AN ASYMPTOTIC FORMULA FOR THE VARIANCE OF THE NUMBER OF ZEROES OF A STATIONARY GAUSSIAN PROCESS

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Abstract. We study the number of zeroes of a stationary Gaussian process on a long interval. We give a simple asymptotic description of the variance of this random variable, under mild mixing conditions. In particular, we give a linear lower bound for any non-degenerate process. We show that a small (symmetrised) atom in the spectral measure at a special frequency does not affect the asymptotic growth of the variance, while an atom at any other frequency results in maximal growth. Our results allow us to analyse a large number of interesting examples. We state some conjectures which generalise our results.

Key words: Gaussian process, stationary process, fluctuations of zeroes, Wiener Chaos
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1. Introduction

1.1. **Background.** Zeroes of Gaussian processes, and in particular stationary Gaussian processes (SGPs), are widely used models in physics and signal processing, and consequently were intensively studied. The expected number of zeroes may be computed by the celebrated Kac-Rice formula, whose origins lie in the independent work of Kac [19, 20] and of Rice [31, 32]. Estimating the fluctuations, however, proved to be a much more difficult task. An exact but somewhat inaccessible formula for the variance was rigourously derived by Cramér and Leadbetter [7, Sections 10.6-7], although such a formula was known to physicists [34] and had been proved mathematically under some more restrictive hypothesis[1] (see the footnote on [38, Page 188]). This formula was based on the ideas of Kac-Rice, and little progress in understanding the variance was made until Slud [35, 36] introduced Multiple Wiener Integral techniques some decades later — these were in turn refined and extended by Kratz and Léon [26, 27], using Wiener chaos expansions. For a far more comprehensive overview of the study of zeroes of SGPs we refer the reader to the survey [25].

The formulas mentioned above were used to prove various properties of the zeroes, such as sufficient conditions for linearity of the variance and for a central limit theorem as detailed below, but extracting the asymptotic growth of the variance under reasonably general conditions has proved fruitless. For example, the only attempt at a systematic study of super-linear growth of the variance that we are aware of is a special family of examples due to Slud [36, Theorem 3.2].

The aim of this paper is to give a simple expression which describes the asymptotic growth[2] of the variance of the number of zeroes in a growing interval. One simple corollary is a linear lower bound on the variance, which holds under (essentially) no conditions. The main idea here is due to Slud, although our result holds in greater generality; this is due to Kratz and Leon proving that a certain expansion holds under (essentially) no assumptions. Our main contribution is perhaps, instead, a matching upper bound, which holds under a very mild hypothesis. In particular we give an asymptotic expression for the variance for any process with decaying correlations, *no matter how slow the decay.* This appears to be novel and, in addition to providing a sharp estimate for previously unknown cases, also allows us to improve upon previous results; for example, in Section 2.5 we remove some of the technical assumptions in Slud's [36, Theorem 3.2]. An intriguing

[1] Specifically, the authors assumed the existence of $r^{(vi)}(0)$, in the notation introduced in this article.
[2] We mean this in a wide sense, the variance might be an oscillating function.

feature of our results is the emergence of a 'special frequency': adding an atom to the spectral measure at this frequency *does not change the order of growth of the fluctuations* (strictly speaking we can only prove this if the strength of the atom is below a certain threshold, but we believe it to be true in general).

1.2. **Results.** Let $f:\mathbb{R}\to\mathbb{R}$ be a stationary Gaussian process (SGP) with continuous *covariance kernel*

$$r(t)=\mathbb{E}[f(0)f(t)].$$

Denote by ρ the *spectral measure* of the process, that is, the unique finite, symmetric measure on $\mathbb{R}$ such that

$$r(t)=\mathcal{F}[\rho](t)=\int_{\mathbb{R}}e^{-i\lambda t}d\rho(\lambda).$$

We normalise the process so that $r(0)=\rho(\mathbb{R})=1$. It is well-known (see, e.g., [7, Section 7.6]) that the distribution of f is determined by ρ, and further that any such ρ is the spectral measure of some SGP.

Our main object of study is the number of zeroes of f in a long 'time' interval $[0,T]$, which we denote by

$$N(\rho;T)=N(T)=\#\{t\in[0,T]:\ f(t)=0\}.$$

The expectation of $N(T)$ may be computed by the Kac-Rice formula to be

$$\mathbb{E}[N(T)]=\frac{\sigma}{\pi}T,$$

where

$$\sigma^2=-r''(0)=\int_{\mathbb{R}}\lambda^2 d\rho(\lambda)$$

need not be finite, following a result of Ylvisaker [41] (see also [16]). Cramér and Leadbetter [7, Equation 10.6.2 or 10.7.5] gave[3] an exact but involved formula for the variance of $N(T)$, and further show that it is finite under the hypothesis

$$\int_0^{\varepsilon}\frac{r''(t)-r''(0)}{t}dt<\infty \tag{1}$$

for some $\varepsilon>0$. The condition (1) is customarily referred to as the Geman condition, since he showed [13] that it is also necessary for finite variance[4], and we will assume it throughout this paper. Qualls [30, Lemma 1.3.4][5] has shown that the Geman condition is equivalent to the spectral condition

$$\int_{\mathbb{R}}\log(1+|\lambda|)\lambda^2 d\rho(\lambda)<+\infty.$$

We shall pay special attention to atoms in the spectral measure. Notice that since the spectral measure is symmetric, these are really *symmetrised atoms* of the form $\delta^*_\alpha=\frac{1}{2}(\delta_\alpha+\delta_{-\alpha})$, and when we speak of atoms in this paper we will always mean symmetrised atoms.

Definition. A SGP f is *degenerate* if its spectral measure consists of a single symmetrised atom $\rho=\delta^*_\sigma$, or equivalently if the covariance is $r(t)=\cos(\sigma t)$.

[3] Strictly speaking they exclude the case of purely atomic spectral measures, but their formula may be extended to this case if one takes care of an isolated set of singularities, provided the support of the spectral measure contains at least four points.

[4] Geman uses the Cramér-Leadbetter formula and accordingly excludes purely atomic spectral measures, but in light of the previous comment this is not necessary.

[5] "[E]ssentially the same proof" can be found in [4, Theorem 3]

Notice that we may represent a degenerate process as

$$f(t) = \sqrt{X}\cos(\sigma t + \Phi),$$

where $X \sim \chi^2(2)$ and $\Phi \sim \mathrm{Unif}([0, 2\pi])$. The zero-set is therefore a random shift of the lattice $\frac{1}{\sigma}\mathbb{Z}$, and a computation yields

$$\mathrm{Var}(N(T)) = \left\{\frac{\sigma T}{\pi}\right\}\left(1 - \left\{\frac{\sigma T}{\pi}\right\}\right),$$

where $\{x\}$ denotes the fractional part of x. We exclude this trivial case from the rest of paper.

We now state our main results. We use the notation $A(T) \asymp B(T)$ to denote that there exist $C_1, C_2 > 0$ such that $C_1 \le \frac{A(T)}{B(T)} \le C_2$ for all $T > 0$, and the notation $A(T) \sim B(T)$ to denote that $\lim_{T\to\infty} \frac{A(T)}{B(T)} = 1$.

Theorem 1. *For any SGP,*

$$\mathrm{Var}[N(T)] \ge \frac{\sigma^2}{\pi^2} T \int_0^T \left(1 - \frac{t}{T}\right)\left(r(t) + \frac{r''(t)}{\sigma^2}\right)^2 \, dt.$$

Remarks.

1. For a degenerate SGP, $r(t) + \frac{r''(t)}{\sigma^2}$ vanishes identically, so the conclusion is trivial.
2. One of the important ideas to emerge from our work is that one needs to estimate the decay of $r(t) + \frac{r''(t)}{\sigma^2}$ in order to understand the variance, and this quantity will appear repeatedly throughout the paper. We emphasise the important rôle of cancellation in this expression, see Section 2.7 for an illustration of this.
3. In Section 1.3 we explain how the quantity $r(t) + \frac{r''(t)}{\sigma^2}$ arises naturally when studying the variance of $N(T)$.
4. It is also interesting to note that $r + \frac{r''}{\sigma^2} = \mathcal{F}[\mu]$ where the signed measure μ is defined by $d\mu(\lambda) = \left(1 - \frac{\lambda^2}{\sigma^2}\right) \, d\rho(\lambda)$ and this is crucial to some of our proofs. This fact also explains, in part, how special atoms arise below.
5. In fact, it follows from Parseval's identity that
$$\int_0^T \left(1 - \frac{t}{T}\right)\left(r(t) + \frac{r''(t)}{\sigma^2}\right)^2 \, dt = \pi \int (\mathcal{S}_T * \mu) \, d\mu \tag{2}$$
where $\mathcal{S}_T(\lambda) = \frac{T}{2\pi} \, \mathrm{sinc}^2\left(\frac{T\lambda}{2}\right)$. For details, see Section 6.2.

Corollary 2. *For any non-degenerate SGP there exists a constant $C = C(\rho) > 0$ such that*

$$\mathrm{Var}[N(T)] \ge CT, \quad \forall T > 0. \tag{3}$$

Remarks.

1. Slud [35, Theorem 3] obtained this result under the hypothesis $r, r'' \in \mathcal{L}^2(\mathbb{R})$. In this case we obtain
$$\liminf_{T\to\infty} \frac{\mathrm{Var}[N(T)]}{T} \ge \frac{\sigma^2}{\pi^2} \int_0^\infty \left(r(t) + \frac{r''(t)}{\sigma^2}\right)^2 \, dt.$$
This is easily seen to be identical to Slud's estimate, which is stated in terms of the spectral density, by an application of Plancherel's theorem
2. Probably the most general CLT for $N(T)$ was proved by Cuzick [8], who built on work of Malevič [29]. Cuzick assumed the Geman condition, the mixing condition $r, r'' \in \mathcal{L}^2(\mathbb{R})$, and the bound (3). For the 15 years between the publication of Cuzick's paper and the emergence of Slud's result there was no effective[6] way to check if (3) holds — emphasising the inaccessibility of the Cramér-Leadbetter formula.

[6] Cuzick [8, Lemma 5] provides a sufficient condition, but acknowledges that it is not very satisfactory, see the remarks before and after the statement of that lemma.

3. It also follows from Cuzick's work that the Geman condition, the mixing condition $r, r'' \in \mathcal{L}^2(\mathbb{R})$, and the bound (3) together imply that
$$\lim_{T\to\infty} \frac{\mathrm{Var}[N(T)]}{T}$$
exists. Ancona and Letendre [1, Proposition 1.11] give an exact expression for this limit (see also [9, Proposition 3.1]).

We next present a matching upper bound for the variance, under slightly weaker conditions. To formulate our results we introduce the function
$$\varphi(t) = \max\left\{ |r(t)| + \frac{|r'(t)|}{\sigma}, \frac{|r'(t)|}{\sigma} + \frac{|r''(t)|}{\sigma^2} \right\} \tag{4}$$
which we use to measure the decay of the correlations of f and its derivative.

Theorem 3. *For a non-degenerate SGP satisfying*
$$\limsup_{|t|\to\infty} \varphi(t) < 1, \tag{5}$$
we have
$$\mathrm{Var}[N(T)] \asymp T \int_0^T \left(1 - \frac{t}{T}\right) \left(r(t) + \frac{r''(t)}{\sigma^2} \right)^2 \, dt \tag{6}$$
where the implicit constants depend on ρ.

Remarks.

1. We remark that, following Arcones [3], many previous results were stated in terms of the function
$$\psi(t) = \max\left\{ |r(t)|, \frac{|r'(t)|}{\sigma}, \frac{|r''(t)|}{\sigma^2} \right\}$$
rather than the function φ that we introduced. It is not difficult to modify the proof of Proposition 14 to yield (6) under the assumption $\limsup_{|t|\to\infty} \psi(t) < \frac{1}{2}$ but this is a weaker result than the one we give.
2. The condition (5) may be viewed as a very mild mixing condition. In fact, the condition[7] $r(t) \to 0$ implies that $\varphi(t) \to 0$, so in particular (5) holds whenever the spectral measure is absolutely continuous (by the Riemann-Lebesgue lemma).

   To see that $r(t) \to 0$ implies that $\varphi(t) \to 0$ note first that r' and r'' are uniformly continuous. Now suppose that there exists an $\varepsilon > 0$ and a sequence $\{t_n\}_{n=1}^\infty$ such that $|r'(t_n)| > 2\varepsilon$ for all n and $t_n \to \infty$. By the uniform continuity of r' we get
$$\left| r'(t) - r'(t_n) \right| < \varepsilon$$
for $|t - t_n| < \delta$. Hence $|r'(t)| \geq |r'(t_n)| - |r'(t) - r'(t_n)| > \varepsilon$ and so
$$\left| \int_{t_n-\delta}^{t_n+\delta} r'(t) \, dt \right| = \int_{t_n-\delta}^{t_n+\delta} |r'(t)| dt \geq 2\varepsilon\delta > 0.$$
But, we can also compute
$$\lim_{n\to\infty} \left| \int_{t_n-\varepsilon}^{t_n+\varepsilon} r'(t) \, dt \right| = \lim_{n\to\infty} |r(t_n+\varepsilon) - r(t_n-\varepsilon)| = 0,$$
which is absurd. The same proof shows that $r'' \to 0$.

[7] One says that ρ is a Rajchman measure if its Fourier transform decays at infinity.

3. We do not believe that the condition (5) is a natural one, but we have not been able to remove it. To emphasise this, consider Theorem 7 below — the existence of a transition at a critical value θ_0 of the weight of the atom does not seem plausible. This, in part, motivates the conjectures below.

A special case of interest is when the variance is asymptotically linear. We believe that proving a version of our next result without assuming the condition (5) would be very desirable; in fact one direction is already clear from Theorem 1.

Corollary 4. *Suppose that condition* (5) *holds. Then*

$$\mathrm{Var}[N(T)] \asymp T \iff r + \frac{r''}{\sigma^2} \in \mathcal{L}^2(\mathbb{R}).$$

Remark. Previously, the only condition for asymptotically linear variance (that we are aware of) was $r, r'' \in \mathcal{L}^2(\mathbb{R})$, which follows from combining the results of Cuzick [8] and Slud [35], as mentioned above. We show in Section 2.7 that the condition $r + \frac{r''}{\sigma^2} \in \mathcal{L}^2(\mathbb{R})$ is strictly weaker, therefore Corollary 4 improves upon their result.

In the case of super-linear variance and a slightly stronger mixing condition than (5) we give a precise estimate.

Theorem 5. *Suppose that* $r + \frac{r''}{\sigma^2} \notin \mathcal{L}^2(\mathbb{R})$, *and* $\lim_{|t|\to\infty} \varphi(t) = 0$. *Then*

$$\mathrm{Var}[N(T)] \sim \frac{\sigma^2}{\pi^2} T \int_0^T \left(1 - \frac{t}{T}\right) \left(r(t) + \frac{r''(t)}{\sigma^2}\right)^2 dt.$$

We finally discuss the presence of atoms in the spectral measure. In this case the process is non-ergodic [15, Sec. 5.10], and has a random periodic component. We show that, generically, this makes $\mathrm{Var}[N(T)]$ quadratic in T. This is the maximal possible growth — by stationarity $\mathrm{Var}[N(T)]$ is always at most quadratic.

Theorem 6. *The spectral measure* ρ *contains an atom at a point different from* σ *if and only if*

$$\mathrm{Var}[N(T)] \asymp T^2.$$

The emergence of a special frequency σ seems to be new, and intriguing. Notice that adding an atom at frequency σ does not change $\mathbb{E}[N(T)]$. The following result shows that the asymptotic growth of $\mathrm{Var}\, N(T)$ remains unchanged as well — at least if the strength of the atom is sufficiently small. As previously remarked, we believe this result should in fact hold for all $\theta \in [0, 1)$.

Theorem 7. *Suppose that* (5) *holds for the spectral measure* ρ. *Define*[8] $\rho_\theta = (1-\theta)\rho + \theta\delta^*_\sigma$ *for* $0 < \theta < 1$. *There exists* $\theta_0 > 0$ *such that*

$$\mathrm{Var}[N(\rho; T)] \asymp \mathrm{Var}[N(\rho_\theta; T)]$$

for any $\theta < \theta_0$ *(and the implicit constants may depend on* θ*). Moreover,* θ_0 *depends only on* $\limsup_{|t|\to\infty} \varphi(t)$.

Remarks.

1. For crossings of non-zero levels, the presence of an atom at any frequency leads to quadratic variance. The existence of a distinguished frequency therefore seems to be a special phenomenon of the zero level.
2. Further, this phenomenon is purely real. No such frequency exists for complex zeroes, see [12].

[8] Notice that $\mathbb{E}[N(\rho_\theta; T)]$ is independent of θ.

We have previously indicated that some of the hypotheses in our results feel unsatisfactory. We are led to the following conjectures.

Conjecture (Weak form)**.** *The estimate* (6) *holds for any non-degenerate SGP satisfying*

$$\limsup_{|t|\to\infty} \max\left\{ r(t)^2 + \frac{r'(t)^2}{\sigma^2}, \frac{r''(t)^2}{\sigma^4} + \frac{r'(t)^2}{\sigma^2} \right\} < 1. \tag{7}$$

Conjecture (Strong form)**.** *The estimate* (6) *holds for any non-degenerate SGP.*

Remarks.

1. It is not difficult to show that $r(t)^2 + \frac{r'(t)^2}{\sigma^2} \le 1$ and $\frac{r''(t)^2}{\sigma^4} + \frac{r'(t)^2}{\sigma^2} \le 1$ for any process. Accordingly, 'most' processes satisfy (7).
2. The weak form of the conjecture would imply that Theorem 7 holds for any $\theta \in [0, 1)$.
3. The weak form of the conjecture would allow one to improve Corollary 4 and completely characterise linear variance.
4. The strong form of the conjecture would simplify the proof of Theorem 6.
5. Let us consider two independent processes f_0 and f_1 with corresponding spectral measures ρ_0 and ρ_1 satisfying
$$\int_{\mathbb{R}} \lambda^2 d\rho_0(\lambda) = \int_{\mathbb{R}} \lambda^2 d\rho_1(\lambda).$$
For $\theta \in [0, 1]$ we form the process $f_\theta = \sqrt{1-\theta} f_0 + \sqrt{\theta} f_1$ with corresponding spectral measure $\rho_\theta = (1-\theta)\rho_0 + \theta\rho_1$ and notice that $\mathbb{E}[N(\rho_\theta; T)]$ does not depend on θ. The strong form of the conjecture implies that
$$\mathrm{Var}[N(\rho_\theta; T)] \asymp \max\{\mathrm{Var}[N(\rho_0; T)], \mathrm{Var}[N(\rho_1; T)]\}$$
for $\theta \in (0, 1)$, where the implicit constants depend on θ.
6. We provide further evidence for the conjectures in Section 5.4.
7. If the strong form of the conjecture is true then the implicit constant in the upper bound in (6) should depend on the spectral measure in such a way that it diverges for processes that fail to satisfy the Geman condition (1). This is because the term $r + \frac{r''}{\sigma^2}$ is integrable on any finite interval as soon as r'' is continuous, which is strictly weaker than the Geman condition.

While we were preparing this article for publication we became aware of [28], where the author also gives a linear lower bound for the variance of the number of zeroes of an SGP. While the author considers *linear statistics* (which generalise the zero count), as well as rigidity and predictability of the zeroes, all of the results pertaining to the growth of the variance of the zero count are implied by our results. We also mention recent work of Ancona and Letendre [1, Prop. 1.11] who, in addition to other interesting problems, also study the variance of linear statistics. They work under the same regularity assumption as Slud but their results are more precise.

We finally mention that our work has parallels in different but related models. In the setting of *complex zeroes* of a random Gaussian analytic $f : \mathbb{C} \to \mathbb{C}$, a linear lower bound for the variance, an L^2-condition that guarantees linearity, and a characterisation of maximal (i.e., quadratic) growth were given in [12]. Analogous results were then proved for the *winding number* of a Gaussian stationary $f : \mathbb{R} \to \mathbb{C}$ in [5].

1.3. **Outline of our methods.** Let us briefly outline our method. We write

$$N(T) = \sum_{q=0}^{\infty} \pi_q(N(T))$$

where π_q denotes the projection onto the q'th Wiener chaos. Explicit expressions for this decomposition are well known, it turns out that only the even chaoses contribute, and so we have

$$\mathrm{Var}[N(T)] = \sum_{q=1}^{\infty} \mathbb{E}[\pi_{2q}(N(T))^2].$$

The diagram formula allows us to compute (see Lemma 10)

$$\mathbb{E}[\pi_{2q}(N(T))^2] = \int_{-T}^{T} (T - |t|)\widetilde{P}_q(t)dt$$

where $\widetilde{P}_q$ is a polynomial expression that involves r, r' and r''. Our lower bound comes from explicitly evaluating the term with $q = 1$. For the upper bound we establish that $\left(r + \frac{r''}{\sigma^2}\right)^2$ divides the polynomial[9] $\widetilde{P}_q$ exactly, see Proposition 13. This yields

$$\mathbb{E}[\pi_{2q}(N(T))^2] \le C_q \int_{-T}^{T} (T - |t|) \left(r(t) + \frac{r''(t)}{\sigma^2}\right)^2 dt$$

for some C_q. The remainder of our proof of the upper bound involves showing that this sequence C_q is summable under the given hypothesis.

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2. Examples

In this section we give a number of examples that expand on or illustrate our results.

2.1. Purely atomic measure.

As mentioned earlier, if ρ consists of a single symmetrised atom $\rho = \delta^*_\sigma$, then $f(t) = \sqrt{X}\cos(\sigma t + \Phi)$ is a degenerate process. However, a superposition of such processes results in a random almost periodic function, with non-trivial behavior. Specifically let $\alpha_i \in \mathbb{R}$ and $w_i > 0$ satisfy $\sum_i w_i = 1$ and $\sum_i w_i \alpha_i^2 \log(1 + |\alpha_i|) < +\infty$ [10]. We consider the measure

$$\rho = \sum_i w_i \delta^*_{\alpha_i}$$

Then the covariance function is $r(t) = \sum_i w_i \cos(\alpha_i t)$ which yields $r''(t) = -\sum_i w_i \alpha_i^2 \cos(\alpha_i t)$ and $\sigma^2 = -r''(0) = \sum_i w_i \alpha_i^2$. By Theorem 6 we have quadratic growth $\mathrm{Var}[N(T)] \asymp T^2$. Using Theorem 1, we may give a concrete lower bound, that is,

$$\liminf_{T\to\infty} \frac{\mathrm{Var}[N(T)]}{T^2} \ge \frac{\sigma^2}{4\pi^2} \sum_{i=1}^{n} w_i^2 \left(1 - \frac{\alpha_i^2}{\sigma^2}\right)^2;$$

we omit the details.

[9] Strictly speaking we first add a small computable quantity, which leads to the difference between P_q and $\widetilde{P}_q$ in Section 3.

[10] There might only be finitely many α_i in which case this second condition is redundant.

2.2. **Cuzick–Slud covariance functions.** We have already discussed in detail the case $r, r'' \in \mathcal{L}^2(\mathbb{R})$; in this case the limit

$$\lim_{T\to\infty} \frac{\mathrm{Var}[N(T)]}{T}$$

exists and we further have a CLT for $N(T)$. We mention that a number of classic kernels satisfy this hypothesis, for example, the Paley-Wiener kernel $\operatorname{sinc} t = \frac{\sin t}{t}$ or the Gaussian kernel (sometimes referred to as the Bargman-Fock kernel) e^{-t^2}.

2.3. **Exponential kernel and approximations.** Consider the Ornstein-Uhlenbeck (OU) process, defined by the covariance function $r(t) = e^{-|t|}$. This process has attracted considerable attention since it arises as a time-space change of Brownian motion. Since the covariance is not differentiable at the origin, none of our results may be directly applied. However, one may approximate the OU process by differentiable processes.

One way to do so is by taking $r_a(t) = e^{a-\sqrt{a^2+t^2}}$, with $a \downarrow 0$. In this case $\sigma_a^2 = \frac{1}{a}$ and

$$r_a(t) + \frac{r_a''(t)}{\sigma_a^2} = \left(1 - \frac{a^3}{(a^2+t^2)^{3/2}} + \frac{at^2}{a^2+t^2}\right) r_a(t) \geq e^{-t}$$

where the inequality holds for $t \geq a^{2/3}$. We deduce from Theorem 1 that for $T \geq a^{2/3}$ we have

$$\mathrm{Var}[N(T)] \geq \frac{T}{\pi^2 a} \int_{a^{2/3}}^{T} \left(1 - \frac{t}{T}\right) e^{-2t}\, dt.$$

As $a \downarrow 0$, we see that the variance is unbounded, and this holds even on certain short intervals that are not 'too short', i.e, such that $T \gg \sqrt{a}$.

Another approximation may be derived using the spectral measure. The OU process has spectral density $\frac{1}{\pi(1+\lambda^2)}$. Thus one may consider the spectral density $\frac{M}{\pi(M-1)}\left(\frac{1}{\lambda^2+1} - \frac{1}{\lambda^2+M^2}\right)$, with $M \to \infty$. The corresponding covariance kernel is $r_M(t) = \frac{Me^{-|t|}-e^{-M|t|}}{M-1}$, with $\sigma_M^2 = M$. In this case we have

$$r_M(t) + \frac{r_M''(t)}{\sigma_M^2} = \frac{M+1}{M-1}\left(e^{-t} - e^{-Mt}\right) \geq e^{-t} - e^{-Mt}$$

for $t > 0$, so that applying Theorem 1 we obtain

$$\mathrm{Var}[N(T)] \geq \frac{MT}{\pi^2} \int_0^T \left(1 - \frac{t}{T}\right)\left(e^{-2t} - 2e^{-(M+1)t} + e^{-2Mt}\right)\, dt.$$

Once more we see that the variance is unbounded, even on short intervals that satisfy $T \gg \frac{1}{\sqrt{M}}$.

2.4. **Bessel function.** Let $J_\alpha(t)$ be the α'th Bessel function of the first kind. If $r(t) = J_0(t)$ then $J_0'(t) = -J_1(t)$ and $J_0''(t) = \frac{J_2(t)-J_0(t)}{2}$, so that $\sigma^2 = \frac{1}{2}$ and

$$J_0(t) + \frac{J_0''(t)}{\sigma^2} = J_2(t) \notin \mathcal{L}^2(\mathbb{R}).$$

Moreover, in this example $r(t) \to 0$, and so Theorem 5 applies and we have

$$\mathrm{Var}[N(T)] \sim \frac{\sigma^2}{\pi^2} T \int_0^T \left(1 - \frac{t}{T}\right) J_2(t)^2\, dt \sim \frac{1}{2\pi^3} T \ln T.$$

2.5. **Intermediate growth.** Using Theorem 5 it is possible to construct more examples where $T \ll \mathrm{Var}[N(T)] \ll T^2$. For instance, consider

$$r(t) = \frac{1}{(1+t^2)^b},$$

with $0 < b < \frac{1}{4}$. One way to see that this is a legitimate covariance is to compute the spectral measure $d\rho_b(\lambda) = \frac{1}{\Gamma(b)\sqrt{\pi}} 2^{\frac{1}{2}-b} |\lambda|^{b-\frac{1}{2}} K_{\frac{1}{2}-b}(|\lambda|)$, where K_α is the modified Bessel function of the second kind of order α. Alternatively[11] note that

$$r(t) = \mathbb{E}[e^{-i(X-\tilde{X})}]$$

where X and $\tilde{X}$ are independent $\Gamma(b,1)$ random variables. It is easy to check that the hypotheses of Theorem 5 are satisfied and so

$$\mathrm{Var}[N(T)] \sim \frac{2b}{\pi^2(1-4b)(2-4b)} T^{2-4b}.$$

More generally let L be a function that 'varies slowly at infinity', that is, for every $x > 0$ we have

$$\frac{L(tx)}{L(t)} \to 1$$

as $t \to \infty$ and suppose that we have

$$r(t) = \frac{L(|t|)}{(1+t^2)^b}.$$

Suppose further that there exist $C > 0$ and $\delta < b$ such that

$$\frac{|r'(t)|}{|r(t)|}, \frac{|r''(t)|}{|r(t)|} \le C(1+t^2)^\delta. \tag{8}$$

Then Slud [36, Theorem 3.2] showed that

$$\mathrm{Var}[N(T)] \sim \frac{\sigma^2}{\pi^2(1-4b)(2-4b)} L(T)^2 T^{2-4b}$$

and moreover that $N(T)$ satisfies a non-CLT.

Theorem 5 allows us to prove Slud's result, without imposing the hypothesis (8). Indeed, Slud uses this hypothesis to show that the higher order chaoses are negligible, that is, that[12]

$$\mathrm{Var}[N(T)] \sim \frac{\sigma^2}{\pi^2} \frac{\mathbb{E}[N_1(T)^2]}{4}.$$

But this is precisely the conclusion of Theorem 5, which applies here since $r(t) \to 0$ as $t \to \infty$. To compute the asymptotic growth of the variance we write, as in (2),

$$\frac{1}{L(T)^2 T^{2-4b}} \int_{-T}^{T} (T-|t|) \left(r(t) + \frac{r''(t)}{\sigma^2} \right)^2 dt = \frac{2\pi T}{L(T)^2 T^{2-4b}} \int (\mathcal{S}_T * \mu) \, d\mu$$

where $d\mu(\lambda) = \left(1 - \frac{\lambda^2}{\sigma^2}\right) d\rho(\lambda)$. We define $d\rho_T(\lambda) = \frac{T^{2b}}{L(T)} d\rho\left(\frac{\lambda}{T}\right)$ and $d\mu_T(\lambda) = \left(1 - \frac{\lambda^2}{T^2\sigma^2}\right) d\rho_T(\lambda)$. Changing variables in the previous equation we see that

$$\frac{1}{L(T)^2 T^{2-4b}} \int_{-T}^{T} (T-|t|) \left(r(t) + \frac{r''(t)}{\sigma^2} \right)^2 dt = 2\pi \int (\mathcal{S}_1 * \mu_T) \, d\mu_T.$$

[11] This argument appears in [36]

[12] $N_1(T)$ will be defined in the next section.

Now [10, Proposition 1] implies that ρ_T converges locally weakly to the measure ρ_0 with density $\frac{|\lambda|^{2b-1}}{2\Gamma(2b)\cos(b\pi)}$. Since $1-\frac{\lambda^2}{T^2\sigma^2}\to 1$ uniformly on bounded sets we have $\mu_T\to\rho_0$ locally weakly also. Further, by [10, Equation 1.10], we have $\hat{\rho}_0(t)=|t|^{-2b}$. This yields

$$\begin{aligned}\frac{1}{L(T)^2T^{2-4b}}\int_{-T}^{T}(T-|t|)\left(r(t)+\frac{r''(t)}{\sigma^2}\right)^2 &\sim 2\pi\int(\mathcal{S}_1*\rho_0)\,d\rho_0\\ &=\int_{-1}^{1}(1-|t|)|\hat{\rho}_0(t)|^2dt\\ &=2\int_0^1\frac{1-t}{t^{4b}}dt=\frac{2}{(1-4b)(2-4b)}.\end{aligned}$$

2.6. **Singular continuous spectral measure.** If the spectral measure is absolutely continuous with respect to Lebesgue measure, then (5) holds by the Riemann-Lebesgue lemma and we may apply Theorem 3. If there are atoms in the spectral measure then Theorem 6 applies. Singular continuous measures fall between these two stools. Here we give a family of examples that show that there is no simple characterisation for this class of processes.

Fix $\alpha_1\geq\alpha_2\geq\ldots>0$ such that $\sum_{n=1}^{\infty}\alpha_n^2<\infty$, and let $r_\alpha(t)=\prod_{n=1}^{\infty}\cos(\alpha_n t)$. In this case the spectral measure ρ_α is the infinite convolution of the atomic measures $\delta^*_{\alpha_n}$ and is usually referred to as a symmetric Bernoulli convolution. By the Jessen–Wintner theorem [18, Theorem 11] it is of pure type, being either absolutely continuous or singular continuous. (In particular it contains *no atoms*.) Moreover, the support of ρ_α (the 'spectrum') is a perfect set and is either compact or all of $\mathbb{R}$, according as $\sum_{n=1}^{\infty}\alpha_n$ converges or diverges.

In the former case we write $R_n=\sum_{i=n+1}^{\infty}\alpha_i$ and the spectrum is a subset of $[-R_0,R_0]$. There are two special cases that are particularly tractable:

1. $\alpha_n>R_n$ for every $n\geq 1$, and
2. $\alpha_n\leq R_n$ for every $n\geq 1$.

In the first case Kershner and Wintner [24, Pages 543–544] showed that the support of ρ_α is nowhere dense (and so is a Cantor-type set) and has total length $\ell=2\lim_{n\to\infty}2^nR_n$. The measure ρ_α is singular if and only if $\ell=0$; if $\ell>0$ then

$$\rho_\alpha(I)=\frac{1}{\ell}\,\mathrm{Leb}(I\cap S_\alpha)$$

where Leb is the usual Lebesgue measure on the real line and S_α is the spectrum.

In Case 2 the spectrum is all of $[-R_0,R_0]$ [24, Page 547], and, as the examples below illustrate, the measure ρ_α may be absolutely continuous or singular.

We now list some examples. One particularly simple choice is $\alpha_n=a^n$ for some $a\in(0,1)$ and we write ρ_a and r_a for the corresponding spectral measure and covariance function.

- If $0<a<\frac{1}{2}$ we are in Case 1 and ρ_a is singular. For rational a, by [23], $r_a(t)\to 0$ as $|t|\to\infty$ if and only if $\frac{1}{a}$ is not an integer. In this case we have $r_a(t)=O(|\log t|^{-\gamma})$ for large $|t|$ where $\gamma=\gamma(a)>0$.

  In the particular case $a=\frac{1}{3}$ we get the usual Cantor middle-third set (shifted to be contained in $[-\frac{1}{2},\frac{1}{2}]$) and the distribution function $\rho_{1/3}((-\infty,x])$ is the usual devil's staircase function (again, shifted).
- If $a\geq\frac{1}{2}$ then we are in Case 2. If $a=\frac{1}{2}$ then $r_{1/2}(t)=\operatorname{sinc}t$ (this formula was discovered by Euler), which was covered in Section 2.2.
- More generally if $a=(\frac{1}{2})^{1/k}$ where $k\in\mathbb{N}$ then ρ_a is absolutely continuous and $r_a(t)=O(|t|^{-k})$ (see [40, Page 836]). Once more we are covered by Section 2.2.
- Erdős [11, Section 2] showed that if $a=\frac{1}{b}$ where $b\neq 2$ is a 'Pisot-Vijayaraghavan number' (a real algebraic integer whose conjugates lie in the unit disc) then $\limsup_{t\to\infty}r_a(t)>0$

and therefore ρ_a is singular. Concrete values of $a > \frac{1}{2}$ are $a = \frac{\sqrt{5}-1}{2}$ (the Fibonacci number) and the positive root of the cubic $a^3 + a^2 - 1$.

- Conversely, Salem [33, Theorem II] showed that if $r_a(t) \not\to 0$ then $\frac{1}{a}$ is a Pisot-Vijayaraghavan number. This, of course, does not rule out the possibility that there are other values of $a > \frac{1}{2}$ for which ρ_a is singular but $r_a(t) \to 0$, but to the best of our knowledge there has been no progress on this question since.

More involved examples give more sophisticated behaviour, unless otherwise indicated the examples are taken from [18, Examples 1–8].

- If $\alpha_{2n-1} = \alpha_{2n} = \frac{1}{3^n}$ then $\rho_\alpha = \rho_{1/3} * \rho_{1/3}$ which is supported on the whole interval $[-1, 1]$ and $r_\alpha(t) = r_{1/3}(t)^2$. We therefore have $\limsup_{t\to\infty} r_\alpha(t) > 0$ and so the spectral measure is singular.
- If the sequence α_n consists of the numbers of the form $2^{-m!}$ repeated exactly $2^{m!}$ times for $m = 1, 2, \ldots$ then the spectrum is all of $\mathbb{R}$ and $r_\alpha(t) \not\to 0$, which means that ρ_α is singular.
- If $\alpha_n = \frac{1}{n2^n} - \frac{1}{(n+1)2^{n+1}}$ then ρ_α is singular but $r_\alpha(t) \to 0$.
- If $\alpha_n = \frac{1}{n!}$ then $\alpha_n > R_n$ for every n and $(-1)^{k+1} r_\alpha(\pi k!) \to 1$ as $k \to \infty$, by[13] [28, Proposition 9].

We now present a general result which applies in Case 1. Its proof appears in Section 6.3.

Proposition 8. *Suppose that $\alpha_n > R_n$ for every $n \geq 1$, and define $M = M_T$ by $R_M < \frac{1}{T} \leq R_{M-1}$. Then*

$$T \int_0^T \left(1 - \frac{t}{T}\right) \left(r(t) + \frac{r''(t)}{\sigma^2}\right)^2 \, dt \asymp T^2 2^{-M}$$

for $T \geq T_0$.

Remark. This is of course a lower bound for $\mathrm{Var}[N(\rho_\alpha; T)]$ in general, and gives the correct order of growth when $r_\alpha(t) \to 0$.

Let us apply this result to some of the examples above.

- If $\alpha_n = a^n$ where $0 < a < \frac{1}{2}$ then $T^2 2^{-M} \asymp T^{2-d}$ where $d = \frac{\log 2}{\log \frac{1}{a}}$ is the Hausdorff dimension of the spectrum. It would be interesting to understand if there is any relation between the dimension of the spectrum and the behaviour of the variance for a general singular measure. (See also the remarks after Lemma 21.)
- If $\alpha_n = \frac{1}{n!}$ then it is not difficult to show that $\alpha_n > R_n$ and $T^2 2^{-M} \asymp T^{2 - \frac{\log 2 + o(1)}{\log\log T}}$. A less precise lower bound for the variance of this process is claimed in [28, Proposition 11], although the proof given there is not entirely convincing. Nonetheless, we do use some of the ideas found there in our proof of Proposition 8.
- By choosing a sequence R_n decaying sufficiently fast, it is clear that we can make the term 2^{-M} decay arbitrarily slowly. We can therefore construct a process with non-atomic spectral measure whose variance grows faster than $T^{2-\phi(T)}$ where $\phi(T) \to 0$ arbitrarily slowly, that is, arbitrarily close to maximal growth.

2.7. Cancellation in the quantity $r + \frac{r''}{\sigma^2}$.

As we indicated previously, an important message of this paper is that the behaviour of the variance is governed by the quantity $r + \frac{r''}{\sigma^2}$. We wish to emphasise the important rôle of cancellation between the two terms here, and we have already presented some examples of this when the spectral measure has an atom at a ‘special frequency’. However this cancellation phenomenon is not just about atoms, and as an illustrative example we will produce a[14] covariance function r such that:

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[13] It is actually stated there that $(-1)^k r_\alpha(\pi k!) \to 1$ but this is easily seen to be incorrect.

[14] In fact we produce a family of such covariance functions.

- The spectral measure ρ has an $\mathcal{L}^1(\mathbb{R})$ density.
- $r + \frac{r''}{\sigma^2} \in \mathcal{L}^2(\mathbb{R})$ where $\sigma^2 = \int_{\mathbb{R}} \lambda^2 \, d\rho(\lambda)$.
- $r, r'' \notin \mathcal{L}^2(\mathbb{R})$.

Writing $d\rho(\lambda) = \phi(\lambda)d\lambda$ and applying the Fourier transform we see that it is equivalent to produce a function $\phi \geq 0$ satisfying:

1. $\int_{\mathbb{R}} \phi(\lambda)d\lambda = 1$ but $\phi \notin \mathcal{L}^2(\mathbb{R})$.
2. $\lambda^2\phi(\lambda) \in \mathcal{L}^1(\mathbb{R})$, but $\lambda^2\phi(\lambda) \notin \mathcal{L}^2(\mathbb{R})$.
3. $\left(1 - \frac{\lambda^2}{\sigma^2}\right)\phi(\lambda) \in \mathcal{L}^2(\mathbb{R})$ where $\sigma^2 = \int_{\mathbb{R}} \lambda^2\phi(\lambda)d\lambda$.

We proceed to produce such a function ϕ.

Let $\alpha \in \left(\frac{1}{2}, 1\right)$. Choose $M > 1$ such that

$$M^2 + M + 1 > 3 + 3(1-\alpha)\left(\frac{1}{3-\alpha} - \frac{2}{2-\alpha}\right), \tag{9}$$

and let $c_1, c_2 \in \mathbb{R}$ be the solution of the linear system

$$\begin{array}{ccccccc} \frac{1}{1-\alpha} & c_1 & + & (M-1) & c_2 & = & \frac{1}{2}, \\ \left(\frac{1}{1-\alpha} - \frac{2}{2-\alpha} + \frac{1}{3-\alpha}\right) & c_1 & + & \frac{M^3-1}{3} & c_2 & = & \frac{1}{2}. \end{array} \tag{10}$$

We note that (9) ensures that the determinant of the matrix associated to (10) is positive, and since we also have $\frac{M^3-1}{3} > M - 1$ and $\frac{2}{2-\alpha} > \frac{1}{3-\alpha}$, it follows that $c_1, c_2 > 0$. Define

$$\phi(\lambda) = \begin{cases} c_1(1-|\lambda|)^{-\alpha}, & \text{for } |\lambda| < 1, \\ c_2, & \text{for } 1 < |\lambda| < M. \end{cases}$$

Then:

- Since $\alpha \in \left(\frac{1}{2}, 1\right)$, it follows that $\phi \in \mathcal{L}^1(\mathbb{R})$ but $\phi \notin \mathcal{L}^2(\mathbb{R})$.
- Integration yields, by the first equation in (10), that $\int_{\mathbb{R}} \phi(\lambda)d\lambda = 1$.
- Similarly $\lambda^2\phi(\lambda) \in \mathcal{L}^1(\mathbb{R})$, but $\lambda^2\phi(\lambda) \notin \mathcal{L}^2(\mathbb{R})$.
- Now the second equation in (10) shows that $\sigma^2 = \int_{\mathbb{R}} \lambda^2\phi(\lambda)d\lambda = 1$.
- Finally note that $\left(1-\lambda^2\right)\phi(\lambda) \in \mathcal{L}^2(\mathbb{R})$.

3. A formula for the variance

The goal of this section is to give an infinite series expansion for $\mathrm{Var}[N(T)]$, each coming from a different component of the Wiener chaos (or Hermite-Itô) expansion of $N(T)$. We begin with some notation. For $q \in \mathbb{N}$ and $l, l_1, l_2, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ write[15]

$$a_q(l) = \frac{1}{l!(q-l)!} \cdot \frac{1}{2l-1} \tag{11}$$

and

$$b_q(l_1, l_2, n) = \frac{(2q-2l_1)!(2l_1)!(2q-2l_2)!(2l_2)!}{(2q-2l_1-2l_2+n)!(2l_1-n)!(2l_2-n)!n!}. \tag{12}$$

Next define the polynomials

$$\widetilde{P}_q(x,y,z) = \sum_{l_1,l_2=0}^{q} a_q(l_1)a_q(l_2) \sum_{n=\max(0,2(l_1+l_2-q))}^{\min(2l_1,2l_2)} b_q(l_1,l_2,n) \cdot x^{2(q-l_1-l_2)+n} y^{2(l_1+l_2-n)} z^n \tag{13}$$

and

$$P_q(x,y,z) = \widetilde{P}_q(x,y,z) + c_q\left(x^{2q-1}z + (2q-1)x^{2q-2}y^2\right) \tag{14}$$

[15]We adopt the standard convention $\frac{1}{n!} = 0$ when n is a negative integer.

where

$$c_q = \frac{2^{4q}(q!)^2}{2q(2q)!} = \frac{2^{4q}}{2q\binom{2q}{q}}. \tag{15}$$

We are now ready to state the expansion.

Proposition 9. *We have*

$$\operatorname{Var} N(T) = \frac{\sigma^2}{\pi^2}\sum_{q=1}^{\infty}\frac{V_q(T)}{4^q} + \frac{\arccos r(T)}{\pi}\left(1 - \frac{\arccos r(T)}{\pi}\right)$$

where

$$V_q(T) = 2\int_0^T (T-t)\,P_q\left(r(t), \frac{r'(t)}{\sigma}, \frac{r''(t)}{\sigma^2}\right) dt. \tag{16}$$

Furthermore

$$\operatorname{Var} N(T) \geq \frac{\sigma^2}{4\pi^2}V_1(T) + \frac{1}{\pi^2}\left(1 - r(T)^2\right). \tag{17}$$

The starting point in our calculations is the following Hermite expansion for $N(T)$ given by Kratz and Léon [27, Proposition 1] assuming only the Geman condition (though they and other authors had considered it previously under more restrictive assumptions). We have (the sum converges in $L^2(\mathbb{P})$)

$$N(T) = \frac{\sigma}{\pi}\sum_{q=0}^{\infty}\frac{(-1)^{q+1}}{2^q}N_q(T)$$

where[16]

$$N_q(T) = \sum_{l=0}^{q} a_q(l)\int_0^T H_{2(q-l)}(f(t))H_{2l}(f'(t)/\sigma)\; dt, \tag{18}$$

and H_l is the l'th Hermite polynomial. Further each $N_q(T)$ belongs to the $2q$'th Wiener chaos which yields

$$\mathbb{E}[N(T)] = \frac{\sigma}{\pi}N_0(T) = \frac{\sigma}{\pi}T,$$

and

$$\operatorname{Var}[N(T)] = \frac{\sigma^2}{\pi^2}\sum_{q=1}^{\infty}4^{-q}\mathbb{E}[N_q(T)^2]. \tag{19}$$

Furthermore

$$\operatorname{Var}[N(T)] \geq \frac{\sigma^2}{\pi^2}\frac{\mathbb{E}[N_1(T)^2]}{4}. \tag{20}$$

The next lemma allows us to evaluate $\mathbb{E}\left[N_q(T)^2\right]$

Lemma 10. *For all $q \in \mathbb{N}$*

$$\mathbb{E}\left[N_q(T)^2\right] = 2\int_0^T (T-t)\,\widetilde{P}_q\left(r(t), \frac{r'(t)}{\sigma}, \frac{r''(t)}{\sigma^2}\right) dt,$$

where $\widetilde{P}_q$ is given by (13).

We now show how this lemma yields the desired expression.

[16]Under the Geman condition, one cannot assume that f is continuously differentiable, and 'conversely' a continuously differentiable process need not satisfy the Geman condition, see [13, Section 4]. However the existence of r'' implies the existence of the derivative in quadratic mean of the process, and this is how the object f' should be understood if the process is not differentiable.

Proof of Proposition 9, assuming Lemma 10. Lemma 10 yields

$$\mathbb{E}\left[N_q(T)^2\right] = V_q(T) - \frac{2c_q}{\sigma^2}\int_0^T (T-t)\left(r(t)^{2q-1}r''(t) + (2q-1)r(t)^{2q-2}r'(t)^2\right)\, dt.$$

Note that $r(t)^{2q-1}r''(t) + (2q-1)r(t)^{2q-2}r'(t)^2 = \frac{d^2}{dt^2}\left[\frac{r(t)^{2q}}{2q}\right]$ and so

$$\begin{aligned}
\int_0^T (T-t)&\Big(r(t)^{2q-1}r''(t) + (2q-1)r(t)^{2q-2}r'(t)^2\Big)\, dt \\
&= \frac{1}{2q}\int_0^T (T-t)\frac{d^2}{dt^2}\left[r(t)^{2q}\right]\, dt \\
&= \frac{1}{2q}\left[(T-t)\cdot 2q\cdot r(t)^{2q-1}r'(t)\Big|_{t=0}^T + \int_0^T \frac{d}{dt}\left[r(t)^{2q}\right]\, dt\right] \\
&= \frac{1}{2q}\left[r(T)^{2q} - 1\right].
\end{aligned}$$

We therefore have

$$\mathbb{E}\left[N_q(T)^2\right] = V_q(T) + \frac{c_q}{q\sigma^2}\left(1 - r(T)^{2q}\right).$$

Applying (20) yields the desired lower bound

$$\mathrm{Var}[N(T)] \geq \frac{\sigma^2}{4\pi^2}\mathbb{E}[N_1(T)^2] = \frac{\sigma^2}{4\pi^2}V_1(T) + \frac{1}{\pi^2}(1 - r(T)^2)$$

while (19) gives

$$\begin{aligned}
\mathrm{Var}\left[N(T)\right] &= \frac{\sigma^2}{\pi^2}\sum_{q=1}^\infty \frac{1}{4^q}\left[V_q(T) + \frac{c_q}{q\sigma^2}\left(1 - r(T)^{2q}\right)\right] \\
&= \frac{\sigma^2}{\pi^2}\sum_{q=1}^\infty \frac{V_q(T)}{4^q} + \frac{1}{2\pi^2}\sum_{q=1}^\infty \frac{2^{2q} - (2r(T))^{2q}}{q^2\binom{2q}{q}}.
\end{aligned}$$

We identify the last series as

$$\arcsin^2(x) = \frac{1}{2}\sum_{q=1}^\infty \frac{2^{2q}}{q^2\binom{2q}{q}}x^{2q} \tag{21}$$

for all $|x| \leq 1$ implying that

$$\begin{aligned}
\mathrm{Var}[N(T)] &= \frac{\sigma^2}{\pi^2}\sum_{q=1}^\infty \frac{V_q(T)}{4^q} + \frac{\arcsin^2(1) - \arcsin^2(r(T))}{\pi^2} \\
&= \frac{\sigma^2}{\pi^2}\sum_{q=1}^\infty \frac{V_q(T)}{4^q} + \frac{\arccos r(T)}{\pi}\left(1 - \frac{\arccos r(T)}{\pi}\right),
\end{aligned}$$

where the last equality follows from $\arccos(x) = \frac{\pi}{2} - \arcsin(x)$. □

We now proceed to prove Lemma 10.

Proof of Lemma 10. Squaring the expression for $N_q(T)$ given in (18) yields

$$N_q(T)^2 = \sum_{l_1,l_2=0}^q a_q(l_1)a_q(l_2)\int_0^T\int_0^T H_{2(q-l_1)}(f(t))H_{2(q-l_2)}(f(s))H_{2l_1}\left(\frac{f'(t)}{\sigma}\right)H_{2l_2}\left(\frac{f'(s)}{\sigma}\right) ds\, dt.$$

and so

$$\begin{aligned}&\mathbb{E}\big[N_q(T)^2\big]\\&\quad=\sum_{l_1,l_2=0}^{q} a_q(l_1)a_q(l_2)\int_0^T\int_0^T \mathbb{E}\left[H_{2(q-l_1)}(f(t))H_{2(q-l_2)}(f(s))H_{2l_1}\left(\frac{f'(t)}{\sigma}\right)H_{2l_2}\left(\frac{f'(s)}{\sigma}\right)\right]ds\,dt.\end{aligned}$$

Applying Lemma 11 below, and using the simple change of variables

$$\int_0^T\int_0^T h(t-s)dt\,ds=\int_{-T}^T(T-|x|)h(x)dx$$

for any $h\in L^1([-T,T])$, we get

$$\mathbb{E}\big[N_q(T)^2\big]=\int_{-T}^T(T-|t|)\,\widetilde{P}_q\left(r(t),\frac{r'(t)}{\sigma},\frac{r''(t)}{\sigma^2}\right)dt.$$

Noting that r is an even function and that only even powers of y appear in $\widetilde{P}_q$ yields Lemma 10. □

Lemma 11. *For all $q\in\mathbb{N}$ and $l_1,l_2\in\mathbb{N}_0$ such that $0\le l_1,l_2\le q$ we have*

$$\begin{aligned}&\mathbb{E}\left[H_{2q-2l_1}(f(t))\,H_{2q-2l_2}(f(s))\,H_{2l_1}\left(\frac{f'(t)}{\sigma}\right)H_{2l_2}\left(\frac{f'(s)}{\sigma}\right)\right]\\&\quad=\sum_{n=\max(0,2l_1+2l_2-2q)}^{\min(2l_1,2l_2)} b_q(l_1,l_2,n)\left(\frac{r''(t-s)}{\sigma^2}\right)^n\left(\frac{r'(t-s)}{\sigma}\right)^{2(l_1+l_2-n)}(r(t-s))^{2(q-l_1-l_2)+n}.\end{aligned}$$

Before proving the lemma we first recall the diagram formula.

Lemma 12 (The diagram formula [6, Page 432; 17, Theorem 1.36])**.** *Let $X_1,\dots,X_k$ be jointly Gaussian random variables, and $n_1,\dots,n_k\in\mathbb{N}$. A Feynman diagram is a graph with $n_1+\ldots+n_k$ vertices such that*

- *There are n_i vertices labelled X_i for each i (and each vertex has a single label). For a vertex a we write $X_{\ell(a)}$ for the label of a.*
- *Each vertex has degree 1.*
- *No edge joins 2 vertices with the same label.*

Let $\mathscr{D}$ be the set of such diagrams. For $\gamma\in\mathscr{D}$ we define the value of γ to be

$$v(\gamma)=\prod_{(a,b)\in E(\gamma)}\mathbb{E}\left[X_{\ell(a)}X_{\ell(b)}\right]$$

where $E(\gamma)$ is the set of edges of γ. Then

$$\mathbb{E}\left[H_{n_1}(X_1)\cdot\dots\cdot H_{n_k}(X_k)\right]=\sum_{\gamma\in\mathscr{D}}v(\gamma).$$

Proof of Lemma 11. We apply the diagram formula to the random variables $f(t),f(s),f'(t)/\sigma$ and $f'(s)/\sigma$ and corresponding integers $2(q-l_1),2(q-l_2),2l_1$ and $2l_2$ and denote by $\mathscr{D}$ the collection of relevant Feynman diagrams. Since $\mathbb{E}\left[f(t)f'(t)\right]=\mathbb{E}\left[f(s)f'(s)\right]=r'(0)=0$, it is enough to consider diagrams whose edges do not join vertices labeled $f(t)$ to $f'(t)/\sigma$ or vertices labeled $f(s)$ to $f'(s)/\sigma$.

Let n be the number of edges joining a vertex labeled $f'(t)/\sigma$ to a vertex labeled $f'(s)/\sigma$, see Figure 1. Then $0\le n\le\min(2l_1,2l_2)$. Moreover, as the other vertices labeled $f'(t)/\sigma$ must be joined to vertices labeled $f(s)$, we see that $2l_1-n\le 2q-2l_2$, so $\max(0,2l_1+2l_2-2q)\le n\le\min(2l_1,2l_2)$. Further, every value of n in this range is attained by some diagram.

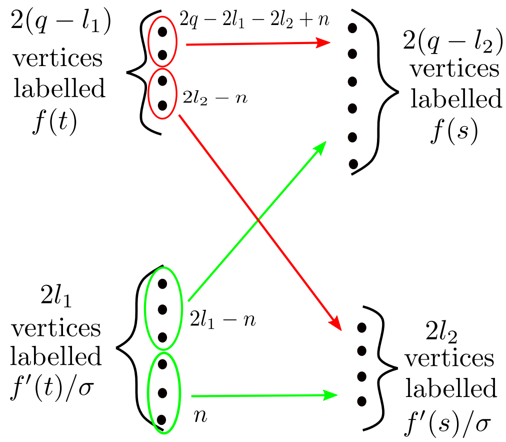

FIGURE 1. Counting the number of Feynman diagrams

We compute the value of such a diagram to be

$$v(\gamma) = \mathbb{E}\left[f'(t)f'(s)/\sigma^2\right]^n \mathbb{E}\left[f'(t)f(s)/\sigma\right]^{2l_1-n} \mathbb{E}\left[f(t)f'(s)/\sigma\right]^{2l_2-n} \mathbb{E}\left[f(t)f(s)\right]^{2q-2l_1-2l_2+n}$$
$$= \left(\frac{r''(t-s)}{\sigma^2}\right)^n \left(\frac{r'(t-s)}{\sigma}\right)^{2(l_1+l_2-n)} (r(t-s))^{2(q-l_1-l_2)+n}.$$

Finally, we count the number of such diagrams. There are

$$\binom{2l_1}{n}\binom{2l_2}{n}n!$$

ways to choose n vertices labeled $f'(t)/\sigma$, to choose n vertices labeled $f'(s)/\sigma$ and to pair them. There are

$$\binom{2q-2l_2}{2l_1-n}(2l_1-n)!$$

ways to choose $2l_1 - n$ vertices labeled $f(s)$ and to pair them with the remaining vertices labeled $f'(t)/\sigma$. There are

$$\binom{2q-2l_1}{2q-2l_1-2l_2+n}(2q-2l_1-2l_2+n)!$$

ways to choose $2q - 2l_1 - 2l_2 + n$ vertices labeled $f(t)$ and to pair them with the remaining ones labeled $f(s)$. There are

$$(2l_2-n)!$$

ways to pair the remaining vertices labeled $f(t)$ and $f'(s)/\sigma$. Since these choices are independent, we multiply these counts to get that there are $b_q(l_1, l_2, n)$ such diagrams, where b_q is given by (12). Applying the diagram formula completes the proof. □

4. LOWER BOUND

In this section we prove Theorem 1 and Corollary 2. From Proposition 9 we have

$$\mathrm{Var}[N(T)] \geq \frac{\sigma^2}{4\pi^2} V_1(T)$$

and Theorem 1 follows simply by computing

$$P_1(x, y, z) = 2(x+z)^2$$

which gives

$$V_1(T) = 4T\int_0^T \left(1-\frac{t}{T}\right)\left(r(t)+\frac{r''(t)}{\sigma^2}\right)^2 dt. \tag{22}$$

To deduce the corollary it is enough to find an interval I such that $\left|r+\frac{r''}{\sigma^2}\right| \geq C > 0$ on I. But this follows from the fact that r'' is continuous and r is not cosine.

5. Upper bound

In this section, we prove Theorems 3 and 5. Our method is to bound each $V_q(T)$ by $V_1(T)$ and apply Proposition 9. We achieve this by proving the following properties of the polynomials P_q (recall (14)).

Proposition 13. *For all $q \geq 1$ we have $(x+z)^2 \mid P_q(x,y,z)$.*

Proposition 14. *Set $M = \max(|x|+|y|,|y|+|z|)$. Then*

$$\frac{|P_q(x,y,z)|}{(x+z)^2} \leq \frac{e^2}{\sqrt{\pi}} q^{3/2} 4^q M^{2q-2}. \tag{23}$$

Proving Proposition 13 amounts to proving some identities for the coefficients of the polynomials P_q, which is deferred to Section 7 where we implement a general method due to Zeilberger [2]. We proceed to prove Proposition 14.

5.1. Proof of Proposition 14.

By Proposition 13, we may prove Proposition 14 by bounding the second derivative of P_q. To achieve this we borrow the main idea from the proof of Arcones' Lemma [3, Lemma 1].

Proof of Proposition 14. Our goal is to bound $\frac{\partial^2 P_q}{\partial x^2}$. For $k \leq 2q-2$, define

$$\alpha_q(k) = \begin{cases} 0, & \text{for odd } k, \\ \frac{1}{q!}\cdot\binom{q}{k/2}\frac{(2q-k)!k!}{k-1}, & \text{for even } k, \end{cases}$$

which yields (recall (11))

$$\alpha_q(2k) = \binom{q}{k}\frac{(2q-2k)!(2k)!}{(2k-1)\cdot q!} = (2q-2k)!(2k)!\cdot a_q(k).$$

Let $0 \leq k,l \leq 2q-2$ and suppose that n is an integer such that $\max(0,l+k-2q+2) \leq n \leq \min(l,k)$. Recalling (12) we have

$$\begin{aligned}\alpha_q(2k)\alpha_q(2l) &= (2q-2k)!(2k)!(2q-2l)!(2l)!\cdot a_q(k)a_q(l)\\ &= (2q-2k-2l+n)!(2k-n)!(2l-n)!n!\cdot a_q(k)a_q(l)b_q(k,l,n)\end{aligned}$$

and so

$$a_q(k)a_q(l)b_q(k,l,n)\frac{\partial^2}{\partial x^2}\left[x^{2(q-k-l)+n}\right] = \frac{\alpha_q(2k)\alpha_q(2l)x^{2q-2k-2l-2+n}}{(2q-2k-2l-2+n)!(2k-n)!(2l-n)!n!}. \tag{24}$$

Let

$$k_1 = 2q-2-k, \quad k_2 = k, \quad l_1 = 2q-2-l, \quad \text{and} \quad l_2 = l,$$

define

$$\begin{aligned}\mathcal{A}(k,l) &= \left\{\begin{pmatrix} 2q-l-k-2+n & l-n \\ k-n & n\end{pmatrix} : \max(0,l+k-2q+2) \leq n \leq \min(l,k)\right\}\\ &= \left\{a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix} : a_{ij}\in\mathbb{N}_0, a_{i1}+a_{i2} = k_i, a_{1i}+a_{2i} = l_i\right\}\end{aligned}$$

and

$$\tilde{\mathcal{A}}(k) = \bigcup_{l=0}^{2q-2} \mathcal{A}(k,l) = \left\{ a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} : a_{ij} \in \mathbb{N}_0, a_{i1} + a_{i2} = k_i \right\}.$$

Then, using (24) and recalling (13), we have

$$\begin{aligned}
\frac{\partial^2 \widetilde{P}_q}{\partial x^2} &= \sum_{k,l=0}^{q-1} \alpha_q(2k)\alpha_q(2l) \sum_{n=\max(0,2k+2l-2q+2)}^{\min(2k,2l)} \frac{x^{2(q-k-l-1)+n}}{(2q-2k-2l-2+n)!} \frac{y^{2k-n}}{(2k-n)!} \frac{y^{2l-n}}{(2l-n)!} \frac{z^n}{n!} \\
&= \sum_{k,l=0}^{q-1} \alpha_q(2k)\alpha_q(2l) \sum_{A\in\mathcal{A}(2k,2l)} \prod_{i,j=1}^{2} \frac{x_{ij}^{a_{ij}}}{a_{ij}!} \\
&= \sum_{k,l=0}^{2q-2} \alpha_q(k)\alpha_q(l) \sum_{A\in\mathcal{A}(k,l)} \prod_{i,j=1}^{2} \frac{x_{ij}^{a_{ij}}}{a_{ij}!}
\end{aligned}$$

where

$$x_{11} = x, \quad x_{12} = x_{21} = y, \quad \text{and} \quad x_{22} = z.$$

We now bound

$$\begin{aligned}
\left| \frac{\partial^2 \widetilde{P}_q}{\partial x^2} \right| &\le \sum_{k,l=0}^{2q-2} |\alpha_q(k)\alpha_q(l)| \sum_{a\in\mathcal{A}(k,l)} \prod_{i,j=1}^{2} \frac{|x_{ij}|^{a_{ij}}}{a_{ij}!} \\
&\le \sum_{k,l=0}^{2q-2} \left(\frac{\alpha_q(k)^2 + \alpha_q(l)^2}{2} \right) \sum_{a\in\mathcal{A}(k,l)} \prod_{i,j=1}^{2} \frac{|x_{ij}|^{a_{ij}}}{a_{ij}!}
\end{aligned}$$

Algebraic manipulation of this last quantity yields

$$\left| \frac{\partial^2 \widetilde{P}_q}{\partial x^2} \right| \le \sum_{k=0}^{2q-2} \alpha_q(k)^2 \sum_{l=0}^{2q-2} \sum_{a\in\mathcal{A}(k,l)} \prod_{i,j=1}^{2} \frac{|x_{ij}|^{a_{ij}}}{a_{ij}!} = \sum_{k=0}^{2q-2} \alpha_q(k)^2 \sum_{a\in\tilde{\mathcal{A}}(k)} \prod_{i,j=1}^{2} \frac{|x_{ij}|^{a_{ij}}}{a_{ij}!}$$

Applying the Binomial Theorem to the last term gives

$$\begin{aligned}
\sum_{k=0}^{2q-2} \alpha_q(k)^2 \prod_{i=1}^{2} \frac{(|x_{i1}| + |x_{i2}|)^{k_i}}{k_i!} &\le M^{2q-2} \sum_{k=0}^{2q-2} \frac{\alpha_q(k)^2}{k!(2q-2-k)!} \\
&\le 4q^2 M^{2q-2} \sum_{k=0}^{2q-2} \frac{\alpha_q(k)^2}{k!(2q-k)!} \\
&= 4q^2 M^{2q-2} \sum_{k=0}^{q-1} \frac{1}{(q!)^2} \binom{q}{k}^2 \frac{(2k)!(2q-2k)!}{(2k-1)^2} = 4q^2 c_q M^{2q-2}.
\end{aligned}$$

where the last identity is due to Lemma 15 below, and we remind the reader of (15). We also have, from (14), that

$$\frac{\partial^2 P_q}{\partial x^2} = \frac{\partial^2 \widetilde{P}_q}{\partial x^2} + (2q-1)(2q-2)c_q \left(x^{2q-3} z + (2q-3) x^{2q-4} y^2 \right).$$

We next bound this final summand. Note that for $q = 1$ this term vanishes. Otherwise, on the domain $D_M = \{|x| + |y| \le M, |y| + |z| \le M\}$, it attains its maximum on the boundary, and a calculation reveals the maximum is attained at $|z| = |x| = M, y = 0$. Therefore

$$|x^{2q-3} z + (2q-3) x^{2q-4} y^2| \le M^{2q-2}.$$

Combining these two estimates we obtain

$$\left|\frac{\partial^2 P_q}{\partial x^2}\right| \le \left(4q^2 + (2q-1)(2q-2)\right) c_q M^{2q-2} \le 8q^2 c_q M^{2q-2}.$$

Using Sterling's bounds [17] we see that $\binom{2q}{q} \ge \frac{2\sqrt{\pi}}{e^2}\frac{2^{2q}}{\sqrt{q}}$ which yields

$$c_q = \frac{2^{4q}}{2q\binom{2q}{q}} \le \frac{e^2}{4\sqrt{\pi}}\frac{4^q}{\sqrt{q}}$$

so that

$$\sup_{D_M}\left|\frac{\partial^2 P_q}{\partial x^2}\right| \le \frac{2e^2}{\sqrt{\pi}} q^{3/2} 4^q M^{2q-2}. \tag{25}$$

By the mean value theorem,

$$P_q(x,y,z) = P_q(-z,y,z) + \frac{\partial P_q}{\partial x}(-z,y,z)(x+z) + \frac{1}{2}\frac{\partial^2 P_q}{\partial x^2}(t,y,z)(x+z)^2$$

for some t between x and $-z$. It follows from Proposition 13 that $P_q(-z,y,z) = \frac{\partial P_q}{\partial x}(-z,y,z) = 0$, so that

$$P_q(x,y,z) = \frac{1}{2}\frac{\partial^2 P_q}{\partial x^2}(t,y,z)(x+z)^2.$$

Note that $|t| \le \max(|x|,|z|) \le M - |y|$ and so by (25) we have

$$\frac{|P_q(x,y,z)|}{(x+z)^2} \le \frac{1}{2}\sup_{(t,y,z)\in D_M}\left|\frac{\partial^2 P_q}{\partial x^2}(t,y,z)\right| \le \frac{e^2}{\sqrt{\pi}} q^{3/2} 4^q M^{2q-2}. \qquad \square$$

In the course of the proof we used the following computation.

Lemma 15. *For all $q \in \mathbb{N}$ we have*

$$c_q = \sum_{l=0}^{q} \binom{2l}{l}\binom{2q-2l}{q-l}\frac{1}{(2l-1)^2}.$$

Proof. For $q \ge 0$, let us denote $T_q = \sum_{l=0}^{q}\binom{2l}{l}\binom{2q-2l}{q-l}\frac{1}{(2l-1)^2}$. Notice that

$$\sum_{q=0}^{\infty} T_q x^{2q} = \phi(x)\psi(x) \tag{26}$$

where

$$\phi(x) = \sum_{l=0}^{\infty}\binom{2l}{l}\frac{x^{2l}}{(2l-1)^2}, \quad \text{and} \quad \psi(x) = \sum_{l=0}^{\infty}\binom{2l}{l}x^{2l} = \frac{1}{\sqrt{1-4x^2}}.$$

We next compute ϕ. We have

$$\frac{d}{dx}\left[\frac{\phi(x)}{x}\right] = \sum_{l=0}^{\infty}\binom{2l}{l}\frac{x^{2l-2}}{2l-1} = -\frac{1}{x^2}\sqrt{1-4x^2} = \frac{d}{dx}\left[\frac{\sqrt{1-4x^2}}{x} + 2\arcsin(2x)\right]$$

and so $\frac{\phi(x)}{x} = \frac{\sqrt{1-4x^2}}{x} + 2\arcsin(2x) + C$ for some constant C. Since all the functions in this equation are odd, it follows that $C = 0$, and so $\phi(x) = \sqrt{1-4x^2} + 2x\arcsin(2x)$. Therefore, using the Taylor

[17] The constants here are not asymptotically optimal, but this is irrelevant for our purposes.

series (21) once more,

$$\begin{aligned}\phi(x)\psi(x) &= 1 + \frac{2x\arcsin(2x)}{\sqrt{1-4x^2}} \\ &= 1 + \frac{x}{2}\frac{d}{dx}\left(\arcsin(2x)\right)^2 \\ &= 1 + \frac{x}{2}\frac{d}{dx}\sum_{q=1}^{\infty}\frac{(4x)^{2q}}{2q^2\binom{2q}{q}} = 1 + \sum_{q=1}^{\infty}\frac{4^{2q}x^{2q}}{2q\binom{2q}{q}}.\end{aligned}$$

Comparing this with (26) we conclude that $T_q = \frac{2^{4q}}{2q\binom{2q}{q}} = c_q$ for $q \geq 1$. □

5.2. **Proof of Theorem 3.** Having Proposition 14 at our disposal, we are ready to prove Theorem 3. Let

$$M' = \limsup_{|t|\to\infty} \varphi(t) < 1$$

and choose $M \in (M', 1)$. Then there exists some $T_0 > 0$ such that $\varphi(t) \leq M$ for all $|t| > T_0$. We can rearrange (16) to obtain

$$\begin{aligned}V_q(T) = V_q(T_0) &+ 2\,(T - T_0)\int_0^{T_0} P_q\left(r(t), \frac{r'(t)}{\sigma}, \frac{r''(t)}{\sigma^2}\right)\,dt \\ &+ 2\int_{T_0}^{T}(T-t)\,P_q\left(r(t), \frac{r'(t)}{\sigma}, \frac{r''(t)}{\sigma^2}\right)\,dt. \end{aligned}\tag{27}$$

Proposition 14 yields

$$\begin{aligned}\left|\int_{T_0}^{T}(T-t)\,P_q\left(r(t), \frac{r'(t)}{\sigma}, \frac{r''(t)}{\sigma^2}\right)\,dt\right| &\leq \frac{e^2}{\sqrt{\pi}}q^{3/2}4^qM^{2q-2}\int_{T_0}^{T}(T-t)\left(r(t)+\frac{r''(t)}{\sigma^2}\right)^2\,dt \\ &\leq \frac{e^2}{\sqrt{\pi}}q^{3/2}4^qM^{2q-2}\int_{0}^{T}(T-t)\left(r(t)+\frac{r''(t)}{\sigma^2}\right)^2\,dt \\ &= \frac{e^2}{\sqrt{\pi}}q^{3/2}4^{q-1}M^{2q-2}V_1(T),\end{aligned}\tag{28}$$

see (22). Since $M < 1$ we see that

$$\sum_{q=1}^{\infty}\frac{1}{4^q}\left|\int_{T_0}^{T}(T-t)\,P_q\left(r(t), \frac{r'(t)}{\sigma}, \frac{r''(t)}{\sigma^2}\right)\,dt\right| < \infty.$$

By Proposition 9, since we are assuming the Geman condition, we have $\sum_{q=1}^{\infty}\frac{V_q(T)}{4^q} < \infty$ for every $T > 0$ and so we may write, from (27)

$$\begin{aligned}\sum_{q=1}^{\infty}\frac{V_q(T)}{4^q} = \sum_{q=1}^{\infty}\frac{V_q(T_0)}{4^q} &+ (T-T_0)\sum_{q=1}^{\infty}\frac{1}{4^q}\int_0^{T_0} P_q\left(r(t), \frac{r'(t)}{\sigma}, \frac{r''(t)}{\sigma^2}\right)\,dt \\ &+ \sum_{q=1}^{\infty}\frac{1}{4^q}\int_{T_0}^{T}(T-t)\,P_q\left(r(t), \frac{r'(t)}{\sigma}, \frac{r''(t)}{\sigma^2}\right)\,dt.\end{aligned}$$

Combining this with (28) we get

$$\sum_{q=1}^{\infty}\frac{V_q(T)}{4^q} \leq C_0 + C_1T + C_2V_1(T)$$

where C_0, C_1 and C_2 depend on T_0 and M. Recalling Proposition 9 we have

$$\mathrm{Var}[N(T)] \leq \frac{\sigma^2}{\pi^2} \sum_{q=1}^{\infty} \frac{V_q(T)}{4^q} + \frac{1}{4} \leq C_3 V_1(T)$$

where we have used the lower bound proved in Section 4 for the final bound.

5.3. **Proof of Theorem 5.** By (22) we need to show that $\mathrm{Var}[N(T)] \sim \frac{\sigma^2}{4\pi^2} V_1(T)$. The lower bound follows immediately from Theorem 1 and so we focus on the upper bound. We proceed as in the previous section, but estimate more carefully. By Proposition 9 we have

$$\mathrm{Var}[N(T)] \leq \frac{\sigma^2}{4\pi^2} V_1(T) + \frac{\sigma^2}{\pi^2} \sum_{q=2}^{\infty} \frac{V_q(T)}{4^q} + \frac{1}{4}.$$

Now fix $\varepsilon > 0$ and choose $T_0 = T_0(\varepsilon)$ such that $\varphi(t) < \varepsilon$ for all $t > T_0$. As in the previous section we write

$$\sum_{q=2}^{\infty} \frac{V_q(T)}{4^q} = \sum_{q=2}^{\infty} \frac{V_q(T_0)}{4^q} + (T - T_0) \sum_{q=2}^{\infty} \frac{1}{4^q} \int_0^{T_0} P_q\left(r(t), \frac{r'(t)}{\sigma}, \frac{r''(t)}{\sigma^2}\right) \, dt$$
$$+ \sum_{q=2}^{\infty} \frac{1}{4^q} \int_{T_0}^{T} (T - t)\, P_q\left(r(t), \frac{r'(t)}{\sigma}, \frac{r''(t)}{\sigma^2}\right) \, dt$$

and estimate

$$\left| \int_{T_0}^{T} (T - t)\, P_q\left(r(t), \frac{r'(t)}{\sigma}, \frac{r''(t)}{\sigma^2}\right) \, dt \right| \leq \frac{e^2}{2\sqrt{\pi}} q^{3/2} 4^q \varepsilon^{2q-2} V_1(T).$$

This yields

$$\sum_{q=2}^{\infty} \frac{V_q(T)}{4^q} = C_0 + C_1 T + \frac{2e^2}{\sqrt{\pi}} \sum_{q=2}^{\infty} q^{3/2} (4\varepsilon^2)^{q-1} V_1(T) \leq C_0 + C_1 T + C_3 \varepsilon^2 V_1(T)$$

and we finally note that since $r + \frac{r''}{\sigma^2} \notin \mathcal{L}^2(\mathbb{R})$ we have

$$\frac{V_1(T)}{T} \to \infty$$

as $T \to \infty$. This completes the proof.

5.4. **Conjectural Bounds.** In this section we give some evidence in favor of the conjectures stated in the Introduction. The precise expression for the variance appearing in Proposition 9 establishes a way to prove even tighter upper bounds, by reducing to combinatorial statements about the polynomials P_q, defined in (14). It is not difficult to see that the vector $(r(t), r'(t)/\sigma, r''(t)/\sigma^2)$ always lies in the domain

$$D = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1, y^2 + z^2 \leq 1\}.$$

By Proposition 13, $R_q(x, y, z) = P_q(x, y, z)/(x + z)^2$ is a homogeneous polynomial and since D contains all segments to the origin, it follows that R_q attains the maximum of its absolute value on the boundary. We expect that the maximum should be obtained at the points where $|x| = |z|$.

When $x = -z$, the same techniques employed in this paper show the value to be

$$\left.\frac{P_q(x, y, z)}{(x + z)^2}\right|_{z=-x} = 2^{2q-1} (x^2 + y^2)^{q-1}$$

and so on this boundary component the value of R_q is 2^{2q-1}. We believe that this bound is the one relevant to Gaussian processes, however numerical computations suggest that R_q can be much larger at the points where $x = z$. We believe that there is some 'hidden' structure that prevents $r(t)$

from being close to $r''(t)/\sigma^2$ in certain subregions of D. For example, if $r(t)$ is close to 1 then we should be close to a local maximum and so we would expect $r''(t)$ to be negative. Understanding the 'true domain' where the vector $(r(t), r'(t)/\sigma, r''(t)/\sigma^2)$ 'lives' already appears to be a quite interesting question.

6. Singular spectral measure

6.1. **Atoms in the spectral measure: proofs of Theorems 6 and 7.** In this section we consider the effect of atoms in the spectral measure, that is, we prove Theorems 6 and 7. Our proof relies on the following proposition.

Proposition 16. *Let μ be a signed-measure with $\int_{\mathbb{R}} d|\mu| < \infty$. Then μ contains an atom if and only if there exists $c > 0$ such that*

$$\int_{-T}^{T} (T - |t|)\, |\widehat{\mu}(t)|^2 dt \geq cT^2$$

for all $T > 0$.

We postpone the proof of Proposition 16 to Section 6.2. We will also need the following result.

Lemma 17. *Let f be a SGP with covariance kernel r, spectral measure ρ and suppose that ρ has a continuous component. Let $\psi(t) = A\cos(\sigma t + \alpha)$, where $A \in \mathbb{R}$, $\alpha \in [0, 2\pi]$ and $\sigma^2 = -r''(0)$. Denote by $N_J(\psi) = \#\{t \in [0, \pi J/\sigma] : f(t) = \psi(t)\}$ the number of crossings of the curve ψ by the process. Then $\mathbb{E}[N_J(\psi)] = J$.*

Proof. Denote the Gaussian density function by φ and by Φ the corresponding distribution function. The generalised Rice formula [7, Equation 13.2.1] gives

$$\begin{aligned}
\mathbb{E}N_J(\psi) &= \sigma \int_0^{\frac{\pi J}{\sigma}} \varphi(\psi(y)) \left[2\varphi\left(\frac{\psi'(y)}{\sigma}\right) + \frac{\psi'(y)}{\sigma}\left(2\Phi\left(\frac{\psi'(y)}{\sigma}\right) - 1\right)\right] dy \\
&= \sigma \int_0^{\frac{\pi J}{\sigma}} \frac{e^{-\frac{A^2}{2}\cos^2(\sigma y+\alpha)}}{\sqrt{2\pi}} \left[\frac{2e^{-\frac{A^2}{2}\sin^2(\sigma y+\alpha)}}{\sqrt{2\pi}} - A\sin(\sigma y + \alpha)\left(2\Phi\left(-A\sin(\sigma y+\alpha)\right) - 1\right)\right] dy \\
&= Je^{-\frac{A^2}{2}} - \frac{\sigma}{\sqrt{2\pi}} \int_0^{\frac{\pi J}{\sigma}} e^{-\frac{A^2}{2}\cos^2(\sigma y+\alpha)} A\sin(\sigma y+\alpha)\left(2\Phi\left(-A\sin(\sigma y+\alpha)\right) - 1\right) dy \\
&= Je^{-\frac{A^2}{2}} - \frac{\sigma}{\sqrt{2\pi}} \int_0^{\frac{\pi J}{\sigma}} e^{-\frac{A^2}{2}\cos^2(\sigma y+\alpha)} |A|\sin(\sigma y+\alpha)\left(2\Phi\left(-|A|\sin(\sigma y+\alpha)\right) - 1\right) dy.
\end{aligned}$$

Write

$$F(y) = e^{-\frac{A^2}{2}\cos^2(y)} |A|\sin(y)\left(2\Phi\left(-|A|\sin(y)\right) - 1\right)$$

and notice that F is periodic with period π. This yields

$$\mathbb{E}N_J(\psi) = J\left(e^{-\frac{A^2}{2}} - \frac{\sigma}{\sqrt{2\pi}} \int_0^{\frac{\pi}{\sigma}} F(\sigma y + \alpha) dy\right) = J\left(e^{-\frac{A^2}{2}} - \frac{1}{\sqrt{2\pi}} \int_0^{\pi} F(y) dy\right). \tag{29}$$

Moreover, since F is even we have

$$\int_0^{\pi} F(y)\, dy = \int_0^{\frac{\pi}{2}} F(y)\, dy + \int_{\frac{\pi}{2}}^{\pi} F(y)\, dy = \int_0^{\frac{\pi}{2}} F(y)\, dy + \int_{-\frac{\pi}{2}}^{0} F(y)\, dy = 2\int_0^{\frac{\pi}{2}} F(y)\, dy.$$

Substituting $u = |A|\cos(y)$ we obtain

$$\begin{aligned}
-\frac{1}{\sqrt{2\pi}}\int_0^{\pi} F(y)dy &= -\sqrt{\frac{2}{\pi}}\int_0^{\pi/2} F(y)dy \\
&= -\sqrt{\frac{2}{\pi}}\int_0^{|A|} e^{-\frac{u^2}{2}} \cdot \left(2\Phi\left(-\sqrt{A^2-u^2}\right) - 1\right) du \\
&= \frac{2}{\pi}\int_0^{|A|}\int_0^{\sqrt{A^2-u^2}} e^{-\frac{u^2+v^2}{2}}\, dv\, du \\
&= \frac{2}{\pi}\int_0^{|A|}\int_0^{\pi/2} e^{-\frac{r^2}{2}} r d\theta dr = 1 - e^{-\frac{A^2}{2}}.
\end{aligned}$$

Inserting this value into (29) yields the result. □

Proof of Theorem 6. First we note that, by stationarity, $\mathrm{Var}[N(T)] \le CT^2$ for some $C > 0$. Assume that ρ has an atom at a point different from σ. By (17) and (22), to show that $\mathrm{Var}[N(T)] \ge \frac{c\sigma^2}{2\pi^2}T^2$ for some $c > 0$ it is enough to see that

$$\int_{-T}^{T} (T - |t|)\left(r(t) + \frac{r''(t)}{\sigma^2}\right)^2 dt \ge cT^2.$$

But this follows from Proposition 16 if we define the signed measure μ by $d\mu(\lambda) = (1 - \frac{\lambda^2}{\sigma^2})d\rho(\lambda)$ and notice that $\hat{\mu} = r + \frac{r''}{\sigma^2}$ and that μ has an atom.

For the converse, notice that it is enough to check that for integer J we have

$$\frac{\mathrm{Var}[N(\frac{\pi}{\sigma}J)]}{J^2} \to 0 \quad \text{as} \quad J \to \infty,$$

since this implies that $\mathrm{Var}[N(T)] = o(T^2)$, by stationarity. Assume first that ρ has no atoms; we adapt the proof of [5, Thm 4]. By the Fomin-Grenander-Maruyama theorem, f is an ergodic process (see, e.g., [15, Sec. 5.10]). By standard arguments, this also implies that the sequence

$$\mathcal{N}_j = \#\left\{t \in \left[(j-1)\frac{\pi}{\sigma}, j\frac{\pi}{\sigma}\right) : \; f(t) = 0\right\}.$$

is ergodic. Recall that we assume the Geman condition, which implies that the first and second moments of

$$N\left(\frac{\pi}{\sigma}J\right) = \sum_{j=1}^{J} \mathcal{N}_j$$

are finite. Thus, by von Neumann's ergodic theorem, we have

$$\lim_{J\to\infty} \frac{N(\frac{\pi}{\sigma}J)}{J} = \mathbb{E}[\mathcal{N}_1] = 1,$$

where the convergence is both in L^1 and L^2 (see [39, Cor. 1.14.1]). We conclude that

$$\lim_{J\to\infty} \frac{\mathrm{Var}[N(\frac{\pi}{\sigma}J)]}{J^2} = 0.$$

Finally suppose that $\rho = \theta\rho_c + (1-\theta)\delta_\sigma^*$ where $0 < \theta < 1$ and ρ_c has no atoms. We may represent the corresponding process as

$$f(t) = \sqrt{\theta} f_c + \sqrt{(1-\theta)X}\cos(\sigma t + \Phi)$$

where f_c is a SGP with spectral measure ρ_c, $X \sim \chi^2(2)$, $\Phi \sim \mathrm{Unif}([0,2\pi])$, and moreover f_c, X and Φ are pairwise independent. By the law of total variance and Lemma 17 we have

$$\begin{aligned}\mathrm{Var}\left[N\Big(\frac{\pi}{\sigma}J\Big)\right] &= \mathbb{E}\Big[\,\mathrm{Var}\left[N\Big(\frac{\pi}{\sigma}J\Big)\Big|X,\Phi\right]\Big] + \mathrm{Var}\left[\mathbb{E}\left[N\Big(\frac{\pi}{\sigma}J\Big)\Big|X,\Phi\right]\right]\\ &= \mathbb{E}\Big[\,\mathrm{Var}\left[N\Big(\frac{\pi}{\sigma}J\Big)\Big|X,\Phi\right]\Big]. \end{aligned} \tag{30}$$

We define, for $A \in \mathbb{R}$ and $\alpha \in [0,2\pi]$,

$$\mathcal{N}_j^{A,\alpha} = \#\left\{t \in \Big[(j-1)\frac{\pi}{\sigma}, j\frac{\pi}{\sigma}\Big) :\ f_c(t) = A\cos(\sigma t + \alpha)\right\}.$$

As before the process f_c is ergodic, and so is the sequence $\mathcal{N}_j^{A,\alpha}$ for fixed A and α. This implies that

$$\lim_{J\to\infty} \frac{\mathrm{Var}[N(\frac{\pi}{\sigma}J)|X,\Phi]}{J^2} = 0$$

(almost surely), exactly as before. Furthermore, using stationarity we have

$$\frac{1}{J^2}\,\mathrm{Var}\left[N\Big(\frac{\pi}{\sigma}J\Big)\Big|X,\Phi\right] \le \mathrm{Var}\left[N\Big(\frac{\pi}{\sigma}\Big)\Big|X,\Phi\right]$$

and using (30) we see that

$$\mathbb{E}\Big[\,\mathrm{Var}\left[N\Big(\frac{\pi}{\sigma}\Big)\Big|X,\Phi\right]\Big] = \mathrm{Var}\left[N\Big(\frac{\pi}{\sigma}\Big)\right] < +\infty,$$

since we assume the Geman condition. It follows from dominated convergence that

$$\lim_{J\to\infty}\frac{1}{J^2}\mathbb{E}\Big[\,\mathrm{Var}\left[N\Big(\frac{\pi}{\sigma}J\Big)\Big|X,\Phi\right]\Big] = 0$$

whence $\lim_{J\to\infty}\frac{\mathrm{Var}[N(\frac{\pi}{\sigma}J)]}{J^2} = 0$. □

Proof of Theorem 7. Let $M = \limsup_{|t|\to\infty}\varphi(t)$, where φ is defined in (4). By assumption we have $M < 1$ and we define

$$\theta_0 = \frac{1-M}{\sqrt{2}-M}.$$

We would like to apply Theorem 3 to the spectral measure ρ_θ. Writing $r_\theta = \mathcal{F}[\rho_\theta]$ and $r = \mathcal{F}[\rho]$ we have $r_\theta(t) = (1-\theta)r(t) + \theta\cos(\sigma t)$, and $\sigma_\theta^2 = -r_\theta''(0) = \sigma^2$. We accordingly compute

$$\begin{aligned}\varphi_\theta(t) &= \max\left\{|r_\theta(t)| + \frac{|r_\theta'(t)|}{\sigma},\ \frac{|r_\theta''(t)|}{\sigma^2} + \frac{|r_\theta'(t)|}{\sigma}\right\}\\ &\le \max\Big\{(1-\theta)\left(|r(t)| + \frac{|r'(t)|}{\sigma}\right) + \theta(|\cos\sigma t| + |\sin\sigma t|),\\ &\qquad\qquad (1-\theta)\left(\frac{|r''(t)|}{\sigma^2} + \frac{|r'(t)|}{\sigma}\right) + \theta(|\cos\sigma t| + |\sin\sigma t|)\Big\}\\ &\le (1-\theta)M + \theta\sqrt{2}\end{aligned}$$

and so

$$\limsup_{|t|\to\infty}\varphi_\theta(t) < 1$$

for $\theta < \theta_0$. Applying Theorem 3 to ρ_θ and to ρ we obtain

$$\begin{aligned}\mathrm{Var}[N(\rho_\theta;T)] &\asymp T\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)\left(r_\theta(t)+\frac{r''_\theta(t)}{\sigma_\theta^2}\right)^2\ dt\\ &= (1-\theta)^2 T\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)\left(r(t)+\frac{r''(t)}{\sigma^2}\right)^2\ dt\\ &\asymp \mathrm{Var}[N(\rho;T)].\end{aligned}$$

□

6.2. **Proof of Proposition 16.** We begin with a review of some elementary harmonic analysis that we will need, for more details and proofs see, e.g., Katznelson's book [21, Ch. VI]. Let $\mathcal{M}(\mathbb{R})$ denote the space of all finite signed measures on $\mathbb{R}$ endowed with the *total mass* norm $\|\mu\|_1 = \int_{\mathbb{R}} d|\mu|$. Recall that the *convolution* of two measures $\mu, \nu \in \mathcal{M}(\mathbb{R})$ is given by $(\mu * \nu)(E) = \int \mu(E-\lambda)d\nu(\lambda)$ for any measurable set E and satisfies $\|\mu*\nu\|_1 \le \|\mu\|_1\|\nu\|_1$ and $\mathcal{F}[\mu*\nu] = \mathcal{F}[\mu]\cdot\mathcal{F}[\nu]$. Moreover, $\mathcal{F}[\cdot]$ is a uniformly continuous map with $\|\mathcal{F}[\mu]\|_\infty \le \|\mu\|_1$. We identify a function $f \in L^1$ with the measure whose density is f.

The following lemma is a version of Parseval's identity, see [21, VI 2.2].

Lemma 18 (Parseval)**.** *If $f, \mathcal{F}[f] \in L^1(\mathbb{R})$ and $\nu \in \mathcal{M}(\mathbb{R})$, then $\int f d\nu = \frac{1}{2\pi}\int \mathcal{F}[f]\overline{\mathcal{F}[\nu]}$.*

A simple application of Parseval's identity proves our next lemma.

Lemma 19. *Suppose that $\mu, \nu \in \mathcal{M}(\mathbb{R})$ and $S, \mathcal{F}[S] \in L^1(\mathbb{R})$. Then*

$$\int (S*\mu)d\nu = \frac{1}{2\pi}\int \mathcal{F}[S]\mathcal{F}[\mu]\overline{\mathcal{F}[\nu]}.$$

Proof. Note that $S*\mu$ is a function and further that

$$\begin{gathered}\|S*\mu\|_1 \le \|\mu\|_1\|S\|_1 < \infty, \qquad \text{and}\\ \|\mathcal{F}[S*\mu]\|_1 = \|\mathcal{F}[S]\mathcal{F}[\mu]\|_1 \le \|\mathcal{F}[\mu]\|_\infty\|\mathcal{F}[S]\|_1 \le \|\mu\|_1\|\mathcal{F}[S]\|_1 < \infty.\end{gathered}$$

A simple application of Lemma 18 finishes the proof. □

We will also use the so-called 'triangle function'

$$\mathcal{T}_T(t) = \left(1-\frac{|t|}{T}\right)\mathbb{1}_{[-T,T]}(t)$$

which satisfies $\mathcal{T}_T = \mathcal{F}[\mathcal{S}_T]$ where[18]

$$\mathcal{S}_T(\lambda) = \frac{T}{2\pi}\ \mathrm{sinc}^2\left(\frac{T\lambda}{2}\right).$$

Notice that applying Lemma 19 to these functions, we obtain

$$\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)|\widehat{\mu}(t)|^2 dt = \int_{\mathbb{R}} \mathcal{T}_T|\mathcal{F}[\mu]|^2 = 2\pi\int (\mathcal{S}_T*\mu)\ d\mu,$$

which is (2).

We are now ready to prove Proposition 16. First suppose that μ contains an atom at α. Write $\mu = \mu_1+\mu_2$ where $\mu_1 = c\delta_\alpha$ for some $c \ne 0$ and $\mu_2(\{\alpha\}) = 0$. Note that

$$|\mu_2([\alpha-\varepsilon,\alpha+\varepsilon])| \le |\mu_2|([\alpha-\varepsilon,\alpha+\varepsilon]) \downarrow 0,\ \text{as } \varepsilon\downarrow 0. \tag{31}$$

We have

$$\begin{aligned}|\mathcal{F}[\mu](t)|^2 &= |\mathcal{F}[\mu_1](t)|^2 + 2\mathrm{Re}\{\mathcal{F}[\mu_1](t)\overline{\mathcal{F}[\mu_2](t)}\} + |\mathcal{F}[\mu_2](t)|^2\\ &\ge |c|^2 + 2\mathrm{Re}\{\mathcal{F}[\mu_1](t)\overline{\mathcal{F}[\mu_2](t)}\}\end{aligned}$$

[18]We use the normalisation $\mathrm{sinc}(x) = \frac{\sin x}{x}$.

Using this and Lemma 19 we obtain

$$\int_{-T}^{T}(T-|t|)|\widehat{\mu}(t)|^2dt = T\int_{\mathbb{R}}\mathcal{T}_T|\mathcal{F}[\mu]|^2 \geq |c|^2T\int_{\mathbb{R}}\mathcal{T}_T + 2T\mathrm{Re}\left\{\int_{\mathbb{R}}\mathcal{T}_T\mathcal{F}[\mu_1]\overline{\mathcal{F}[\mu_2]}\right\}$$
$$= |c|^2T^2 + 4\pi T\,\mathrm{Re}\left\{\int_{\mathbb{R}}\mathcal{S}_T*\mu_1\ d\mu_2\right\}.$$

It is therefore enough to show that $\int_{\mathbb{R}}(\mathcal{S}_T*\mu_1)\ d\mu_2 = o(T)$. We bound

$$\left|\int(\mathcal{S}_T*\mu_1)(\lambda)\ d\mu_2(\lambda)\right| = \left|\frac{cT}{2\pi}\int_{\mathbb{R}}\mathrm{sinc}^2\left(\tfrac{T}{2}(\lambda-\alpha)\right)\ d\mu_2(\lambda)\right| \leq \frac{|c|T}{2\pi}\int_{\mathbb{R}}\mathrm{sinc}^2\left(\tfrac{T}{2}(\lambda-\alpha)\right)\ d|\mu_2|(\lambda).$$

Let $I_\alpha(T) = \left[\alpha - \frac{\log T}{T}, \alpha + \frac{\log T}{T}\right]$. By (31) we have

$$\int_{I_\alpha(T)}\mathrm{sinc}^2\left(\tfrac{T}{2}(\lambda-\alpha)\right)d|\mu_2|(\lambda) \leq |\mu_2|(I_\alpha(T)) \to 0, \quad \text{as } T\to\infty.$$

On $\mathbb{R}\setminus I_\alpha(T)$ we have $\frac{T}{2}|\lambda-\alpha| \geq \frac{\log T}{2}$, so that

$$\int_{\mathbb{R}\setminus I_\alpha(T)}\mathrm{sinc}^2\left(\tfrac{T}{2}(\lambda-\alpha)\right)d|\mu_2|(\lambda) \leq \frac{4}{(\log T)^2}|\mu_2|(\mathbb{R}) \to 0, \quad \text{as } T\to\infty.$$

This concludes the first part of the proof.

Conversely, suppose that μ contains no atoms. Recall that

$$\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)|\widehat{\mu}(t)|^2dt = 2\pi\int(\mathcal{S}_T*\mu)\ d\mu.$$

We will show that $|(\mathcal{S}_T*\mu)(\lambda)| = o(T)$, uniformly in λ, which will conclude the proof. As before, denoting $I_\lambda(T) = \left[\lambda-\frac{\log T}{T}, \lambda+\frac{\log T}{T}\right]$ we have

$$|(\mathcal{S}_T*\mu)(\lambda)| = \left|\int_{\mathbb{R}}\frac{T}{2\pi}\mathrm{sinc}^2\left(\tfrac{T}{2}(\lambda-\tau\right)d\mu(\tau)\right| \leq \frac{T}{2\pi}\left(|\mu|(I_\lambda(T)) + \frac{4|\mu|(\mathbb{R})}{(\log T)^2}\right).$$

It therefore suffices to prove the following claim.

Claim 20. *Let ν be a non-negative, finite measure on $\mathbb{R}$ that contains no atoms. Then*

$$\sup_{x\in\mathbb{R}}\nu\big([x-\varepsilon,x+\varepsilon]\big) \to 0, \quad \textit{as } \varepsilon\downarrow 0.$$

Proof. Denote $B(x,\varepsilon) = [x-\varepsilon, x+\varepsilon]$ and $m(\varepsilon) = \sup_{x\in\mathbb{R}}\nu\big(B(x,\varepsilon)\big)$. It is clear that $m(\varepsilon)$ decreases with ε so $m(\varepsilon)$ must converge as $\varepsilon\downarrow 0$ to some non-negative limit, $2\delta\geq 0$. Suppose that $\delta>0$ and choose $N>0$ such that $\nu(\mathbb{R}\setminus[-N/2,N/2])<\delta$. Fix $n\in\mathbb{N}$ and divide $[-N,N]$ into disjoint 'dyadic' intervals

$$D_n = \left\{[kN2^{-n},(k+1)N2^{-n}) : k\in\mathbb{Z}\cap[-2^n,2^n)\right\}.$$

For any $x\in\mathbb{R}$, either $B(x,\frac{N}{2^n})\subseteq\mathbb{R}\setminus[-N/2,N/2]$, which implies that $\nu(B(x,\frac{N}{2^n}))<\delta$, or $B(x,\frac{N}{2^n})\subseteq I\cup I'$ for some $I,I'\in D_{n-1}$. Therefore,

$$m\left(\tfrac{N}{2^n}\right) \leq \max\left(\delta, 2\sup_{I\in D_{n-1}}\nu(I)\right).$$

Recall that by definition of δ we have $m\left(\frac{N}{2^n}\right)\geq 2\delta$. We conclude that for every $n\in\mathbb{N}$ we can find $I_n\in D_n$ such that

$$\nu(I_n)\geq\delta. \tag{32}$$

Next we shall construct a sequence of nested dyadic intervals $\{J_n\}_{n=0}^{\infty}$ such that, for all n,

$$J_n\in D_n, \quad J_{n+1}\subseteq J_n, \quad \nu(J_n)\geq\delta.$$

This will imply, by Cantor's lemma, that $\bigcap_n J_n = \{x\}$ for some $x \in \mathbb{R}$, and further that $\nu(\{x\}) = \lim_{n\to\infty} \nu(J_n) \geq \delta > 0$. This contradicts the assumption that ν has no atoms, which will end our proof.

We start by setting $J_0 = [-N, N]$. Suppose that we have constructed $J_0 \supset J_1 \supset J_2 \supset \cdots \supset J_m$ such that for every $n > m$ we can find $I'_n \in D_n$ that satisfies

$$I'_n \subset J_m, \quad \text{and} \quad \nu(I'_n) \geq \delta; \tag{33}$$

that is, the interval J_m has a descendant of any generation whose ν-measure is at least δ. Notice that this holds for $m = 0$ by (32). Notice that if (33) fails for both descendants of J_m in the generation D_{m+1}, then it also fails for J_m, since $\nu(J) \geq \nu(J')$ for every descendant $J' \subseteq J$. This completes the inductive construction of J_m and consequently the proof. □

6.3. **Singular continuous measures: Proof of Proposition 8.** Throughout this section we assume the notation of Section 2.6 and that $\alpha_n > R_n$ for every $n \geq 1$. We begin with the following observation.

Lemma 21. *(i) Suppose that I is an interval and $|I| < R_n$. Then $\rho(I) \leq 2^{-n}$.*
(ii) Let $\delta > 0$ and suppose that $R_n < \frac{\delta}{4}$. Then $\rho((\lambda-\delta, \lambda+\delta)) \geq 2^{-n}$ for any λ in the support of ρ.

Remarks.

1. A probability measure μ is said to be *exact dimensional* if there exists d such that
$$\lim_{\delta\to 0} \frac{\log \mu((\lambda-\delta, \lambda+\delta))}{\log \delta} = d \tag{34}$$
for μ-almost every λ. It follows from this lemma that the measure ρ_a is exact dimensional for $0 < a < \frac{1}{2}$, with $d = \log 2 / \log \frac{1}{a}$, and moreover
$$\left| \frac{\log \rho_a((\lambda-\delta, \lambda+\delta))}{\log \delta} - d \right| = O\left(\frac{1}{\log \frac{1}{\delta}} \right)$$
uniformly for every λ in the support of ρ_a.
2. If $\frac{1}{2} \leq a < 1$ then the measure ρ_a is also exact dimensional, but understanding more detailed properties seems to be a difficult question, related to certain notions of entropy. We refer the reader to the surveys [14, 37] for a more thorough discussion.
3. For any compactly supported exact dimensional spectral measure ρ, of dimension d, such that (34) converges uniformly for ρ-almost every λ, we may imitate the proof of Proposition 8 to yield
$$cT^{2-d-\varepsilon} \leq T \int_0^T \left(1 - \frac{t}{T}\right) \left(r(t) + \frac{r''(t)}{\sigma^2}\right)^2 \, dt \leq CT^{2-d+\varepsilon}$$
for any $\varepsilon > 0$ and some $c, C > 0$.

Proof. (i) We begin by recalling Kershner and Wintner's proof that the spectrum is a Cantor type set. Set $C_0 = [-R_0, R_0]$. We form C_1 by deleting an interval of length $2(\alpha_1 - R_1)$ from the centre of C_0, which yields $C_1 = [-R_0, -\alpha_1 + R_1] \cup [\alpha_1 - R_1, R_0]$. We inductively construct C_n to consist of 2^n (closed) intervals formed by deleting an interval of length $2(\alpha_n - R_n)$ from the centre of each of the intervals in the previous generation. Then $S = \cap_{n=0}^{\infty} C_n$ is the spectrum.

Let $\rho_n = \delta^*_{\alpha_1} * \delta^*_{\alpha_2} * \cdots * \delta^*_{\alpha_n}$ which is supported on the set

$$S_n = \left\{ \sum_{j=1}^n \varepsilon_j \alpha_j : \varepsilon_j = \pm 1 \right\}$$

and assigns a mass of 2^{-n} to each of the points of S_n. Furthermore, the elements of S_n are the midpoints of the intervals that make up C_n, and if $s, s' \in S_n$ are distinct then $|s - s'| \geq 2\alpha_n > 2R_n$.

We conclude that if $|I| < R_n$ then I contains at most one element of S_n and so $\rho_n(I) \le 2^{-n}$. Moreover I can only intersect with the descendants of one of the intervals of C_n, and so $\rho_m(I) \le 2^{-n}$ for every $m \ge n$.

It remains to note that ρ_m converges weakly to ρ, and that an interval is always a continuity set of the measure ρ, whence

$$\rho(I) = \lim_{m\to\infty} \rho_m(I) \le 2^{-n}.$$

(ii) In fact the measure ρ_n introduced above is the law of the random variable

$$X_n = \sum_{j=1}^{n} \varepsilon_j \alpha_j$$

where ε_j denotes a sequence of i.i.d. Rademacher[19] random variables, while ρ is the law of the infinite sum

$$X = \sum_{j=1}^{\infty} \varepsilon_j \alpha_j. \tag{35}$$

For λ in the support of ρ we can write $\lambda = \sum_{j=1}^{\infty} \widetilde{\varepsilon}_j \alpha_j$ for some fixed sequence $\widetilde{\varepsilon}_j \in \{-1, 1\}$ and we write $\lambda_n = \sum_{j=1}^{n} \widetilde{\varepsilon}_j \alpha_j$.

Notice that $|X - \lambda| \le |X_n - \lambda_n| + 2R_n$. If $R_n < \frac{\delta}{4}$ we therefore have

$$\mathbb{P}\big[|X-\lambda| < \delta\big] \ge \mathbb{P}\big[|X_n - \lambda_n| < 2R_n\big] = \mathbb{P}[\varepsilon_j = \widetilde{\varepsilon}_j \text{ for } 1 \le j \le n] = 2^{-n}. \qquad \square$$

We now proceed to prove Proposition 8.

Proof of Proposition 8. Using (2) we see that we wish to prove that

$$\iint_{\mathbb{R}^2} \operatorname{sinc}^2\left(\frac{T}{2}(\lambda - \lambda')\right) d\mu(\lambda) d\mu(\lambda') \asymp 2^{-M} \tag{36}$$

where, as before, $d\mu(\lambda) = \left(1 - \frac{\lambda^2}{\sigma^2}\right) d\rho(\lambda)$. Fix $A > 4$ (to be chosen large) and $\frac{1}{2} < \beta < 1$ and define $\widetilde{M}$ by $R_{\widetilde{M}} < \frac{2A}{T} \le R_{\widetilde{M}-1}$. We claim that[20]

$$\eta 2^{-M} - \frac{CA}{T} 2^{-\widetilde{M}} \le \iint_{|\lambda-\lambda'|<\frac{A}{T}} \operatorname{sinc}^2\left(\frac{T}{2}(\lambda-\lambda')\right) d\mu(\lambda) d\mu(\lambda') \le C' 2^{-\widetilde{M}}$$

for some constant $\eta > 0$ (depending only on ρ) and that

$$\iint_{\frac{2A}{T} \le |\lambda-\lambda'| < 2^{\beta M} \frac{2A}{T}} \operatorname{sinc}^2\left(\frac{T}{2}(\lambda-\lambda')\right) d|\mu|(\lambda) d|\mu|(\lambda') \le \frac{C 2^{-\widetilde{M}}}{A^2}.$$

Let us first see how the claims imply the proposition. Since $R_n < \frac{1}{2} R_{n-1}$ we see that

$$\frac{1}{T} \le R_{M-1} \le 2^{\widetilde{M}-M-1} R_{\widetilde{M}} < 2^{\widetilde{M}-M} \frac{A}{T}$$

which yields $2^{M-\widetilde{M}} < A$. Inserting this into our claims and combining them we get

$$\left(\eta - C\Big(\frac{1}{A} + \frac{A^2}{T}\Big)\right) 2^{-M} \le \iint_{|\lambda-\lambda'|<2^{\beta M}\frac{2A}{T}} \operatorname{sinc}^2\left(\frac{T}{2}(\lambda-\lambda')\right) d\mu(\lambda) d\mu(\lambda') \le C'\Big(A + \frac{1}{A}\Big) 2^{-M}.$$

[19] That is, $\mathbb{P}[\varepsilon_j = 1] = \mathbb{P}[\varepsilon_j = -1] = \frac{1}{2}$.

[20] Throughout this proof c, c', C and C' denote positive constants whose exact value is irrelevant and which may vary from one occurrence to the next. They may depend on the measure ρ but are independent of A and T.

We now fix A so large that the lower bound in this expression is at least $\left(\frac{\eta}{2}-\frac{CA^2}{T}\right)2^{-M}$. We then have

$$\frac{\eta}{4}2^{-M} \leq \iint\limits_{|\lambda-\lambda'|<2^{\beta M}\frac{2A}{T}} \operatorname{sinc}^2\left(\frac{T}{2}(\lambda-\lambda')\right)d\mu(\lambda)d\mu(\lambda') \leq C'\Big(A+\frac{1}{A}\Big)2^{-M}$$

for sufficiently large T.

We finally bound

$$\iint\limits_{|\lambda-\lambda'|\geq 2^{\beta M}\frac{2A}{T}} \operatorname{sinc}^2\left(\frac{T}{2}(\lambda-\lambda')\right)d|\mu|(\lambda)d|\mu|(\lambda') \leq \frac{2^{-2\beta M}}{A^2}\iint\limits_{\mathbb{R}^2} d|\mu|(\lambda)d|\mu|(\lambda') \leq \frac{C}{A^2}2^{-2\beta M} = o(2^{-M}),$$

since $\beta > \frac{1}{2}$ and A has been fixed. This yields (36).

It remains only to establish the two claims. We begin with the terms near the diagonal. Since the spectrum is compact we note that $1-\frac{\lambda^2}{\sigma^2}$ is bounded on the support of ρ, and using part (i) of Lemma 21 we see that

$$\begin{aligned}\iint\limits_{|\lambda-\lambda'|<\frac{A}{T}} \operatorname{sinc}^2\left(\frac{T}{2}(\lambda-\lambda')\right)d\mu(\lambda)d\mu(\lambda') &\leq C \iint\limits_{|\lambda-\lambda'|<\frac{A}{T}} d\rho(\lambda)d\rho(\lambda') \\ &= C\int_{\mathbb{R}} \rho\left(\left(\lambda-\frac{A}{T},\lambda+\frac{A}{T}\right)\right)d\rho(\lambda) \leq C2^{-\widetilde{M}}.\end{aligned}$$

In the other direction we note that

$$\begin{aligned}\iint\limits_{|\lambda-\lambda'|<\frac{A}{T}} \operatorname{sinc}^2\left(\frac{T}{2}(\lambda-\lambda')\right)d\mu(\lambda)d\mu(\lambda') &= \iint\limits_{|\lambda-\lambda'|<\frac{A}{T}} \operatorname{sinc}^2\left(\frac{T}{2}(\lambda-\lambda')\right)\left(1-\frac{\lambda^2}{\sigma^2}\right)^2 d\rho(\lambda)d\rho(\lambda') \\ &+ \iint\limits_{|\lambda-\lambda'|<\frac{A}{T}} \operatorname{sinc}^2\left(\frac{T}{2}(\lambda-\lambda')\right)\frac{\lambda^2-(\lambda')^2}{\sigma^2}d\mu(\lambda)d\rho(\lambda').\end{aligned}$$

The integrand of the first term is positive and so it is bounded from below by

$$\iint\limits_{\substack{|\lambda-\lambda'|<\frac{A}{T}\\ ||\lambda|-\sigma|>c}} \operatorname{sinc}^2\left(\frac{T}{2}(\lambda-\lambda')\right)\left(1-\frac{\lambda^2}{\sigma^2}\right)^2 d\rho(\lambda)d\rho(\lambda') \geq c' \iint\limits_{\substack{|\lambda-\lambda'|<\frac{A}{T}\\ ||\lambda|-\sigma|>c}} d\rho(\lambda)d\rho(\lambda') \geq c'2^{-M}$$

where we have used part (ii) of Lemma 21. We also estimate, similar to before,

$$\left|\iint_{|\lambda-\lambda'|<\frac{A}{T}} \operatorname{sinc}^2\left(\frac{T}{2}(\lambda-\lambda')\right)\frac{\lambda^2-(\lambda')^2}{\sigma^2}d\mu(\lambda)d\rho(\lambda')\right| \leq C\frac{A}{T}\iint\limits_{|\lambda-\lambda'|<\frac{A}{T}} d\rho(\lambda)d\rho(\lambda') \leq C\frac{A}{T}2^{-\widetilde{M}}$$

and we have therefore shown that

$$\eta 2^{-M} - \frac{CA}{T}2^{-\widetilde{M}} \leq \iint\limits_{|\lambda-\lambda'|<\frac{A}{T}} \operatorname{sinc}^2\left(\frac{T}{2}(\lambda-\lambda')\right)d\mu(\lambda)d\mu(\lambda') \leq C'2^{-\widetilde{M}}$$

where $\eta > 0$, as desired.

We finally estimate the off-diagonal contribution. We have

$$\iint\limits_{2^j\frac{A}{T}\leq|\lambda-\lambda'|<2^{j+1}\frac{A}{T}} \operatorname{sinc}^2\left(\frac{T}{2}(\lambda-\lambda')\right)d|\mu|(\lambda)d|\mu|(\lambda') \leq \frac{C}{2^{2j}A^2}\iint\limits_{2^j\frac{A}{T}\leq|\lambda-\lambda'|<2^{j+1}\frac{A}{T}} d\rho(\lambda)d\rho(\lambda').$$

For $0 \le j \le \beta M$ we notice that $\frac{2^{\beta M}}{T} < 2^{\beta M} R_M = o(2^M R_M) = o(1)$ and, since $R_n < \frac{1}{2} R_{n-1}$, we see that $R_{\widetilde{M}-j} \ge 2^{j-1} R_{\widetilde{M}-1} \ge \frac{2^j A}{T}$. Applying part (i) of Lemma 21 once more we get

$$\iint_{2^j \frac{A}{T} \le |\lambda - \lambda'| < 2^{j+1} \frac{A}{T}} d\rho(\lambda) d\rho(\lambda') \le 2^{j-\widetilde{M}}.$$

Summing we get

$$\iint_{\frac{2A}{T} \le |\lambda-\lambda'| < 2^{\beta M} \frac{2A}{T}} \operatorname{sinc}^2\left(\frac{T}{2}(\lambda - \lambda')\right) d|\mu|(\lambda) d|\mu|(\lambda') \le \frac{C 2^{-\widetilde{M}}}{A^2} \sum 2^{-j} = \frac{C 2^{-\widetilde{M}}}{A^2}$$

as claimed. □

7. Proof of Proposition 13

7.1. **Dehomogenisation.** Our first step is based on the following lemma.

Lemma 22. *Let $P(x, y, z)$ be a homogeneous polynomial. Then $(x + z)^2 \mid P(x, y, z)$ if and only if $P(-1, y, 1) = 0$ and $\frac{\partial P}{\partial x}(-1, y, 1) = 0$.*

Proof. Consider the polynomial $f(x, y) = P(x, y, 1)$ and write f as a polynomial in $x + 1$ to obtain $f(x, y) = \sum_{j=0}^{d} a_j(y) \cdot (x + 1)^j$. Suppose first that

$$a_0(y) = f(-1, y) = P(-1, y, 1) = 0 \tag{37}$$

and

$$a_1(y) = \frac{\partial f}{\partial x}(-1, y) = \frac{\partial P}{\partial x}(-1, y, 1) = 0. \tag{38}$$

It follows that $(x + 1)^2 \mid f(x, y)$, and we write $f(x, y) = (x + 1)^2 g(x, y)$.

As $P(x, y, z)$ is homogeneous, one has

$$\begin{aligned} P(x, y, z) &= z^{\deg P} P\left(\frac{x}{z}, \frac{y}{z}, 1\right) = z^{\deg P} f\left(\frac{x}{z}, \frac{y}{z}\right) \\ &= z^{\deg P} \left(\frac{x}{z} + 1\right)^2 g\left(\frac{x}{z}, \frac{y}{z}\right) = (x + z)^2 \cdot z^{\deg P - 2} g\left(\frac{x}{z}, \frac{y}{z}\right). \end{aligned}$$

Finally $z^{\deg P-2} g\left(\frac{x}{z}, \frac{y}{z}\right)$ is a homogeneous polynomial, and we are done.

For the converse, note that if $(x + z)^2 \mid P(x, y, z)$, then $(x + 1)^2 \mid f(x, y)$, hence equations (37) and (38) hold. □

In light of Lemma 22, Proposition 13 is equivalent to the next proposition.

Proposition 23. *For all $q \ge 1$ we have*

(a) $P_q(-1, y, 1) = 0$, *and*
(b) $\frac{\partial P_q}{\partial x}(-1, y, 1) = 0$.

We shall therefore concentrate on proving Proposition 23.

7.2. **Reduction to a combinatorial identity.** For $z \in \mathbb{R}$ and $k \in \mathbb{Z}$, we use the standard notation $(z)_k$ for the rising factorial Pochhammer symbol

$$(z)_k = z(z + 1) \cdot \cdots \cdot (z + k - 1) = \frac{\Gamma(z + k)}{\Gamma(z)}$$

where the second equality holds for z not a non-positive integer. We next reformulate Proposition 23 in terms of the purely hypergeometric terms

$$H_q(l_1,l_2,k) = \frac{(-1)^{l_1+l_2}\left(-\frac{1}{2}\right)_{l_1}\cdot\left(-\frac{1}{2}\right)_{l_2}\cdot\left(\frac{1}{2}\right)_{q-l_1}\cdot\left(\frac{1}{2}\right)_{q-l_2}}{(2q-l_1-l_2-k)!(l_2-l_1+k)!(l_1-l_2+k)!(l_1+l_2-k)!}$$

and

$$H'_q(l_1,l_2,k) = (2q-l_1-l_2-k)H_q(l_1,l_2,k),$$

in order to be able to apply Zeilberger's algorithm in Section 7.3. We note that H_q, H'_q are defined for every $k, l_1, l_2 \in \mathbb{Z}$, by expressing everything in terms of the Gamma function.

Proposition 24. *For all $q \geq 1$ we have*

(a)

$$\sum_{l_1,l_2} H_q(l_1,l_2,k) = \begin{cases} 0, & \text{for } k \geq 2,\\ 2^{-4q}(2q-1)c_q, & \text{for } k=1,\\ 2^{-4q}c_q, & \text{for } k=0.\end{cases}$$

(b)

$$\sum_{l_1,l_2} H'_q(l_1,l_2,k) = \begin{cases} 0, & \text{for } k \geq 2,\\ 2^{-4q}(2q-1)(2q-2)c_q, & \text{for } k=1,\\ 2^{-4q}(2q-1)c_q, & \text{for } k=0.\end{cases}$$

Proof that Proposition 24 is equivalent to Proposition 23. A rearrangement of the terms in (14) yields

$$P_q(-1,y,1) = \sum_{k=0}^{q}(-1)^k d_q(k)\cdot y^{2k} + c_q\left((2q-1)y^2-1\right), \tag{39}$$

where

$$d_q(k) = \sum_{\substack{k\leq l_1+l_2\leq 2q-k\\ |l_1-l_2|\leq k}} a_q(l_1)a_q(l_2)b_q(l_1,l_2,l_1+l_2-k)\cdot(-1)^{l_1+l_2}.$$

Similarly, one obtains

$$\frac{\partial P_q}{\partial x}(-1,y,1) = \sum_{k=0}^{q}(-1)^k d'_q(k)\cdot y^{2k} + c_q(2q-1)\left(1-(2q-2)y^2\right)$$

where

$$d'_q(k) = \sum_{\substack{k\leq l_1+l_2\leq 2q-k\\ |l_1-l_2|\leq k}} (2q-l_1-l_2-k)\cdot a_q(l_1)a_q(l_2)b_q(l_1,l_2,l_1+l_2-k)\cdot(-1)^{l_1+l_2-1}.$$

It is therefore enough to prove that

$$(-1)^{l_1+l_2}a_q(l_1)a_q(l_2)b_q(l_1,l_2,l_1+l_2-k) = 2^{4q}H_q(l_1,l_2,k),$$

which is easily verified by standard algebraic manipulations. □

7.3. **Proof of Proposition 24** (a)**.** We will use the multivariate Zeilberger algorithm for multi-sum recurrences of hypergeometric terms (see [2] and [22, Chapters 6 and 7]). For convenience we write

$$S_q(k) = \sum_{l_1,l_2} H_q(l_1,l_2,k) = 2^{-4q}d_q(k).$$

First, we will handle the case where $k=q$.

Lemma 25. *For all $q \geq 2$ we have $S_q(q) = 0$.*

Proof. We have

$$d_q(q) = \sum_{l=0}^{q} a_q(l)a_q(q-l)b_q(l,q-l,0) = \frac{1}{(q!)^2}\sum_{l=0}^{q}\binom{q}{l}^2 \cdot \frac{(2q-2l)!(2l)!}{(2l-1)(2q-2l-1)}$$
$$= \sum_{l=0}^{q}\binom{2q-2l}{q-l}\frac{1}{2q-2l-1}\cdot\binom{2l}{l}\frac{1}{2l-1}.$$

We write $\phi(x) = \sum_{l=0}^{\infty}\binom{2l}{l}\frac{1}{2l-1}x^l = -\sqrt{1-4x}$. Then $\sum_{q=0}^{\infty} d_q(q)x^q = \phi(x)^2 = 1-4x$, showing that $d_q(q) = 0$ for all $q \geq 2$, whence the claim. □

Next, we prove a recurrence relation for $S_q(k)$.

Lemma 26. *For all $q \geq 1$ and all $k \neq q+2$ we have*

$$\frac{q^2}{8(2k-2q-3)(k-q-2)}S_q(k) + \frac{4kq-4q^2+2k-7q-4}{4(2k-2q-3)(k-q-2)}\cdot S_{q+1}(k) + S_{q+2}(k) = 0.$$

Proof. Let us begin by defining some rational functions in 4 variables. Let

$$\begin{aligned}Q_q^{(1)}(l_1,l_2,k) &= 4q^2(l_1-l_2-k) + 4qk^3 + 8qk^2(4-l_1-l_2)\\ &\quad + 4qk\,(l_1-l_2)^2 + 2qk(4l_2+14l_2-11) + 2q(2l_1l_2+3l_1-9l_2+3)\\ &\quad + 4k(2k+1)(3-2l_1)(2l_2-1) + 12l_1l_2 - 6l_1 - 18l_2 + 9,\\ Q_q^{(2)}(l_1,l_2,k) &= 8l_1^2k - 4l_1^2q + 4l_1l_2q - 12l_1kq + 4l_1q^2 - 4l_2q^2 + 4kq^2 - 8l_1^2 + 4l_1l_2 - 12l_1k\\ &\quad + 10l_1q - 6l_2q + 10kq + 6l_1 - 2l_2 + 4k - 2q - 1\end{aligned}$$

and

$$Q_q(l_1,l_2,k) = 32k(2k-2q-3)(k-q-2)\cdot(2q-l_1-l_2-k+1)_4.$$

Define also

$$R_q^{(1)}(l_1,l_2,k) = \frac{Q_q^{(1)}(l_1,l_2,k)(1/2+q-l_1)(l_1+l_2-k)(l_1-l_2+k)}{Q_q(l_1,l_2,k)}$$

and

$$R_q^{(2)}(l_1,l_2,k) = -\frac{Q_q^{(2)}(,l_1,l_2,k)(1/2+q-l_2)(l_1+l_2-k)(l_2-l_1+k)}{Q_q(l_1,l_2,k)}.$$

Applying Zeilberger's algorithm yields the following identity of rational functions, which can be verified directly by expanding (and should be interpreted in the usual way at the poles):

$$\begin{aligned}&\frac{q^2}{8(2k-2q-3)(-q+k-2)} + \frac{4kq-4q^2+2k-7q-4}{4(2k-2q-3)(-q+k-2)}\cdot\frac{H_{q+1}(l_1,l_2,k)}{H_q(l_1,l_2,k)} + \frac{H_{q+2}(l_1,l_2,k)}{H_q(l_1,l_2,k)}\\ &\qquad = R_q^{(1)}(l_1+1,l_2,k)\cdot\frac{H_q(l_1+1,l_2,k)}{H_q(l_1,l_2,k)} - R_q^{(1)}(l_1,l_2,k)\\ &\qquad\quad + R_q^{(2)}(l_1,l_2+1,k)\cdot\frac{H_q(l_1,l_2+1,k)}{H_q(l_1,l_2,k)} - R_q^{(2)}(l_1,l_2,k).\end{aligned}$$

Therefore, after multiplying both sides by $H_q(l_1,l_2,k)$, one gets

$$\begin{aligned}&\frac{q^2H_q(l_1,l_2,k)}{8(2k-2q-3)(-q+k-2)} + \frac{4kq-4q^2+2k-7q-4}{4(2k-2q-3)(-q+k-2)}H_{q+1}(l_1,l_2,k) + H_{q+2}(l_1,l_2,k)\\ &\qquad = G_q^{(1)}(l_1+1,l_2,k) - G_q^{(1)}(l_1,l_2,k) + G_q^{(2)}(l_1,l_2+1,k) - G_q^{(2)}(l_1,l_2,k),\end{aligned}$$

where $G_q^{(1)}(l_1,l_2,k) = R_q^{(1)}(l_1,l_2,k)\cdot H_q(l_1,l_2,k)$, and $G_q^{(2)}(l_1,l_2,k) = R_q^{(2)}(l_1,l_2,k)\cdot H_q(l_1,l_2,k)$. Tedious but routine manipulations show that $G_q^{(1)}$ and $G_q^{(2)}$ are well-defined at the poles of $R_q^{(1)}$

and $R_q^{(2)}$. We can now sum over all l_1, l_2 on both sides, noting that H_q (and therefore $G_q^{(1)}$ and $G_q^{(2)}$) vanish for $|l_1|$ or $|l_2|$ sufficiently large, and get

$$\frac{q^2}{8(2k-2q-3)(-q+k-2)} S_q(k) + \frac{4kq-4q^2+2k-7q-4}{4(2k-2q-3)(-q+k-2)} \cdot S_{q+1}(k) + S_{q+2}(k) = 0,$$

as claimed. □

Now Proposition 24 easily follows from Lemma 26, by induction.

Proof of Proposition 24 (a). We proceed by induction on q. For the base case note that

$$P_1(x,y,z) = 2(x+z)^2$$

whence, recalling (39) and the relation $S_q(k) = 2^{-4q} d_q(k)$,

$$\left(S_1(0) - \frac{4}{2^4}\right) + \left(\frac{4}{2^4} - S_1(1)\right) y^2 + \sum_{k\geq 2} S_1(k) y^{2k} = \frac{1}{2^4} P_1(-1,y,1) = 0.$$

This implies that

$$S_1(k) = \begin{cases} 0, & \text{for } k \geq 2, \\ \frac{1}{4}, & \text{for } k = 1, \\ \frac{1}{4}, & \text{for } k = 0, \end{cases}$$

which is exactly the case $q = 1$. Similarly, one verifies the formula for $q = 2$.

Using now Lemma 26, it is clear that we have $S_{q+2}(k) = 0$ for all $2 \leq k < q+2$. By Lemma 25, this also holds for $k = q+2$. By definition, $S_{q+2}(k) = 0$ for $k > q+2$. It remains to consider the cases $k = 0, 1$. Assume that

$$\begin{aligned} S_q(0) &= 2^{-4q} c_q, & S_q(1) &= 2^{-4q}(2q-1)c_q \\ S_{q+1}(0) &= 2^{-4(q+1)} c_{q+1}, & S_{q+1}(1) &= 2^{-4(q+1)}(2q+1)c_{q+1}. \end{aligned}$$

Then from Lemma 26 we have

$$\begin{aligned} -S_{q+2}(0) &= \frac{q^2}{8(2q+3)(q+2)} \cdot \frac{1}{2q \cdot \binom{2q}{q}} - \frac{4q^2+7q+4}{4(2q+3)(q+2)} \cdot \frac{1}{(2q+2) \cdot \binom{2q+2}{q+1}} \\ &= -\frac{1}{2(q+2) \cdot \binom{2q+4}{q+2}} \end{aligned}$$

and similarly

$$\begin{aligned} -S_{q+2}(1) &= \frac{q^2}{8(2q+1)(q+1)} \cdot \frac{2q-1}{2q \cdot \binom{2q}{q}} - \frac{4q^2+3q+2}{4(2q+1)(q+1)} \cdot \frac{2q+1}{(2q+2) \cdot \binom{2q+2}{q+1}} \\ &= -\frac{1}{4\binom{2q+2}{q+1}} \end{aligned}$$

as claimed. □

7.4. **Proof of Proposition 24** (b)**.** The development is very similar to that of the previous section, and we shall accordingly give less detail. We define

$$S'_q(k) = \sum_{l_1,l_2} H'_q(l_1,l_2,k)$$

and notice that $S'_q(k) = 2^{-4q} d'_q(k)$. We begin with a recurrence relation, similar to before.

Lemma 27. *For all $q \geq 1$ and all $k \neq 2q-1$, we have*

$$\frac{q(k-2q-1)}{2(2k-2q-1)(k-2q+1)} S'_q(k) + S'_{q+1}(k) = 0.$$

Proof. This time we define

$$\begin{aligned} Q_1'(q,l_1,l_2,k) &= 2l_2k - 4kq + 4l_2q - k + 2l_2 - 2q - 1, \\ Q_2'(q,l_1,l_2,k) &= 2k^2 - 2l_2k - 4l_2q + 3k - 2l_2 + 2q + 1 \end{aligned}$$

and

$$Q'(q,l_1,l_2,k) = 4k(2k-2q-1)(k-2q+1)(2q-l_1-l_2-k)(2q-l_1-l_2-k+1).$$

Define also

$$R_1'(q,l_1,l_2,k) = \frac{Q_1'(q,l_1,l_2,k)(1/2+q-l_1)(l_1+l_2-k)(l_1-l_2+k)}{Q'(q,l_1,l_2,k)}$$

and

$$R_2'(q,l_1,l_2,k) = \frac{Q_2'(q,l_1,l_2,k)(1/2+q-l_2)(l_1+l_2-k)(l_2-l_1+k)}{Q'(q,l_1,l_2,k)}.$$

Applying Zeilberger's algorithm again yields

$$\begin{aligned} \frac{q(k-2q-1)}{2(2k-2q-1)(k-2q+1)} H_q'(l_1,l_2,k) + H_{q+1}'(l_1,l_2,k) &= G_1'(q,l_1+1,l_2,k) - G_1'(q,l_1,l_2,k) \\ &\quad + G_2'(q,l_1,l_2+1,k) - G_2'(q,l_1,l_2,k) \end{aligned}$$

where $G_1'(q,l_1,l_2,k) = R_1'(q,l_1,l_2,k) \cdot H_q'(l_1,l_2,k)$, and $G_2'(q,l_1,l_2,k) = R_2'(q,l_1,l_2,k) \cdot H_q'(l_1,l_2,k)$. We can now sum over all l_1, l_2 on both sides and get the result. □

Proposition 24 (b) now follows by induction, as before.

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