An Asymptotic Formula for the Variance of the number of Zeroes of a Stationary Gaussian Process

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Setup

random $f:\mathbb{R}\to\mathbb{R}$ such that $\forall t_1,\dots,t_N\in\mathbb{R}$

(G) $\quad (f(t_1),\dots,f(t_N)) \sim N(0,\Sigma_{t_1\dots t_N})$

(S) $\quad (f(t_1),\dots,f(t_N)) \overset{d}{\sim} (f(t_1+s),\dots,f(t_N+s)) \ \forall s\in\mathbb{R}$

cov. kernel: $\quad r(t)=\mathbb{E}[f(0)f(t)]=\mathbb{E}[f(s)f(s+t)]$

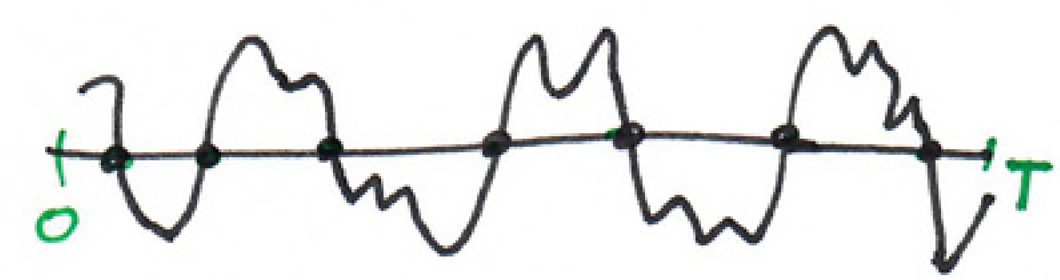
$\boxed{r(0)=1}$ $\quad$ ↑ contin., pos. def.

spectral measure: $\quad \underset{r(0)=\rho(\mathbb{R})<\infty}{\rho}\left(\text{finite}, \rho\geq 0,\ \rho(-I)=\rho(I)\right)$

such that $\quad r(t)=\mathcal{F}[\rho](t)=\int_{\mathbb{R}} e^{-i\lambda t}\,d\rho(\lambda)$

zero-counting:

$N(T)=\#\{x\in[0,T]: f(x)=0\}$



$\mathbb{E}[N(T)]$

Kac-Rice (1940's)

Kac: $N(T) = \int_0^T \delta_0(f(t))\,|f'(t)|\,dt$

$$\Rightarrow \mathbb{E}[N(T)] = \int_0^T \mathbb{E}\left[\,|f'(t)|\;\middle|\; f(t)=0\right]dt$$

For stationary f: (Rice)

$$\boxed{\begin{gathered}\mathbb{E}[N(T)] = \frac{\sigma}{\pi}T \\ \sigma^2 = \int \lambda^2\,d\rho(\lambda) = -r''(0) = \mathbb{E}[f'(t)^2].\end{gathered}}$$

$Var[N(T)]$

- Cramer & Leadbetter (1967), Geman (1972)

  $[G]$ $\int_0^\delta \frac{|r''(t)-r''(0)|}{t}\,dt < \infty \underset{G}{\overset{CL}{\rightleftarrows}} Var[N(T)] < \infty$

- Cuzick (1976)

  $[G] + [r, r'' \in L^2(\mathbb{R})] + \left[\frac{1}{T} Var\, N(T) \xrightarrow{T\to\infty} c > 0\right] \Rightarrow CLT$

$Var[N(T)]$

- <u>Slud (1991, 1994)</u>

(a) $r, r'' \in \mathcal{L}^2(\mathbb{R}) \Longrightarrow \frac{1}{T} Var\, N(T) \xrightarrow{T\to\infty} C > 0$.

(b) $r(t) \sim \frac{L(t)}{t^\alpha}$ where $\begin{cases} \frac{L(th)}{L(t)} \xrightarrow{t\to\infty} 1 \quad \forall h \\ 0 < \alpha < \frac{1}{2} \end{cases}$

$\Downarrow$

$$Var\, N(T) \sim C_\alpha L(T)^2\, T^{2-2\alpha}.$$

- <u>Kratz & Léon</u> (2006, 2010) Wiener Chaos decomp.

$$N(T) = \frac{\sigma}{\pi} \sum_{q=0}^{\infty} \frac{(-1)^{q+1}}{2^q} N_q(T)$$

$$N_q(T) = \sum_{\ell=0}^{q} \frac{1}{\ell!\,(q-\ell)!\,(2\ell-1)} \int_0^T H_{2(q-\ell)}(f(t))\, H_{2\ell}\left(\frac{f'_t}{\sigma}\right) dt$$

Examples:

- $r(t) = \frac{\sin(t)}{t}$, $r(t) = e^{-t^2}$

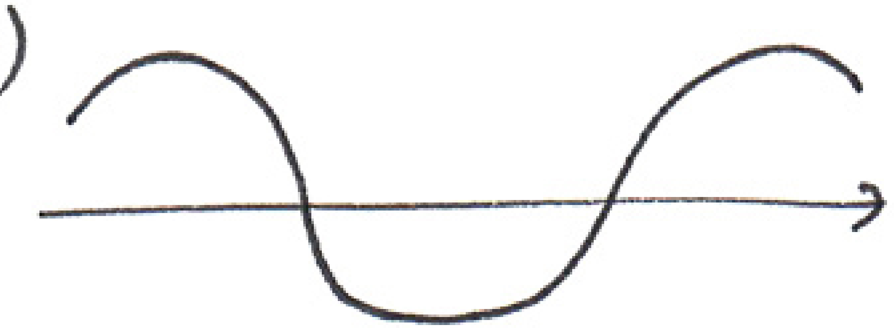
  $r, r'' \in L^2(\mathbb{R}) \Longrightarrow \exists \lim_{T\to\infty} \frac{\mathrm{Var}\, N(T)}{T}$ + CLT.

- $r(t) = e^{-|t|} \Rightarrow \mathrm{var}\, N(T) = \infty$

- $r(t) = \cos(\sigma t) \longleftrightarrow \rho = \frac{1}{2}(\delta_\sigma + \delta_{-\sigma}) = \delta_\sigma^*$

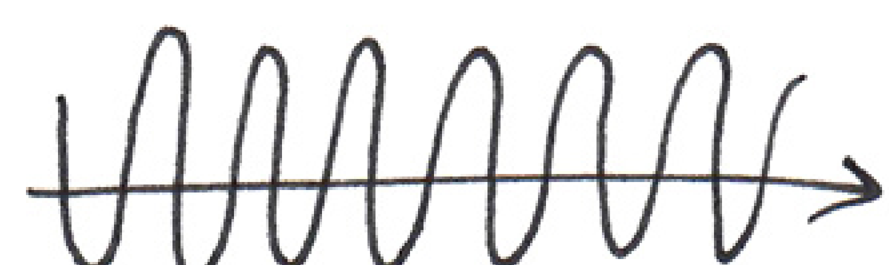
  $\mathrm{Var}\, N(T) = \{\frac{\sigma}{\pi} T\}(1 - \{\frac{\sigma}{\pi} T\}) = O(1)$

  "degenerate"



- $\rho = \sum_j c_j \delta_{\sigma_j}^*$

  $\mathrm{Var}\, N(T) \simeq T^2$



Results:

LB **Thm 1:** $\mathrm{Var}\, N(T) \geq \frac{\sigma^2}{\pi^2} T \int_0^T \left(1-\frac{t}{T}\right)\left(r(t)+\frac{r''(t)}{\sigma^2}\right)^2 dt$

Cor: $\boxed{\mathrm{Var}\, N(T) \geq C \cdot T}$ if non-degenerate

UB **Thm 2:** $\boxed{\mathrm{Var}\, N(T) \asymp T \int_0^T \left(1-\frac{t}{T}\right)\left(r(t)+\frac{r''(t)}{\sigma^2}\right)^2 dt}$

if $\limsup_{|t|\to\infty} \left\{ |r(t)| + \frac{|r'(t)|}{\sigma}, \frac{|r'(t)|}{\sigma} + \frac{|r''(t)|}{\sigma^2} \right\} < 1$. [M]

Cor A: $\boxed{\mathrm{Var}\, N(T) \asymp T \iff r + \frac{r''}{\sigma^2} \in L^2(\mathbb{R})}$, under [M].

Cor*B: If the lim sup in [M] $= 0$, and $r + \frac{r''}{\sigma^2} \notin L^2(\mathbb{R})$,

then $\mathrm{Var}\, N(T) \underset{\text{ratio}\to 1}{\sim} \frac{\sigma^2}{\pi^2} T \int (\;--\;)\, dt$.

rmk: Thm 1 + cor also by R. Lachièze-Rey [2020+]

atoms **Thm 3:** $\boxed{\mathrm{Var}\, N(T) \asymp T^2 \iff \rho = \delta^*_\lambda + \dots \quad \lambda \neq \sigma}$

Thm 4: $\rho_\theta = (1-\theta)\rho + \theta\, \delta^*_\sigma, \quad [M]$

$$\Longrightarrow \boxed{\mathrm{Var}\, N_{\rho_\theta}(T) = \mathrm{Var}\, N_\rho(T)} \quad \forall \theta < \theta_0 .$$

(note that σ is such that $\mathbb{E} N_{\rho_\theta}(T) = \mathbb{E} N_\rho(T)$).

Rmks: • "special freq." does not exist for other levels, nor for GSP $f: \mathbb{C} \to \mathbb{C}$, $f: \mathbb{R} \to \mathbb{C}$.

• the rôle of σ is clear from Thm 2.

Proof Ideas

Kac: $N(T) = \sigma \int_0^T \delta_0(f_t) \left| \frac{f_t'}{\sigma} \right| dt$

Kratz-Léon: "Expand" $\delta_0(x)$ and $|x|$ to get:

$$N(T) = \frac{\sigma}{\pi} \sum_{q=0}^{\infty} \frac{(-1)^{q+1}}{2^q} N_q(T)$$

Hermite

$$N_q(T) = \sum_{l=0}^{q} \frac{1}{l!(q-l)!(2l-1)} \int_0^T H_{2q-2l}(f_t)\, H_{2l}\left(\frac{f_t'}{\sigma}\right) dt$$

Orthogonality: $\operatorname{Var} N(T) = \frac{\sigma^2}{\pi^2} \sum_{q=1}^{\infty} \frac{\mathbb{E}[N_q(T)^2]}{4^q}$

__stationarity:__ $\mathbb{E}\left[N_q(T)^2\right] = \int_{-T}^{T} (T-|t|)\, P_q\!\left(r_t, \frac{r'_t}{\sigma}, \frac{r''_t}{\sigma^2}\right) dt$

$P_q(x,y,z)$ is "explicit", but involves a double sum with hypergeometric coefs.

Thm 1 __Lower bd:__ $q=1$.

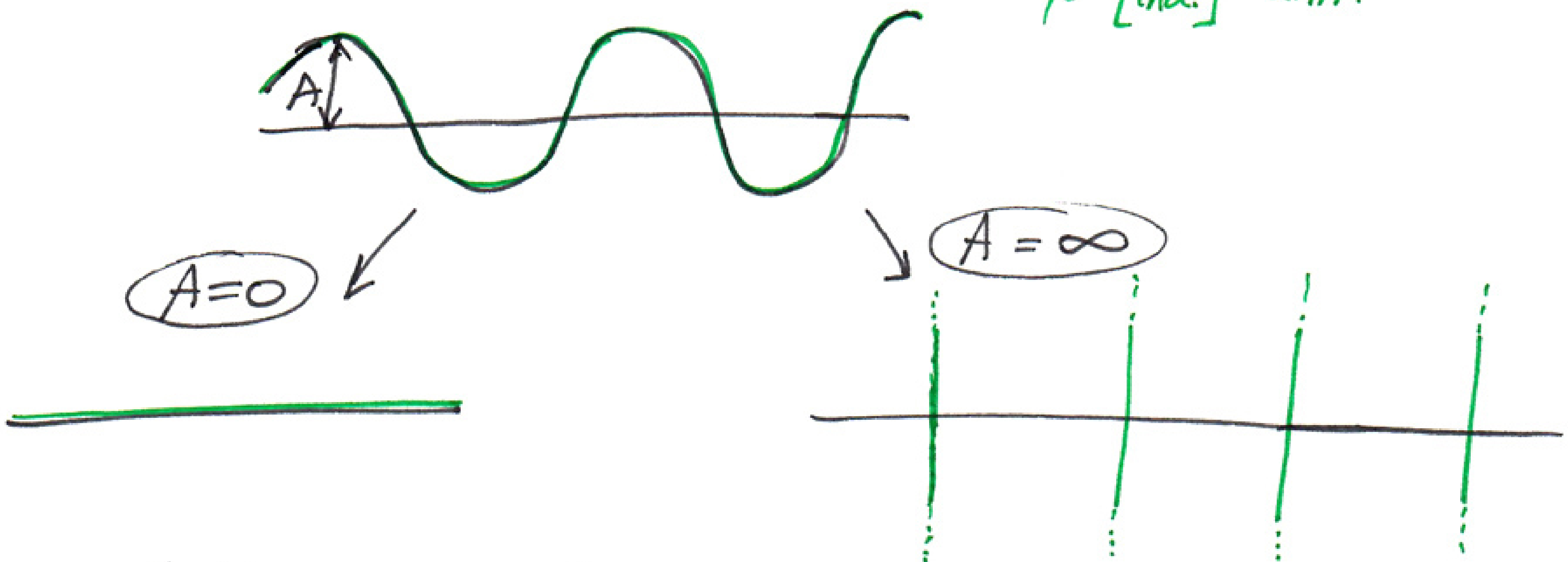
Thm 2 __Upper bd:__

- prove that $\overbrace{(x+z)^2}^{\left(r_t + \frac{r''_t}{\sigma^2}\right)^2} \mid P_q(x,y,z)$.
- bound the quotient.

Prove a recurrence relation using Zeilberger's method for multisum of hypergeometric terms.

<u>special atom</u>: $\delta_\sigma^* \rightsquigarrow \Psi(t) = \mathbf{A}\cos(\sigma t + \alpha)$

χ [ind.] unif.



<u>condition</u> on Ψ.

$$\mathbb{E}\,\#\{\underset{[0,T]}{t}: f(t) = \Psi(t)\} = \frac{\sigma}{\pi} T \quad \text{(for any instance of } \Psi\text{!)}$$

$$\Rightarrow \mathrm{Var}[N_{\underbrace{\rho_0 + \delta_\sigma^*}_{\rightsquigarrow f + \Psi}}(T)] = \mathbb{E}\,\mathrm{Var}(N_{f+\Psi} \mid \Psi) + \underbrace{\mathrm{Var}\,\underbrace{\mathbb{E}(N_{f+\Psi} \mid \Psi)}_{\text{const}}}_{0}$$

Thm 3a <u>Ergodicity</u>: if ρ_0 has no atoms,

$$\mathbb{E}\,\mathrm{Var}(N_{f+\Psi} \mid \Psi) \ll T^2.$$

<u>Ergodicity</u>: if ρ_0 has no atoms,

$$\mathbb{E}\, \mathrm{Var}(N_{f+\psi}^{(T)} \mid \psi) \ll T^2$$

<u>why?</u>

$$\frac{(N_{f+\psi}^{(T)} \mid \psi)}{T} \xrightarrow{L^1 \cap L^2} \mathbb{E}(N_{f+\psi}^{(1)} \mid \psi)$$

[Ergodic Thm]

$$\Rightarrow \frac{1}{T^2} \mathrm{Var}(N_{f+\psi}^{(T)} \mid \psi) \xrightarrow{a.s.} 0$$

+dominance

$$\Rightarrow \mathbb{E}\left(\frac{1}{T^2} \mathrm{Var}(\dots)\right) \longrightarrow 0$$

Thm 3a

Harmonic Analysis:

$$r + \frac{r''}{\sigma^2} = \mathcal{F}\Big[\overbrace{\left(1-\frac{\lambda^2}{\sigma^2}\right) d\rho(\lambda)}^{\mu \text{ (signed meas.)}}\Big]$$

$$\Rightarrow \operatorname{Var} N(T) \simeq T \int_{-T}^{T} \left(1-\frac{|t|}{T}\right)\left(r_t + \frac{r''_t}{\sigma^2}\right)^2 dt$$

$$\overset{\mathcal{F}}{=} T^2 \int_{\mathbb{R}} \left(S_T * \mu\right) d\mu \qquad S_T(\lambda) = \operatorname{sinc}^2\left(\frac{T\lambda}{2}\right)$$

$$= T^2 \underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}} S_T(\lambda_1-\lambda_2)\, d\mu(\lambda_1)\, d\mu(\lambda_2)}_{I_\mu(T)}$$

Thm 3b **Lemma:** $\exists c\ \forall T > T_0:$ $I_\mu(T) \geq c > 0 \iff \mu$ contains an atom

(otherwise, $I_\mu(T) = o(1)$ as $T \to \infty$)

Thm 4: apply Thm 2 to $\rho_\theta = (1-\theta)\rho + \theta\, \delta_\sigma^*$.

Merci Beaucoup

pour votre attention !