

$$\mathbb{P}(\sup_{[0,T]} |U_n^{\mu} - U_n^{\nu}| > \delta) \le 2\exp\left(-\frac{\delta^2}{2^{11}L(\tau^2 m+\varepsilon L/\tau)}T\right). \tag{5.17}$$

We now select $\tau = \left(\frac{\varepsilon L}{m}\right)^{1/3}$ and obtain

$$\mathbb{P}(\sup_{[0,T]} |U_n^{\mu} - U_n^{\nu}| > \delta) \le 2\exp\left(-\frac{\delta^2}{2^{12}(mL^7\varepsilon^2)^{1/3}}T\right). \tag{5.18}$$

Applying Lemma 2.23, there exists $M \in (0,\infty)$ such that $\mathcal{P}_{\mu}^{\ell}(T) > e^{-MT}$ for all $T > 1$. Using (5.13) and (5.18), we reduce (5.12) into showing that

$$e^{-MT} \ge 2e^{-\frac{\delta^2}{2^{12}(L^7\varepsilon^2 m)^{1/3}}T} + 4e^{-\frac{\delta^2}{4(L^7\varepsilon^2 m)^{1/3}}T}.$$

Setting $\delta^2 = 2^{13}M(L^7\varepsilon^2 m)^{1/3}$, this indeed holds for all large enough T. □

5.4 An example of non-existence

In this section we prove Remark 1. More precisely, we show that, for some values of $0 < a < b < \infty$, the absolutely continuous measure $\rho = \rho_{a,b}$ whose density is

$$w_{a,b}(\lambda) = \left(\frac{b+a}{2} + \frac{b-a}{2}\cos\left(\tfrac{1}{\lambda}\right)\right)\mathbb{1}_{|\lambda|\le 1}$$

does not admit existence of the 0-level persistence exponent θ_ρ^0.

Clearly $a \le w_{a,b}(\lambda) \le b$ for all $|\lambda| \le 1$. We observe that

$$\liminf_{\lambda\to 0}\frac{\rho_{a,b}([0,\lambda])}{\lambda} = a, \quad \limsup_{\lambda\to 0}\frac{\rho_{a,b}([0,\lambda])}{\lambda} = b. \tag{5.19}$$

Next, we make use of results from [28], simplifying many constants due to our specific form of spectral density. For instance, due to the fact that $d\rho(\lambda) \ge a\mathbb{1}_{[-1,1]}(\lambda)d\lambda$, we may take $|E| = 2$ and $\nu = a$ in the notations of that paper. Applying [28, Thm. 5.1 and Rmk. 2] with the appropriate parameters, namely

$$\gamma = 1 \text{ (implying } k=0, s=1, r=\tfrac{1}{2}, \theta=1), |E|=2, \nu=a, q=1,$$

we have, for all fixed $\ell > 0$ and all $T > T_0(\ell)$, that

$$\rho\left([0,\tfrac{1}{T}]\right) \le \frac{a+\varepsilon}{T} \quad \Rightarrow \quad \mathcal{P}^0_\rho(T) \le 2\mathbb{P}\left(\sqrt{a+\varepsilon}\, Z > \ell\sqrt{T}\right) + 2\mathbb{P}\left(c_1\sqrt{1-\tfrac{\varepsilon}{a+\varepsilon}}\,|Z| < \ell\right)^{c_2 T}, \quad (5.20)$$

where c_1 and c_2 are universal constants and $Z \sim \mathcal{N}(0,1)$ is a standard normal random variable. Next by [28, Theorem 4.1] we have, for all $\ell > \ell_0(b)$ and all $T > 0$, that

$$\rho\left([0,\tfrac{1}{T}]\right) \ge \frac{b-\varepsilon}{T} \quad \Rightarrow \quad \mathcal{P}^0_\rho(T) \ge \mathbb{P}\left(\sqrt{b-\varepsilon}\, Z > \ell\sqrt{T}\right) \cdot \mathbb{P}\Big(\beta(b)\ |Z| < \ell\Big)^T, \quad (5.21)$$

where $\ell_0(b)$ and $\beta(b)$ are constants which depends only on b.

We proceed by applying the above with $\varepsilon = \frac{a}{2}$. For a given $b > 0$, fix $\ell_1 > \ell_0(b)$. By Lemma 2.1, there exists $\theta_1 > 0$ such that

$$\mathbb{P}\left(\sqrt{b/2}\, Z > \ell_1\sqrt{T}\right) \cdot \mathbb{P}\Big(\beta(b)\ |Z| < \ell_1\Big)^T \ge e^{-\theta_1 T},$$

for all large enough T. Now fix $\theta_2 > \theta_1$, and choose $\ell_2 > 0$ such that

$$2\mathbb{P}\left(c_1\sqrt{\tfrac{2}{3}}\,|Z| < \ell_2\right)^{c_2 T} \le \tfrac{1}{2}e^{-\theta_2 T}.$$

Again by Lemma 2.1 we may choose $a \in (0, \frac{b}{3})$ so small that

$$2\mathbb{P}\left(\sqrt{\tfrac{3a}{2}}\, Z > \ell_2\sqrt{T}\right) \le \tfrac{1}{2}e^{-\theta_2 T}.$$

Combining these choices with (5.20) and (5.21), we obtain that

$$\begin{aligned} \rho\left([0,\tfrac{1}{T}]\right) \ge \frac{b}{2T} \quad &\Rightarrow \quad \mathcal{P}^0_\rho(T) \ge e^{-\theta_1 T}, \\ \rho\left([0,\tfrac{1}{T}]\right) \le \frac{3a}{2T} \quad &\Rightarrow \quad \mathcal{P}^0_\rho(T) \le e^{-\theta_2 T}, \end{aligned}$$

for all large enough T. Recalling (5.19) and the definition of $\theta^0_\rho(T)$ in (1.3), we conclude that

$$\liminf_{T\to\infty} \theta^0_\rho(T) \le \theta_1 < \theta_2 \le \limsup_{T\to\infty} \theta^0_\rho(T),$$

which implies that the persistence exponent θ^0_ρ does not exist.

6 Monotonicity of the ball and persistence exponents

6.1 Proof of Theorem 3–(I)

Let $\beta, B > 0$ be such that $\rho, \nu \in \mathcal{L}_{\beta,B}$. We may assume that $\ell > 0$, since otherwise $\psi^\ell_{\rho+\nu} = \psi^\ell_\rho = \infty$. By Lemma 2.11 we have $f_{\rho+\nu} \stackrel{d}{=} f_\rho \oplus f_\nu$. An application of Anderson's inequality (Proposition 2.16) gives

$$\psi^\ell_\rho \leq \psi^\ell_{\rho+\nu}.$$

Next assume that ν is purely-singular. Using Lemma 2.12(a) we get $\psi^\ell_{\rho+\nu} \leq \psi^{\ell-\delta}_\rho + \psi^\delta_\nu$. Using Theorem 2, we obtain that $\psi^\delta_\nu = 0$. By Theorem 4–(I) we have $\psi^{\ell-\delta}_\rho \leq \psi^\ell_\rho + C_\delta$, where $\lim\limits_{\delta\to 0} C_\delta = 0$. We conclude that

$$\psi^\ell_\rho \leq \psi^\ell_{\rho+\nu} \leq \psi^\ell_\rho + C_\delta,$$

where upon taking $\delta \to 0$ yields $\psi^\ell_{\rho+\nu} = \psi^\ell_\rho$.

6.2 Proof of Theorem 3–(II)

Let $\alpha, A, \beta, B > 0$ be such that $\rho \in \mathcal{M} \cap \mathcal{M}_{\alpha,A} \cap \mathcal{L}_{\beta,B}$ and $\nu \in \mathcal{L}_{\beta,B}$. Assume that $\nu'(0) = 0$. Fixing $\eta, \varepsilon > 0$ and using Corollary 3.2 with $L = \varepsilon T$, there exists $T_0(\eta)$ such that for all $T > \max\left\{T_0, \frac{1}{A}\right\}$ we have

$$\theta^\ell_{\rho+\nu_\varepsilon}(T(1-\eta)) \leq \theta^\ell_{\rho+\nu}(T) + C_\eta,$$

where $\lim_{\eta\to 0} C_\eta = 0$. Letting $T \to \infty$ and using Theorem 1 we get

$$\theta^\ell_{\rho+\nu_\varepsilon} \leq \theta^\ell_{\rho+\nu} + C_\eta.$$

Letting $\eta \to 0$ we then get

$$\theta^\ell_{\rho+\nu_\varepsilon} \leq \theta^\ell_{\rho+\nu}. \tag{6.1}$$

Finally note that

$$d_{\mathrm{TV}_0}(\rho + \nu_\varepsilon, \rho) = \nu'(0) + \nu([-\varepsilon, \varepsilon]) \to 0, \quad \text{as } \varepsilon \to 0,$$

as $\nu'(0) = 0$. Thus, invoking Theorem 4 we get $\lim\limits_{\varepsilon\to 0} \theta^\ell_{\rho+\nu_\varepsilon} = \theta^\ell_\rho$. Along with (6.1), we obtain

$$\theta^\ell_\rho \leq \theta^\ell_{\rho+\nu},$$

thus verifying the inequality of Part (II).

Assume now that ν is purely singular. Lemma 2.12(b) yields that, for any $\delta > 0$,

$$\theta^\ell_{\rho+\nu} \leq \theta^{\ell+\delta}_\rho + \psi^\delta_\nu. \tag{6.2}$$

Since ν is purely singular, Theorem 2 yields that $\psi^\delta_\nu = 0$; while from Lemma 1.1 we deduce that $\lim\limits_{\delta\to 0} \theta^{\ell+\delta}_\rho = \theta^\ell_\rho$. Using these two facts in (6.2) yields $\theta^\ell_{\rho+\nu} \leq \theta^\ell_\rho$, which completes the proof.

7 Exponents under sampling

In this section we prove Theorem 5 concerning the convergence of the ball and persistence exponents of fine mesh sampling of a continuous-time process, to its continuous-time exponent. This is done in Sections 7.1 and 7.2. Then, in Section 7.3 we establish the tightness of our criterion by providing an instructive example of non-convergence.

7.1 Proof of Theorem 5–(I): ball exponent under sampling

Let $\rho \in \mathcal{L}$ and $\ell > 0$. Fix $\Delta > 0$. Complementing the definition in (2.1), we set

$$\psi^{\ell}_{\rho;\Delta}(T) := -\frac{1}{T}\log \mathbb{P}\left(\sup_{n\in\mathbb{Z},\, n\Delta\in[0,T]} |f_\rho(n\Delta)| < \ell\right).$$

Define the centered Gaussian (non-stationary) process $X_\Delta(t) := f_\rho\Big(\Delta\lceil \frac{t}{\Delta}\rceil\Big)$, and note that

$$\psi^{\ell}_{\rho;\Delta}(T) = -\frac{1}{T}\log \mathbb{P}\left(\sup_{[0,T]} |X_\Delta| < \ell\right) \le -\frac{1}{T}\log \mathbb{P}\left(\sup_{[0,T]} |f_\rho| < \ell\right) = \psi^{\ell}_{\rho}(T).$$

By Corollary 2.5 (valid also for discrete processes by Remark 7), we may take limits as $T\to\infty$ to obtain $\psi^{\ell}_{\rho;\Delta} \le \psi^{\ell}_{\rho}$. It remains to show that

$$\psi^{\ell}_{\rho} \le \liminf_{\Delta\to 0} \psi^{\ell}_{\rho;\Delta}. \tag{7.1}$$

To this end, fix $\delta > 0$ and apply Khatri-Sidak Inequality (Proposition 2.3) to get

$$\begin{aligned}\mathbb{P}\left(\sup_{[0,T]} |f_\rho| < \ell+\delta\right) &\ge \mathbb{P}\left(\sup_{[0,T]} |X_\Delta| < \ell, \sup_{[0,T]} |X_\Delta - f_\rho| < \delta\right)\\ &\ge \mathbb{P}\left(\sup_{[0,T]} |X_\Delta| < \ell\right)\mathbb{P}\left(\sup_{[0,1]} |X_\Delta - f_\rho| < \delta\right)^{\lceil T\rceil}.\end{aligned}$$

Upon taking log, dividing by T and letting $T\to\infty$ we have

$$\psi^{\ell+\delta}_{\rho} \le \psi^{\ell}_{\rho;\Delta} + \log \mathbb{P}\Big(\sup_{[0,1]} |X_\Delta - f_\rho| < \delta\Big).$$

The sample path continuity of $f_\rho(.)$ yields that $\sup_{[0,1]} |X_\Delta - f_\rho| \overset{a.s.}{\to} 0$ as $\Delta \to 0$, and so

$$\psi^{\ell+\delta}_{\rho} \le \liminf_{\Delta\to 0} \psi^{\ell}_{\rho;\Delta}.$$

Finally, letting $\delta \to 0$ and using Lemma 1.1 gives (7.1), as required.

7.2 Proof of Theorem 5–(II): persistence exponent under sampling

For $\ell \in \mathbb{R}$, $\Delta > 0$ and a spectral measure ρ, we complement the definitions in (1.3) by setting

$$\mathcal{P}^{\ell}_{\rho;\Delta}(T) := \mathbb{P}\left(\inf_{n\in\mathbb{Z},\, n\Delta\in[0,T]} f_\rho(n\Delta) > \ell\right), \qquad \theta^{\ell}_{\rho;\Delta}(T) := -\frac{1}{T}\log \mathcal{P}^{\ell}_{\rho;\Delta}(T). \tag{7.2}$$

Theorem 5–(II) is a consequence of the following.

Proposition 7.1. *Suppose $\rho \in \mathcal{M}$ has density $\rho' \in C^2(\mathbb{R})$ which is compactly supported. Then*

$$\limsup_{\substack{T\to\infty\\ \Delta\to 0}} \left|\theta^{\ell}_{\rho;\Delta}(T) - \theta^{\ell}_{\rho}(T)\right| = 0. \tag{7.3}$$

Proposition 7.1 yields Theorem 5–(II). Let $\rho \in \mathcal{L}\cap\mathcal{M}$ be compactly supported, and let $\varepsilon > 0$. Write ν for the smooth measure with $d_{\mathrm{TV}}(\rho,\nu) < \varepsilon$ from Claim 5.4. By the triangle inequality we have

$$\left|\theta^{\ell}_{\rho} - \theta^{\ell}_{\rho;\Delta}\right| \le \left|\theta^{\ell}_{\rho} - \theta^{\ell}_{\nu}\right| + \left|\theta^{\ell}_{\nu} - \theta^{\ell}_{\nu;\Delta}\right| + \left|\theta^{\ell}_{\nu;\Delta} - \theta^{\ell}_{\rho;\Delta}\right| \tag{7.4}$$

By Proposition 5.5 we have

$$\left|\theta^{\ell}_{\rho} - \theta^{\ell}_{\nu}\right| < C_\varepsilon, \tag{7.5}$$

where $\lim_{\varepsilon\to 0} C_\varepsilon = 0$ and C_ε depends on ρ.

By Observation 2.8, for any $\Delta > 0$, the spectral measure of the discrete-time process $\{f(j\Delta)\}_{j\in\mathbb{Z}}$ is $\rho^*_\Delta(I) = \rho\left(\bigcup_{n\in\mathbb{Z}}\left\{I + 2\pi\frac{n}{\Delta}\right\}\right)$ (supported in $[-\frac{\pi}{\Delta},\frac{\pi}{\Delta}]$, the dual space of $\Delta\mathbb{Z}$). Since ρ and ν are compactly supported, $\rho^*_\Delta = \rho$ and $\nu^*_\Delta = \nu$ for all small enough Δ. Applying Proposition 5.5 to the sequence $\{f_\rho(j\Delta)\}_{j\in\mathbb{Z}}$ (possible by Remark 7), we conclude that, for small enough Δ,

$$\left|\theta^{\ell}_{\rho;\Delta} - \theta^{\ell}_{\nu;\Delta}\right| < C_\varepsilon. \tag{7.6}$$

By Proposition 7.1 we have

$$\lim_{\Delta\to 0}\left|\theta^{\ell}_{\nu} - \theta^{\ell}_{\nu;\Delta}\right| = 0. \tag{7.7}$$

Taking lim sup as Δ tends to 0 on (7.4), and plugging in (7.5), (7.6) and (7.7), yields

$$\limsup_{\Delta\to 0}\left|\theta^{\ell}_{\rho} - \theta^{\ell}_{\rho;\Delta}\right| < 2C_\varepsilon.$$

As $\lim_{\varepsilon\to 0} C_\varepsilon = 0$, the proposition follows. □

Proof of Proposition 7.1. The proof is an application of Lemma 5.7. Fixing $\Delta > 0$ define a (non-stationary) Gaussian process $X_\Delta(\cdot)$ by setting

$$X_\Delta(t) = f_\rho(\Delta\lfloor \tfrac{t}{\Delta}\rfloor).$$

Then fixing $M > 0$ we have

$$\mathbb{P}\left(\inf_{t\in[s,s+M]} X_\Delta(t) > u\right) = \mathbb{P}\left(\min_{\lfloor\frac{s}{\Delta}\rfloor \le i \le \lfloor\frac{s+M}{\Delta}\rfloor} f_\rho(i\Delta) > u\right) = \mathbb{P}\left(\min_{0\le i\le \lfloor\frac{s+M}{\Delta}\rfloor - \lfloor\frac{s}{\Delta}\rfloor} f_\rho(i\Delta) > u\right),$$

which converges to $\mathbb{P}\left(\inf_{t\in[0,M]} f_\rho(t) > u\right)$ uniformly in $s \geq 0$ and in $u \in [\ell-1, \ell+1]$ as $\Delta \to 0$, by stationarity and continuity of sample paths. Therefore

$$\xi_{M,\ell}^{\Delta} := \sup_{u\in[\ell-1,\ell+1]} \sup_{s\geq 0} \left| \frac{\mathbb{P}\left(\inf_{t\in[s,s+M]} X_\Delta(t) > u\right)}{\mathbb{P}\left(\inf_{t\in[0,M]} f_\rho(t) > u\right)} - 1 \right|$$

satisfies $\lim\limits_{\Delta\to 0} \xi_{M,\ell}^{\Delta} = 0$. Moreover, for some $c = c(\rho) > 0$ we have $|r(t)| = |\widehat{\rho}(t)| \leq \frac{c}{|t|^2}$, since ρ is compactly supported with C^2-density. This implies

$$|\mathbb{E}X_\Delta(s)X_\Delta(s+t)| = \widehat{\rho}\left(\Delta\lfloor \tfrac{s}{\Delta}\rfloor - \Delta\lfloor \tfrac{s+t}{\Delta}\rfloor\right) \leq c\left|\Delta\lfloor \tfrac{s}{\Delta}\rfloor - \Delta\lfloor \tfrac{s+t}{\Delta}\rfloor\right|^{-2} \leq \frac{c'}{|t|^2}.$$

The convergence in (7.3) then follows from Lemma 5.7. □

7.3 An example of non-convergence under sampling

In this section we give an example of an absolutely continuous spectral measure ρ, such that the limit of the sampled process' exponents $\theta^0_{\rho;\frac{1}{k}}$ for $k \in \mathbb{N}$ tends to 0 as $k \to \infty$, while θ^0_ρ exists, and is strictly positive.

Let ρ be the the absolutely continuous measure with density

$$\rho'(\lambda) = \frac{(1 - e^{|\lambda|}\mathrm{dist}(\lambda, 2\pi\mathbb{Z}))_+}{|\lambda|} 1\!\!1_{|\lambda|>\pi} + 1\!\!1_{|\lambda|<\pi}.$$

It is clear that ρ' is non-negative and symmetric. To see that $\rho' \in L^1(\mathbb{R})$, note that for any $n \in \mathbb{N}$ we have: $\int_{2\pi n-\pi}^{2\pi n+\pi} \rho'(\lambda)d\lambda \leq \frac{2}{2\pi n-\pi}e^{-2\pi n}$. It is also clear that $\rho \in \mathcal{L} \cap \mathcal{M}$ (in fact, ρ has a finite exponential moment). Thus Theorem 1 implies the existence of $\theta^0_\rho \in (0,\infty)$. Now let us consider the sampled process. By Observation 2.8, for any $k \in \mathbb{N}$, the discrete-time process $j \mapsto f\left(\frac{j}{k}\right)$ has the spectral measure $\rho^*_k(I) = \rho\left(\bigcup_{n\in\mathbb{Z}} \{I + 2\pi nk\}\right)$. The local density of this measure at 0 is

$$\liminf_{\varepsilon\to 0} \frac{\rho^*_k((-\varepsilon,\varepsilon))}{2\varepsilon} \geq \sum_{n\in\mathbb{Z}} \liminf_{\varepsilon\to 0} \frac{\rho((2\pi nk - \varepsilon, 2\pi nk + \varepsilon))}{2\varepsilon} = \sum_{n\in\mathbb{Z}} \frac{1}{2\pi nk} = \infty.$$

Applying Proposition 2.25 to ρ^*_k, we obtain that $\theta^0_{\rho;\frac{1}{k}} = \theta^0_{\rho^*_k} = 0$. As we have seen that $\theta^0_\rho > 0$, we conclude that $\lim\limits_{k\to\infty} \theta^0_{\rho;\frac{1}{k}} \neq \theta^0_\rho$.

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