Limits on the Power of Indistinguishability Obfuscation

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Limits on the Power of iO

- Limits on the Power of Indistinguishability Obfuscation (and Functional Encryption)
  - **FOCS 2015**

- On Constructing One-Way Permutations from Indistinguishability Obfuscation
  - **TCC 2016A**

Obfuscation

- Makes a program "unintelligible" while preserving its functionality

```
for (i=0; i < M.length; i++) {
// Adjust position of clock hands
   var ML=(ns)?document.layers['nsMinutes'+i]:ieMinutes[i].style;
   ML.top=y[i]+HandY+(i*HandHeight)*Math.sin(min)+scrll;
   ML.left=x[i]+HandX+(i*HandWidth)*Math.cos(min);
}
```

```
for(O79=0;O79<l6x.length;O79++){var O63=(l70)?document.layers
["nsM\151\156u\164\145s"+O79]:ieMinutes[O79].style;
O63.top=l61[O79]+O76+(O79*O75)*Math.sin(O51)+l73;
O63.left=l75[O79]+l77+(O79*l76)*Math.cos(O51);}
```

Obfuscation

- [BarakGoldreichImpagliazzoRudichSahaiVadhanYang01] :
  - **Virtual black-box obfuscation (VBB)**
    Obfuscated program reveals no more than a black box implementing the program
    Impossible
  - **Indistinguishability obfuscation (iO)**
    Obfuscations of any two functionally-equivalent programs be computationally indistinguishable
    May be possible?
- [GargGentryHaleviRaykovaSahaiWaters12] :
  A candidate **indistinguishability obfuscator** (iO)

Indistinguishability Obfuscation

- An efficient algorithm $i\mathcal{O}$
  Receives a circuit C, outputs an obfuscated circuit Ĉ
  - **Preserves functionality**: $C(x)=\hat{C}(x)$ for all x
  - **Indistinguishability**: For every PPT distinguisher D, for every pair of functionally-equivalent circuits C_1 and C_2

$$\left| \Pr\left[D(\, i\mathcal{O}(C_1)\,)=1 \right] - \Pr\left[D(\, i\mathcal{O}(C_2)\,)=1 \right] \right| < \mathrm{negl}(n)$$

- What can be constructed using iO?

The Power of Indistinguishability Obfuscation

- Public-key encryption, short "hash-and-sign" signatures, CCA-secure public-key encryption, non-interactive zero-knowledge proofs, Injective trapdoor functions, oblivious transfer [SW14]
- Deniable encryption scheme [SW14]
- One-way functions [KMN+14]
- Trapdoor permutations [BPW15]
- Multiparty key exchange [BZ14]
- Efficient traitor tracing [BZ14]
- Full-domain hash without random oracles [HSW14]
- Multi-input functional encryption [GGG+14, AJ15]
- Functional encryption for randomized functionalities [GJK+15]
- Adaptively-secure multiparty computation [GGH+14a, CGP15, DKR15, GP15]
- Communication-efficient secure computation [HW15]
- Adaptively-secure functional encryption [Wat14]
- Polynomially-many hardcore bits for any one-way function [BST14]
- ZAPs and non-interactive witness-indistinguishable proofs [BP15]
- Constant-round zero-knowledge proofs [CLP14]
- Fully-homomorphic encryption [CLT+15]
- Cryptographic hardness for the complexity class PPAD [BPR14]

(Last update: April 2015)

The Power of Indistinguishability Obfuscation

Is there a natural task that **cannot** be solved using indistinguishability obfuscation?

Yes

(probably…)

Black-Box Separations

- The main technique for proving lower bound in cryptography [IR89]:
  Black Box Separations

- The vast majority of constructions in cryptography are “black box”

  “Building a primitive X from
  ***any implementation** of a primitive Y”*

  - The construction and security proof rely only on the input-output behavior of **Y** and of **X**'s adversary
  - The construction ignores the internal structure of **Y**

- **Examples**:
  - PRF from PRG [GGM86], PRG from OWFs [HILL93]

Black-Box Separations

- Impossibility of black-box constructions
- Typically, show impossibility of "X ⇒Y" by:

"There exists an oracle relative to which Y exists but X does not exist"

- **Examples**:
  - No key agreement from OWFs [IR89]
  - No CRHF from OWFs [Sim98]

Our Challenge: Non-Black-Box Constructions

- **Constructions that are based on *iO*, almost always have some *non-black-box* ingredient**
- **Typical example**
  From private-key to public-key encryption [SW14] **(simplified)**
  - Private-key scheme: $Enc(K,m) = (r, \mathrm{PRF}(K,r) \oplus m)$
  - Public-key scheme: $SK = K,\ PK = iO(Enc(K,\cdot))$

Non-black-box ingredient:

Need the specific evaluation circuit of the PRF

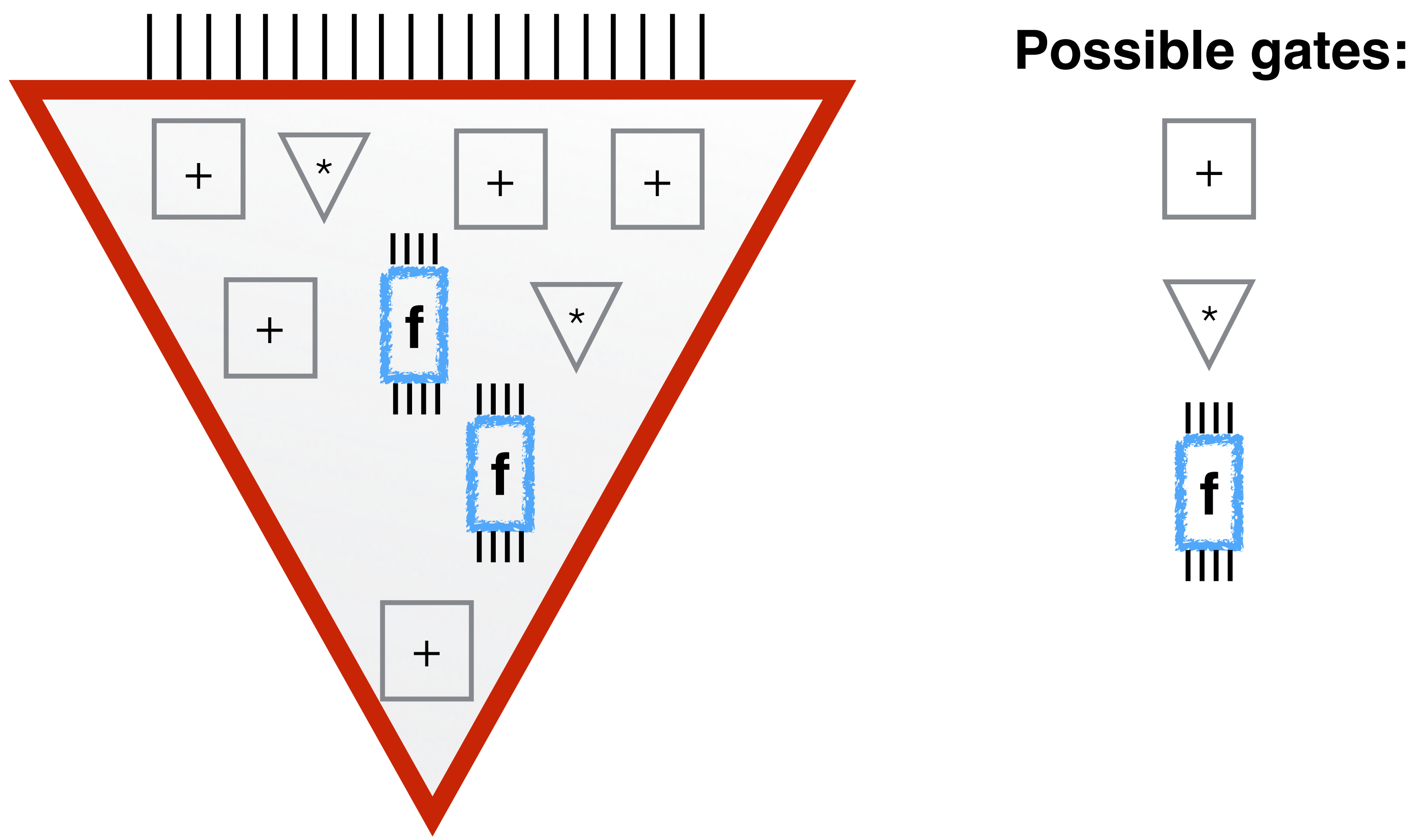
How can one reason about such non-black-box techniques?

Our Solution

- Overcome this challenge by considering $i\mathcal{O}$ for a richer class of circuits:

oracle-aided circuits

(circuits with oracle gates)



Our Solution

- Transform **almost all** iO-based constructions from non-black-box to black-box

$$iO(r, \mathrm{PRF}(K, r) \oplus m))$$

$$\Downarrow$$

$$iO(r, C^{OWF}(K, r) \oplus m)$$

(possible due to [GGM86]+[HILL89])

- Constructing iO for **oracle-aided** circuits
  is clearly **as hard as than**
  constructing iO for standard circuits

- Limits on the power of iO for **oracle-aided** circuits
  thus **imply**
  limits on the power of iO for standard circuits

Techniques We Don't Capture

- Constructions that use NIZK proofs for languages that are defined relative to a computational primitive
- **NIZK proof** $L = \{(d,r) \mid \exists r \text{ s.t. } d = Enc(i;r)\}$
  - Uses Cook-Levin reduction to SAT
  - This reduction uses the circuit for deciding L (representing its computation state as boolean formula) - ***non-black-box***
- [BKSY11] seems as a promising approach for extending our framework to capture such constructions
- Other (less common) techniques (so far not used with iO)

On Constructing One-Way Permutations from Indistinguishability Obfuscation

One-Way Permutation

- One of the most fundamental primitives in cryptography
- Enabling elegant constructions of a wide variety of cryptographic primitives
  - Universal one-way hash function
  - Pseudorandom generators

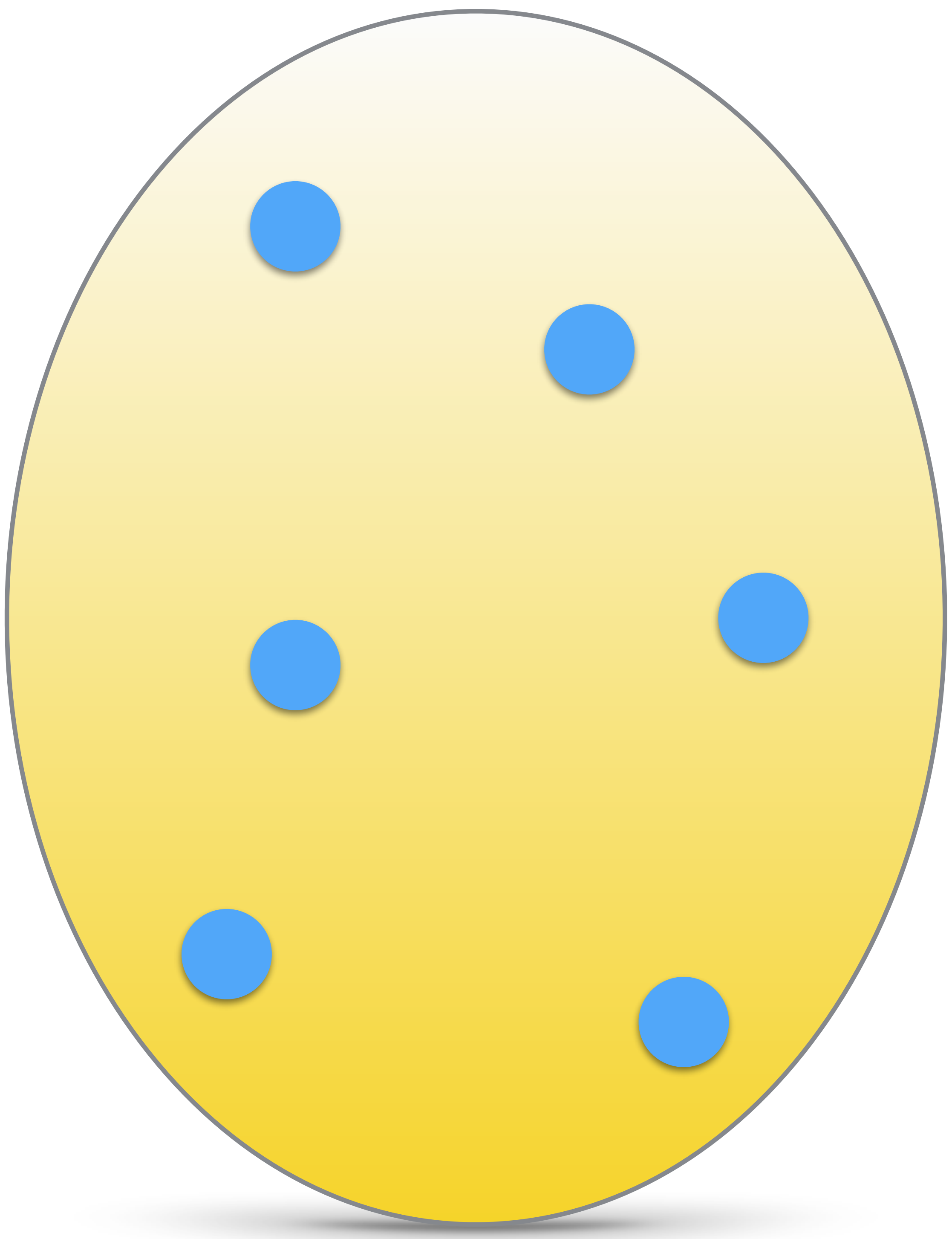
One-Way Permutation

- **One-Way Functions:** Many candidates
- **One-Way Permutations:** Only few candidates
  - Based on hardness of problems related to discrete logarithms and factoring

- [Rudich88,…]:
  No black-box construction of a one-way permutation from a one-way function

TDP from iO+OWF

[BitanskyPanethWichs15]



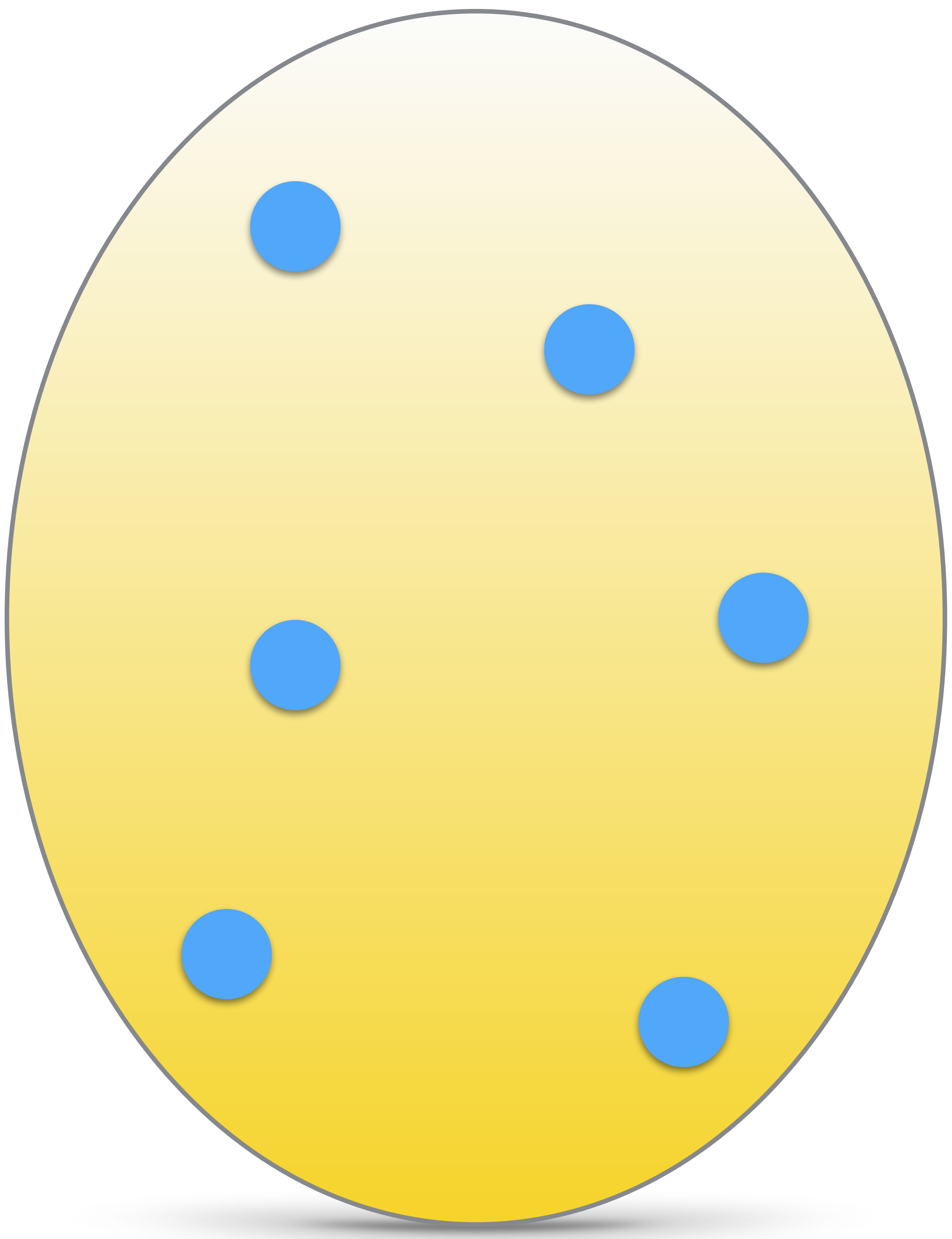
$(i, \mathrm{PRF}_K(i))$ $(i+1, \mathrm{PRF}_K(i+1))$

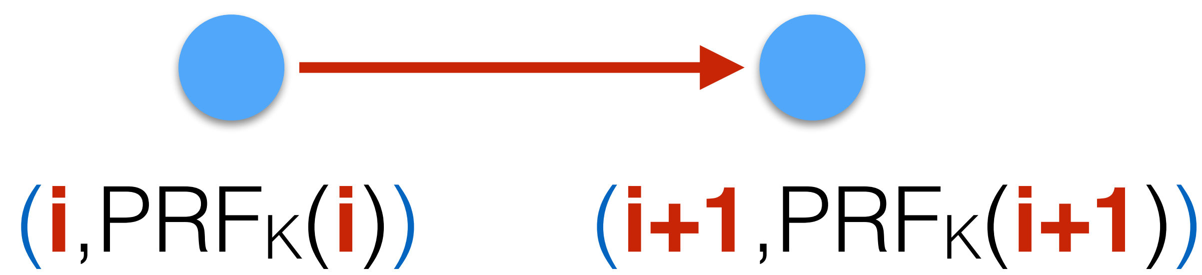
Elements:

 $(i, \mathrm{PRF}_K(i))$

TDP from iO+OWF

[BitanskyPanethWichs15]





Next(x):

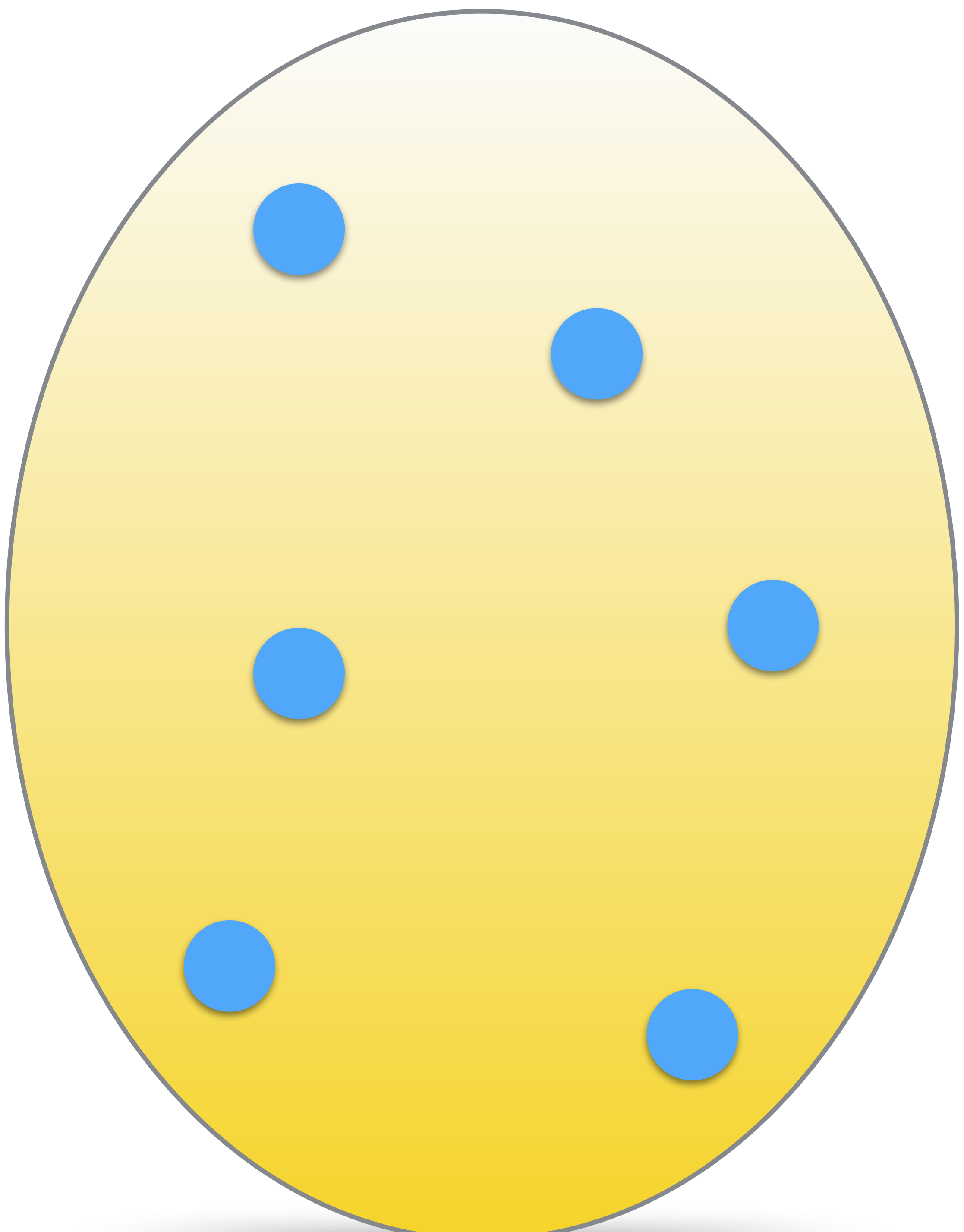
If $x=(i, PRF_K(i))$

Output $(i+1, PRF_K(i+1))$

Output $\perp$

TDP from iO+OWF

[BitanskyPanethWichs15]



$(\mathbf{i}, \mathrm{PRF}_K(\mathbf{i}))$ $(\mathbf{i+1}, \mathrm{PRF}_K(\mathbf{i+1}))$

The obfuscated program:
The Index of the permutation

Next(x):

If $X = (\mathbf{i}, \mathrm{PRF}_K(\mathbf{i}))$

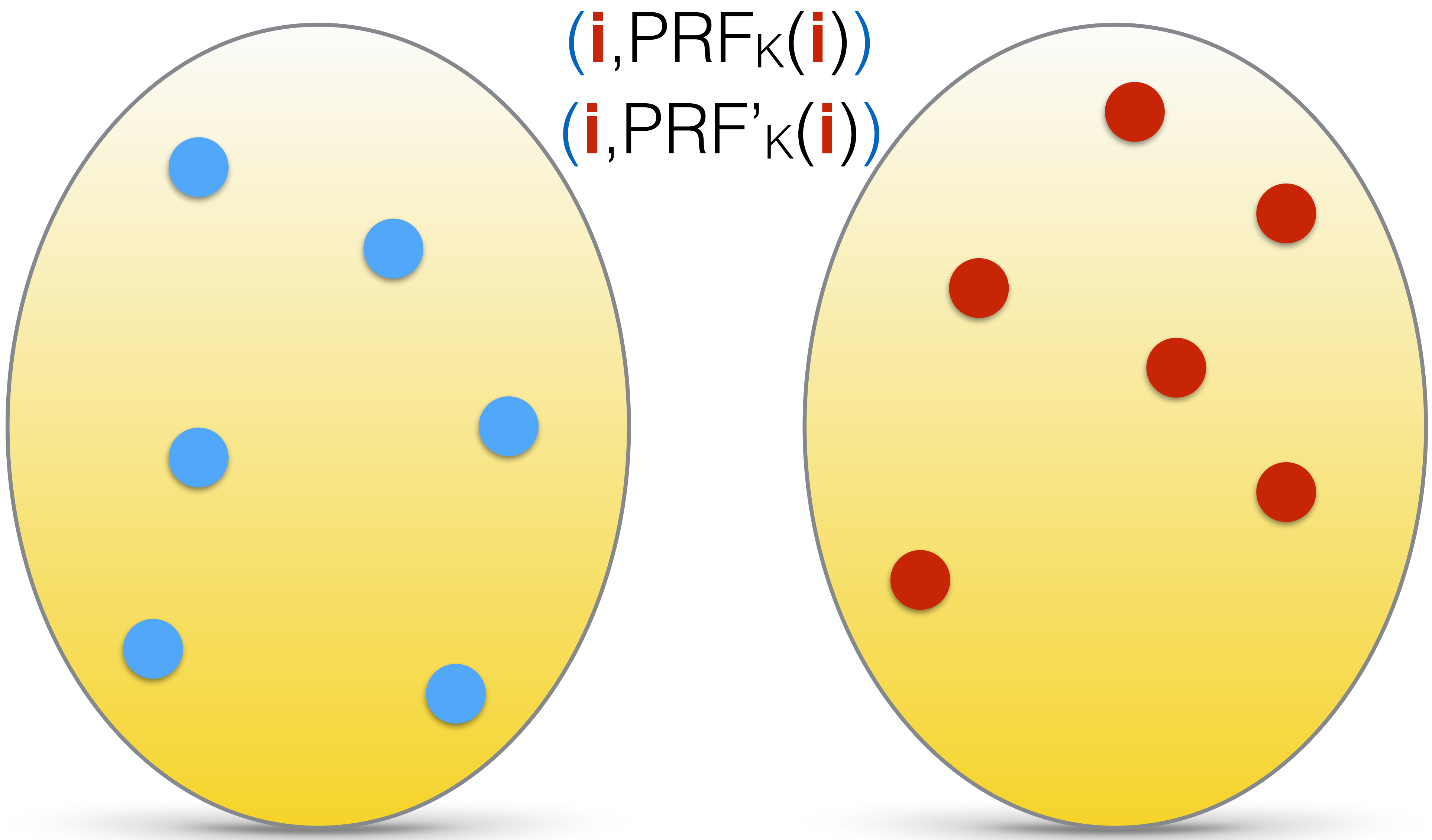
Output $(\mathbf{i+1}, \mathrm{PRF}_K(\mathbf{i+1}))$

Output $\perp$

Question 1:

Can we construct a ***single*** one-way permutation over $\mathbf{\{0,1\}^n}$ from iO+OWF?

The [BPW15] Domain



The domain depends on the specific PRF

For the same K, different underlying PRF - different domain!

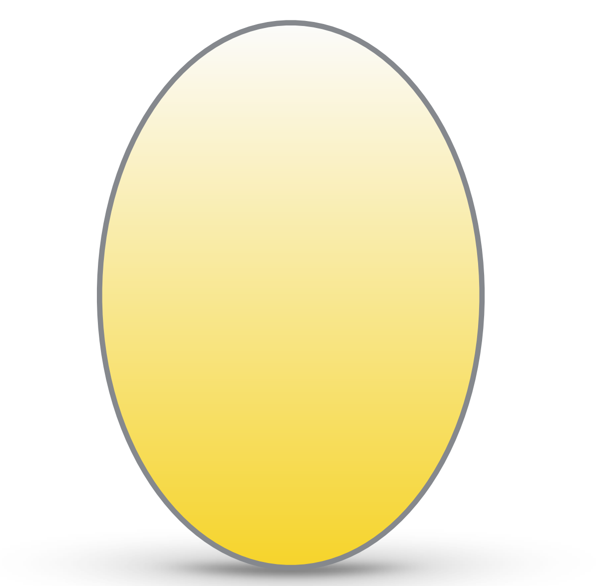
Question 2:

Can we construct a **family** where the domain **does not depend** on the underlying building blocks (iO+OWF)?

We call a construction where the domain does not depend on the underlying building blocks as **"domain invariant"**

Back to [Rudich88,...]

- Separation of OWP from OWF
- Rules out only a ***single*** ***domain-invariant*** permutation
  - Rudich assumes that the domain is independent of the OWF



Question 3:

Can we construct a
non-domain-invariant
OWP (family) from a OWF?

Our Results

Can we construct a ***single*** one-way permutation over **$\{0,1\}^n$** from iO+OWF?

Using the known techniques

NO.

Can we construct a **family** where the domain **does not depend** on the underlying building blocks (iO+OWF)?

NO.

Can we construct a **non-domain-invariant** OWP (family) from a OWF?

NO.

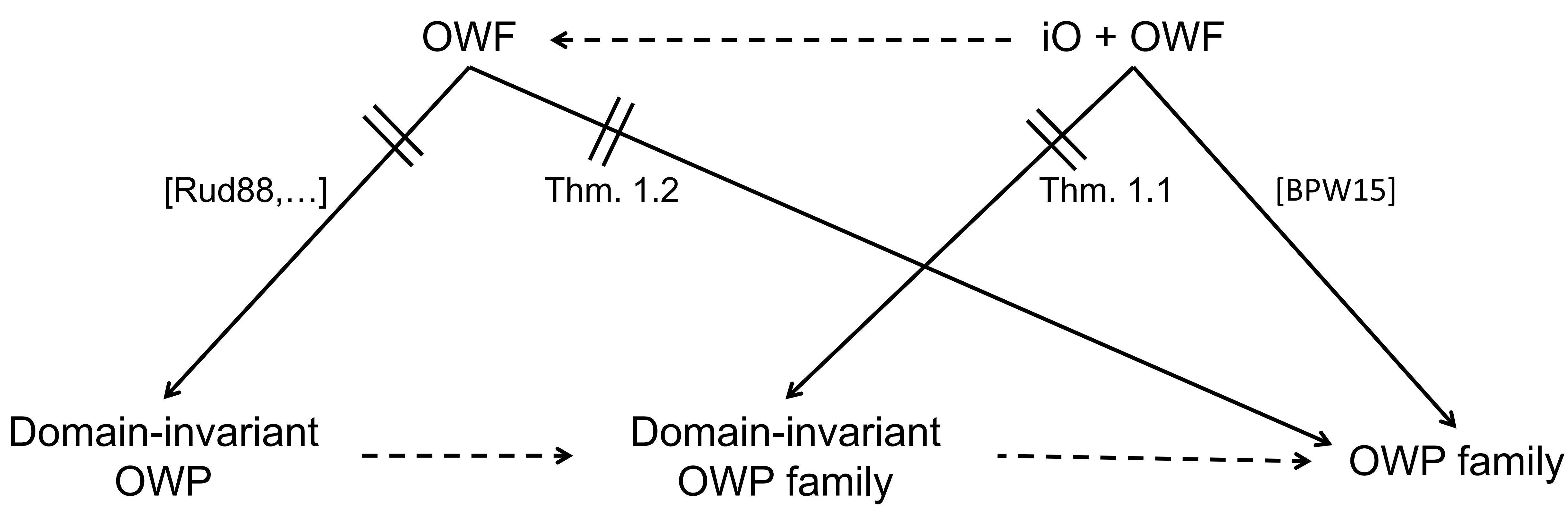
iO+OWF $\not\Rightarrow$ DI-OWPs

- **Theorem 1:**
  There is no fully black-box construction of
  a domain-invariant one-way permutation family
  from
  - a one-way function **f** and
  - an **indistinguishability obfuscator** for all oracle-aided circuits $\mathbf{C^f}$
- Unless with an **exponential** security loss
  (rules out sub-exponential hardness as well!)

OWF $\not\Rightarrow$ DNI-OWPs

- **Theorem 2:**
  There is no fully black-box construction of
  a non-domain-invariant one-way permutation family from
  - a one-way function **f**
- Unless with an **exponential** security loss
  (rules out sub-exponential hardness as well!)

So.. What do we have?



Proof Sketch

- Builds upon and generalizes
  [Rudich88, MatsudaMatsuura11, AsharovSegev15]

- We define an oracle **Γ** such that relative to it:

  1. There exists a **one-way function f**
  2. There exists an **indistinguishability obfuscator** for all oracle-aided circuits $\mathbf{C}^{\mathbf{f}}$
  3. There does not exist a **domain-invariant one-way permutation family**

The Oracle ℾ

The one-way function f

$f = \{f_n\}_n$, where each $f_n : \{0,1\}^n \to \{0,1\}^n$ is a uniformly chosen function

O and Eval

$O = \{O_n\}_{n\in\mathbb{N}}$, where each O_n is a uniformly chosen ***injective*** function $\{0,1\}^{2n} \to \{0,1\}^{10n}$

$Eval(\tilde{C}, a)$ with $|\tilde{C}| = 10n,\ |a| = n$

Looks for the pair $(C, r) \in \{0,1\}^{2n}$ such that $O_n(C,r) = \tilde{C}$

If exists, returns $C^f(a)$

Otherwise, returns $\perp$

- **We implement iO as follows:** $\hat{C}(\cdot) = iO(C)$
  - On input oracle-aided circuit **C** (with |C|=n), choose a random **r**
  - Outputs $\tilde{C} = O_n(C, r)$

We Need to Show

- We define an oracle Γ such that relative to it:

1. There exists a **one-way function f**

   (somewhat similar to [AS15])

2. There exists an **indistinguishability obfuscator** for all oracle-aided circuits $\mathbf{C}^{\mathbf{f}}$

   (somewhat similar to [AS15])

3. There does not exist a **domain-invariant one-way permutation family**

Warm-up: Rudich's Attack in the Random-Oracle Model

f Random oracle

$P^{\mathbf{f}}$ One-Way Permutation over domain $\mathcal{D}$ for every function **f**

Theorem:

There exists an oracle-aided adversary $\mathcal{A}$ that makes polynomially many queries, such that for every $\mathbf{f},\mathbf{x^*}$

$$\Pr[\mathcal{A}^{\mathbf{f}}(\mathbf{y^*}) = \mathbf{x^*}] = 1$$

where $\mathbf{y^*} = P^{\mathbf{f}}(\mathbf{x^*})$

The Adversary

- **Input:** some element $\mathbf{y^*} \in \mathcal{D}$
- **Oracle access:** the random oracle $\mathbf{f}$
  - Initializes a set of queries $\mathbf{Q}$
    (initially empty. always consistent with $\mathbf{f}$)
  - Repeats the following for polynomially many times:
    - **Simulation:** $\mathcal{A}$ finds an input $\mathbf{x'} \in \mathcal{D}$ and a set of oracle/queries $\mathbf{f'}$ that is consistent with $\mathbf{Q}$, such that $P^{\mathbf{f'}}(\mathbf{x'})=\mathbf{y^*}$
    - **Evaluation:** $\mathcal{A}$ evaluates $P^{\mathbf{f}}(\mathbf{x'})$. If $\mathbf{y^*}$ - found!
    - **Update**: $\mathcal{A}$ asks $\mathbf{f}$ for all queries in $\mathbf{f'}$ that are not in $\mathbf{Q}$, and update $\mathbf{Q}$

The Claim

- **Input:** some element $\mathbf{y^*} \in \mathcal{D}$
- **Oracle access:** **f**
  - Initializes a set of queries **Q**
    (initially empty. always consistent with f)
  - Repeats the following for polynomially many times:
    - **Simulation:** $\mathcal{A}$ finds an input $\mathbf{x'} \in \mathcal{D}$ and a set of oracle/queries **f'** that is consistent with **Q**, such that $P^{f'}(\mathbf{x'})=\mathbf{y^*}$
    - **Evaluation:** $\mathcal{A}$ evaluates $P^{f}(\mathbf{x'})$. If $\mathbf{y^*}$ - found!
    - **Update**: $\mathcal{A}$ asks **f** for all queries in **f'** that are not in **Q**, and update **Q**

- In every iteration, one of the following:
  - $\mathcal{A}$ finds $\mathbf{x^*}$, (i.e., $\mathbf{x'}=\mathbf{x^*}$ where $P^{\mathbf{f}}(\mathbf{x^*})=\mathbf{y^*}$) or
  - In the update phase, $\mathcal{A}$ queries **f** with at least one query that is made in the computation of $P^{\mathbf{f}}(\mathbf{x^*})=\mathbf{y^*}$

Otherwise

- In every iteration, one of the following:
  - $\mathcal{A}$ finds $\mathbf{x^*}$, or
  - **In the update phase, $\mathcal{A}$ queries f with at least one query that is made in the computation of $P^f(x^*)=y^*$**

$P^{f'}(x')=y^*$ **Q** $P^f(x^*)=y^*$

$$\Downarrow$$

Define f''

a in **Q**:	$f''(a) := f(a)$
a appears in $P^{f'}(x')$:	$f''(a) := f'(a)$
a appears in $P^f(x^*)$:	$f''(a) := f(a)$

$P^{f''}(x')=y^*$ $P^{f''}(x^*)=y^*$

Otherwise

- In every iteration, one of the following:
  - **$\mathcal{A}$ finds x*, or**
  - In the update phase, $\mathcal{A}$ queries **f** with at least one query that is made in the computation of $P^{f}(x^*)=y^*$

$P^{f'}(x')=y^*$ **Q** $P^{f}(x^*)=y^*$

Define f”

a in **Q**:	$f''(a) := f(a)$
a appears in $P^{f'}(x')$:	$f''(a) := f'(a)$
a appears in $P^{f}(x^*)$:	$f''(a) := f(a)$

$P^{f''}(x')=y^*$ $P^{f''}(x^*)=y^*$

$x' \neq x^*$

In Our Setting

- **Challenges:**
  - Family and not just a single permutation
  - Our oracle Γ is much more structured than just a random oracle

- **Γ consists of:**
  - Length preserving function **f**
  - I*njective* length-increasing function **O**
  - "Evaluation" oracle **Eval**

Recall [BPW15]:

Relative to Γ **there exists** a construction of

a non-domain invariant one-way permutation family!!

Regarding **O**

- **Γ consists of:**
  - length preserving function **f**
  - ***injective*** length-increasing function **O**
  - “evaluation” oracle **Eval**

Q

$P^{\Gamma'}(\mathbf{x'})=\mathbf{y^*}$ $P^{\Gamma}(\mathbf{x^*})=\mathbf{y^*}$

$\mathbf{O'}(\alpha)=\beta$ $\mathbf{O}(\delta)=\beta$

$\boxed{\mathbf{O''}(\alpha)=\beta \qquad \mathbf{O''}(\delta)=\beta}$

Non-injective!

Regarding **O** and **Eval**

- **Γ consists of:**
  - length preserving function **f**
  - ***injective*** length-increasing function **O**
  - "evaluation" oracle **Eval**

Q

$P^{\Gamma'}(\mathbf{x'}) = \mathbf{y^*}$ $\quad$ $P^{\Gamma}(\mathbf{x^*}) = \mathbf{y^*}$

$\mathbf{O'}(C,r) = \hat{\mathbf{C}}$ $\quad$ $\mathbf{Eval}(\hat{\mathbf{C}}, d) = \perp$

$$\boxed{\mathbf{O''}(C,r) = \hat{\mathbf{C}} \qquad \mathbf{Eval''}(\hat{\mathbf{C}}, d) = C^f(d)}$$

incorrect!

The Proof

- Very subtle
- Carefully define the dependencies between oracles in order to avoid the above scenarios
- Regarding **O**: choose the oracle **O’** uniformly at random from the set of all oracles that are consistent with **Q**
  - We show that with high probability
    - **O’** avoids the image of **O**
    - **O’** avoids the invalid **Eval** calls
    - It is possible to construct the hybrid oracle **ℾ”**
    - Relies on the fact that **O** is length-increasing

Further details: see the paper

OWF $\not\Rightarrow$ DNI-OWPs

- **Theorem:**
  There is no fully black-box construction of
  a non-domain-invariant one-way permutation family from
  - a one-way function **f**
- Unless with an **exponential** security loss
  (rules out sub-exponential hardness as well!)

Non-Domain-Invariant Family

$\alpha \leftarrow \mathbf{Gen}^f(1^n)$ $x \leftarrow \mathbf{Samp}^f(\alpha)$ $y \leftarrow \mathbf{P}^f(\alpha, x)$

Different f:
completely different set of indices
(different family)

The domain D_α^f:
depends both on α, f

Careful!
$\boldsymbol{\alpha}$ may be invalid w.r.t **f**
x may not be in $\mathbf{D}_\alpha^f$

Example [BPW15]

A non-domain-invariant family (uses both OWF and iO):
The index depends on iO+OWF
The domain depends on OWF only (and not on the index)

Challenges:
Constructing the Hybrid Oracle

$P^{f'}(a, x') = y^*$ **Q** $P^{f}(a, x^*) = y^*$

Define f''

a in **Q**:	$f''(a) := f(a)$
a appears in $P^{f'}(a, x')$:	$f''(a) := f'(a)$
a appears in $P^{f}(a, x^*)$:	$f''(a) := f(a)$

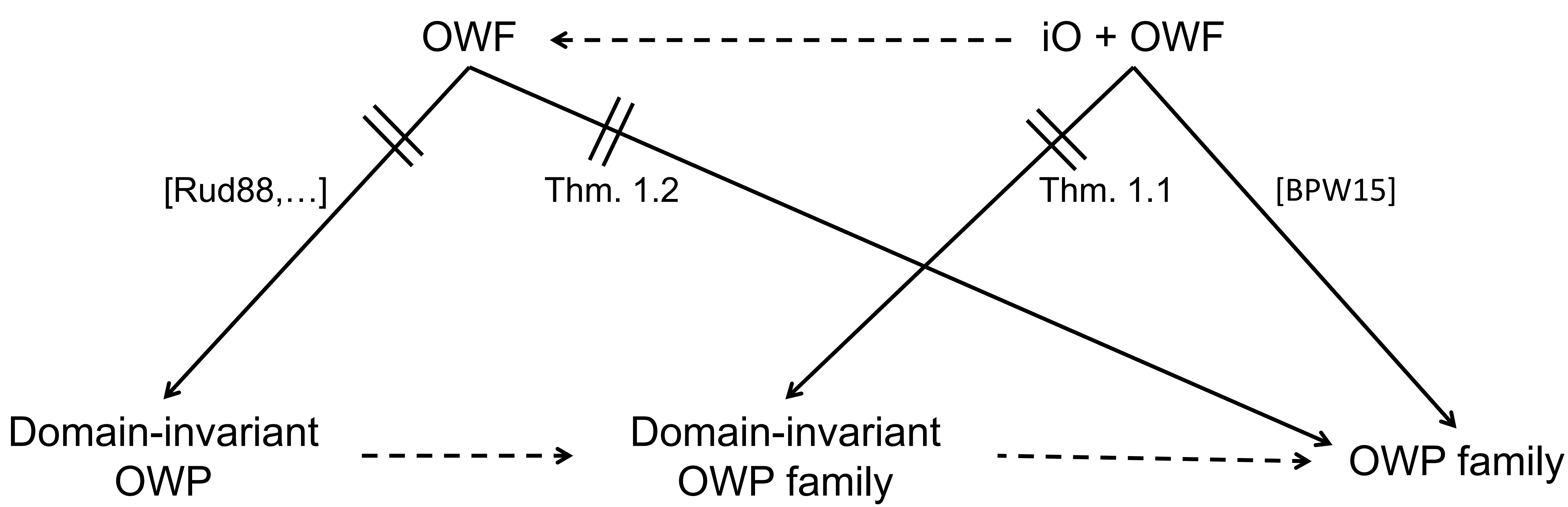
(1) No guarantee that a is a valid index relative to f''
(2) No guarantee that y^* is in the domain of $D_a^{f''}$
(3) The same for x' and x^*

Solutions

- Adversary is given α, $\mathbf{y^*}$
  - Sample in addition to $\mathbf{f'}$:
    - A "certificate" that $\boldsymbol{\alpha}$ is a valid index respectively to $\mathbf{f'}$
    - A "certificate" that $\mathbf{x'}$ is a valid element in the domain of $\boldsymbol{\alpha}$ respective to $\mathbf{f'}$
  - For $\boldsymbol{\alpha}$, $\mathbf{x^*}$ there also exist certificates such that
    - $\boldsymbol{\alpha}$ is a valid index respectively to $\mathbf{f}$
    - $\mathbf{x^*}$ is a valid element in the domain of α respective to $\mathbf{f}$
  - Using these certificate, build $\mathbf{f''}$
    - Guarantees that $\boldsymbol{\alpha}$, $\mathbf{x'}$, $\mathbf{x^*}$, $\mathbf{y^*}$ are valid respective to $\mathbf{f''}$

Further details: see the paper

Conclusions



Thank You!