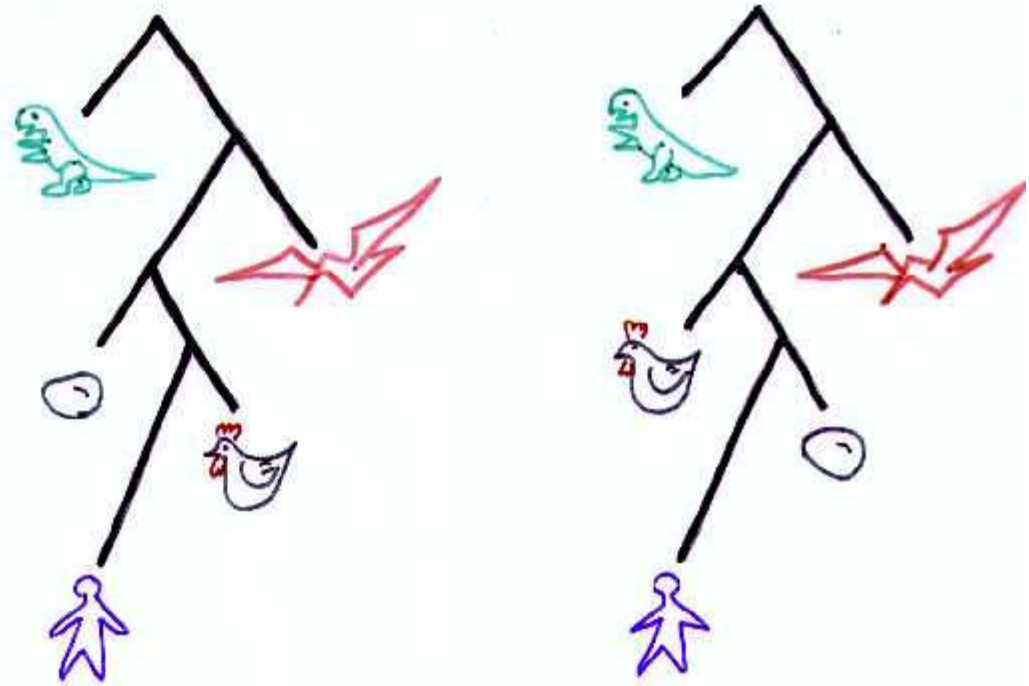
USES OF MATCHING

Famous Open Problem ...



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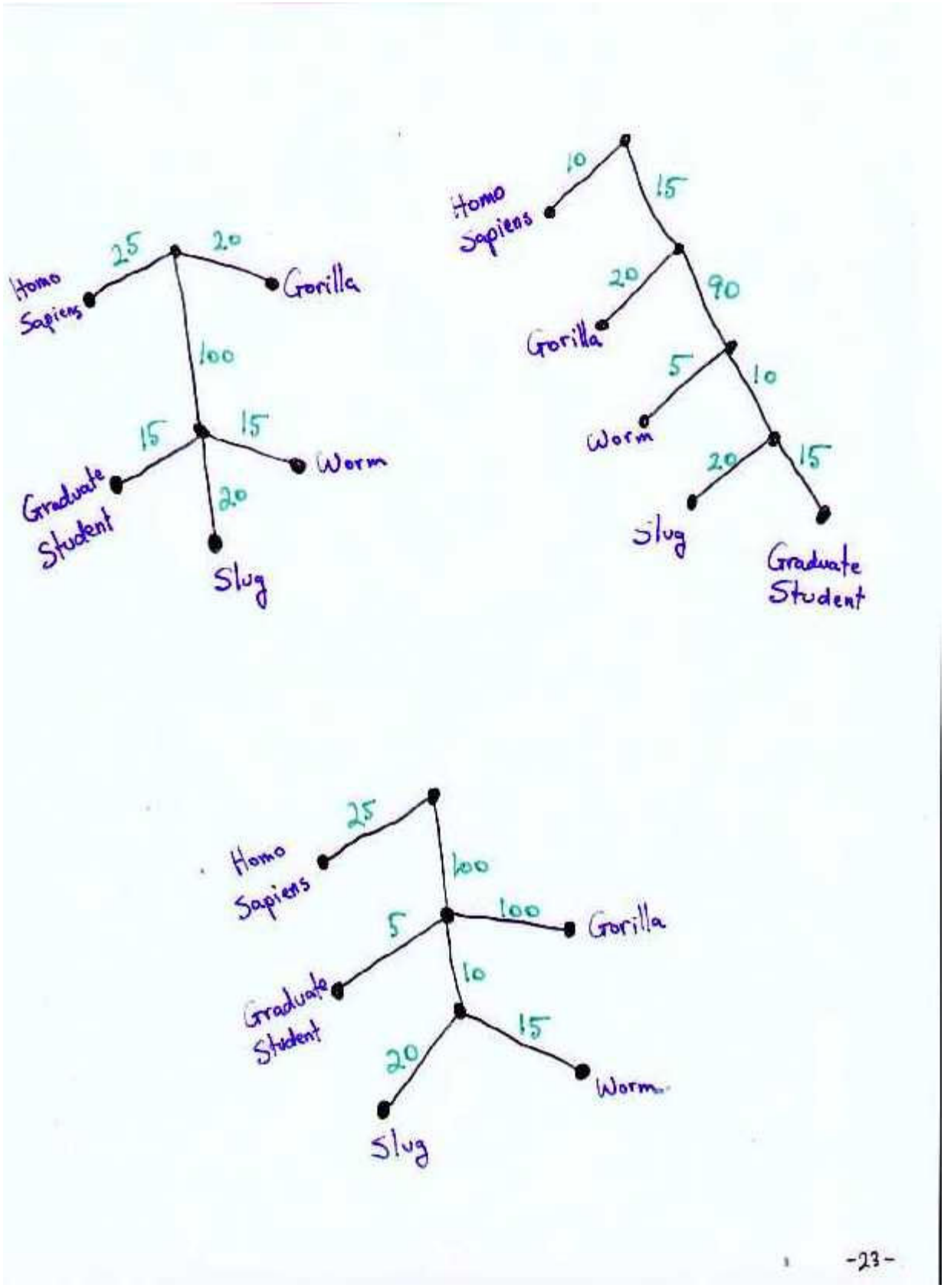

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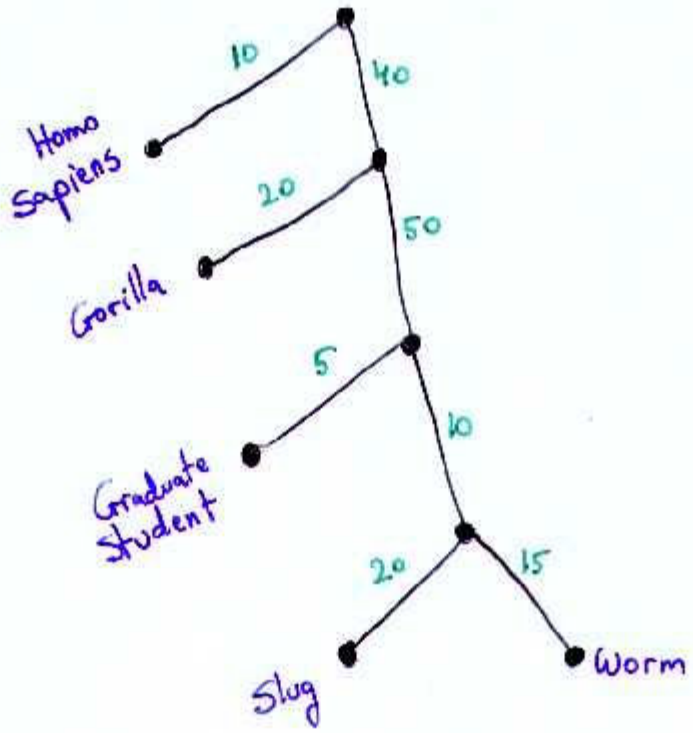
How do we reconcile?

Construct tree "closest" to all.

(e.g. Neumann 83)



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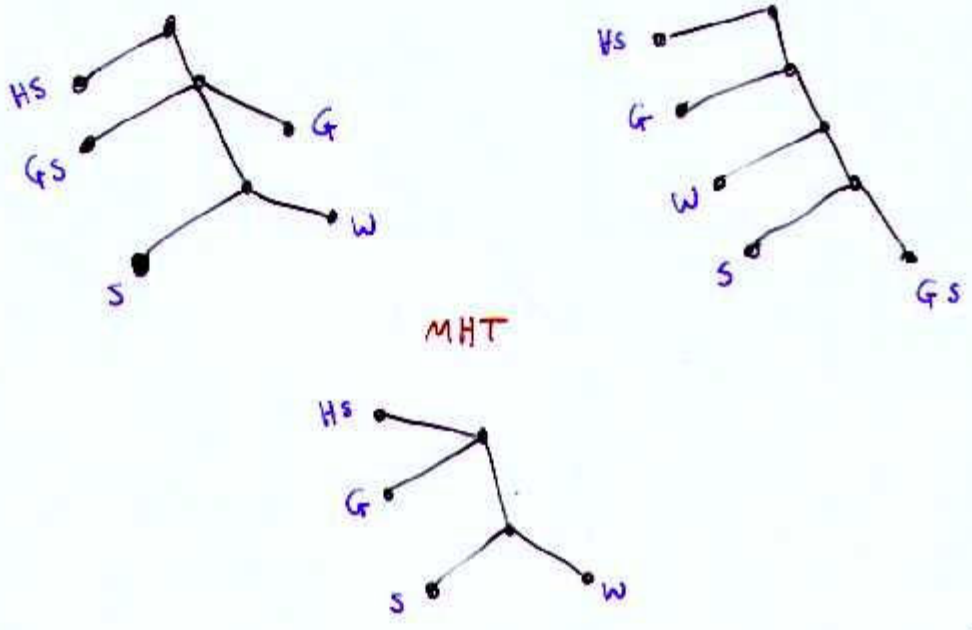
MAXIMUM HOMEOMORPHIC AGREEMENT SUBTREE (MHT)

(Finden-Gordon 85)

Find maximum set of leaves such that minimal homeomorphic subtrees (contract degree-2 nodes) with those leaves are all equal.



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FORMALLY:

INPUT: k trees.

Each has n leaves

labeled by elements from

$S = \{s_1, s_2, \ldots, s_n\}$.

No 2 leaves in a tree have equal label.

OUTPUT: Maximum $S' \subseteq S$ such that the restrictions of all trees to subtree with labels from S' are homeomorphic.

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Naive Solution

- For every subset $S' \subseteq S$ and every tree, construct restriction, check if homeomorphic.
- For all agreements, choose maximum.

Time: $O(n2^n)$

AWFUL !!!

Finden & Gordon (1985) $O(n^3)$ heuristic. Optimal tree not guaranteed. MHT of 2 trees.

Kubicka, Kubicki & McMorris (1992) $O(n^{1/2 + \varepsilon \log_2 n})$ MHT of 2 trees.

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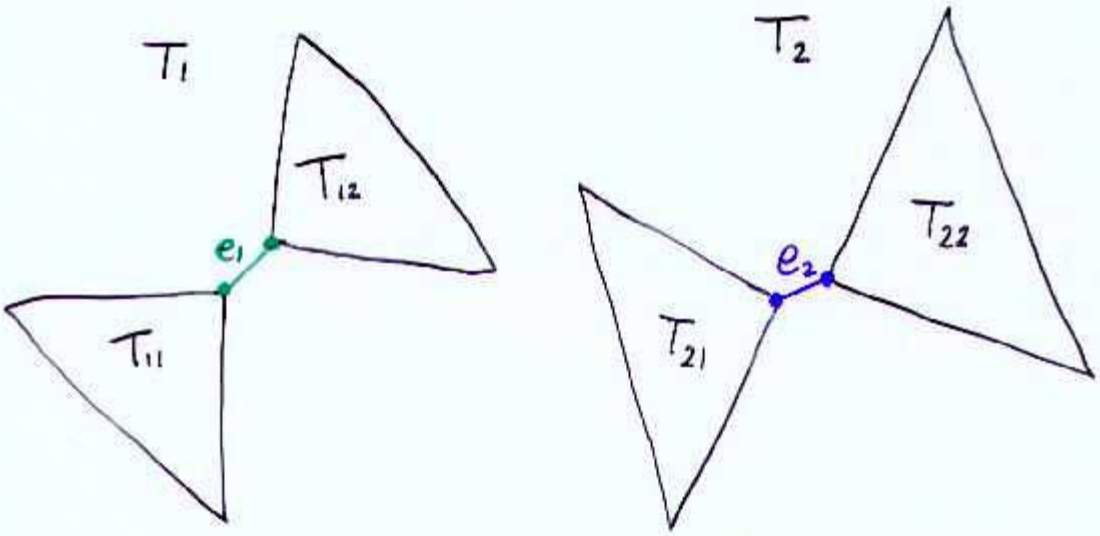
MHT of 2 Trees

(Steel & Warnow, 1993)

IDEA: Dynamic Programming

choose edge $e_1 \in T_1$ $e_2 \in T_2$



Consider $M = \max($

$MHT(T_{11}, T_{21}) \cup MHT(T_{12}, T_{22}),$

$MHT(T_{11}, T_{22}) \cup MHT(T_{12}, T_{21}))$

We will prove that an appropriate choice of e_1, e_2 makes $M = MHT(T_1, T_2)$.

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We need to: order all rooted subtrees (a root, its 2 children & all their descendants.)

How many such rooted subtrees?

Lemma: Let T be an n-leaf binary tree with the property that every node has either degree 1 (leaf) or 3.
Then T has at most $2n-1$ nodes.

Proof: By induction on # of leaves (exercise) ∎

In our case: n leaves $\Rightarrow$
$2n-1$ nodes $\Rightarrow$
$O(n)$ trees.

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Natural Order: $A \leq B$ iff A is a subtree of B.

Do topological sort on trees to get a total order.

We are ready to construct the dynamic programming table.

MHT	T_1^1	T_1^2	T_1^3	...	$T_1^{n_1}$
T_2^1					
T_2^2					
T_2^3					
⋮					
$T_2^{n_2}$					

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We fill up the matrix as follows:

- For every (P,Q) where $|P|=1$ $(P=\{p\})$

$$MHT(P,Q) \leftarrow \begin{cases} 1 & \text{if } p \in Q \\ 0 & \text{otherwise} \end{cases}$$

So we can easily fill first n rows.

- Cell (P,Q) where P has 2 children p_1, p_2 and Q has 2 children q_1, q_2

is filled as follows:

Let P_i be the subtree rooted at p_i
Q_i the subtree rooted at q_i
$i=1,2$.

$$MHT(P,Q) \leftarrow \max\left(MHT(P_1,Q_1) \cup MHT(P_2,Q_2),\ MHT(P_1,Q_2) \cup MHT(P_2,Q_1)\right).$$

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- For the entire trees T_1, T_2 (unrooted):

  For every pair of edges $e_1 \in T_1$
  $e_2 \in T_2$

  which split T_1 to T_{11}, T_{12}
  and T_2 to T_{21}, T_{22}

  compute

  $$\max\left(MHT(T_{11}, T_{21}) \cup MHT(T_{12}, T_{22}),\right.$$
  $$\left.MHT(T_{11}, T_{22}) \cup MHT(T_{12}, T_{21})\right)$$

  choose maximum over all pairs of edges.

Implementation Note:

Both the table and entire tree handling may just keep track of the sizes of the MHT's with pointers that allow computing MHT at the end.

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Time:

1). Matrix size is $O(n^2)$.

- Cells where both trees have more than one element are clearly computed in constant time.
- One-element trees require a "find" operation. Since matrix size is $O(n^2)$ we can afford construction of bit-vector for every set (space & time $O(n^2)$). "finds" are now also constant time.

2) In final stage, constant work per pair of edges, $O(n^2)$ pairs.

3) Constructing $O(n)$ subtrees:
Start from singletons and do "unions".

$O(n \log n)$	straightforward	Union-find algorithm.
$O(n \log^* n)$	Ullman	
$O(n \alpha(n))$	Tarjan	

Total Time: $O(n^2)$

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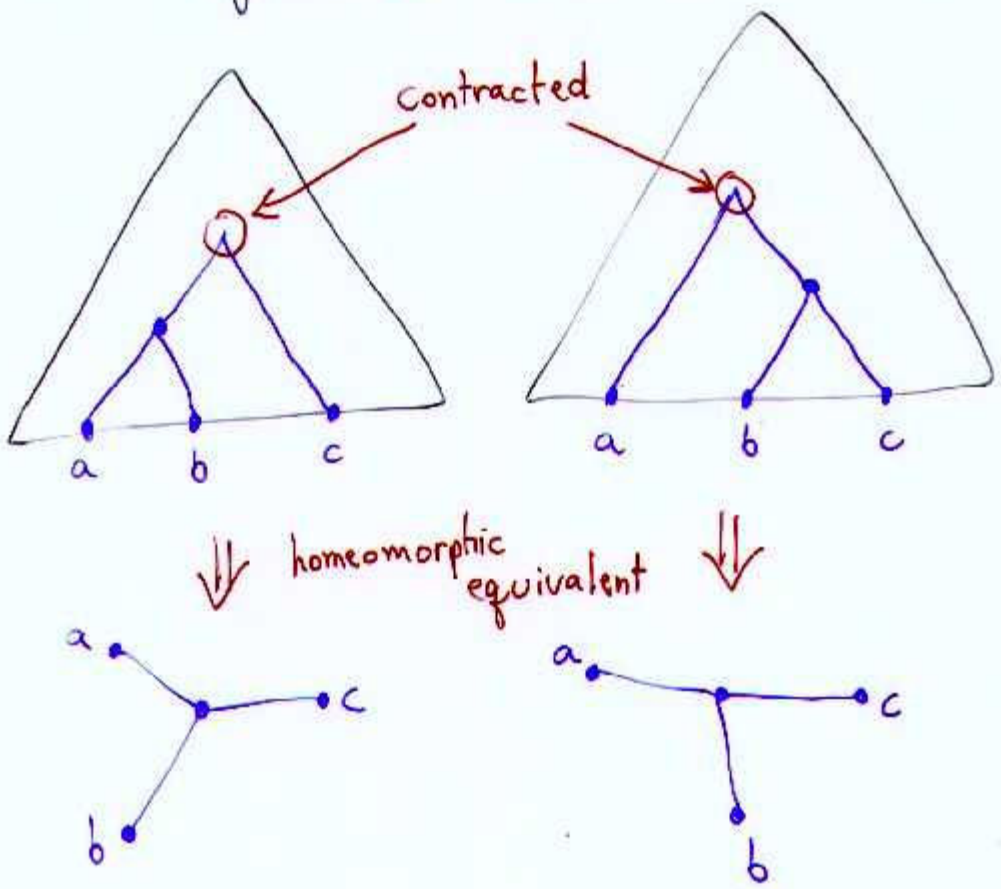
Correctness: Need to show that MHT is found.

Note: For any two trees with $n \geq 3$, the MHT has at least 3 leaves.

Reason: Any triple creates a homeomorphic agreement subtree.



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Consider $A = MHT(T_1, T_2)$

- By note, there is at least one edge e that splits A into two subtrees, creating a bipartition of A's labels.

- But A is an MHT of T_1, T_2.
  So $\exists\, e_1 \in T_1$, $e_2 \in T_2$ that create a bipartition of the labels that preserve A's bipartition.

  Since our algorithm tries all pairs of edges, it will try (e_1, e_2) among the others.
  Since it chooses maximum, it will choose (e_1, e_2) or an equivalent pair and produce an MHT. ∎

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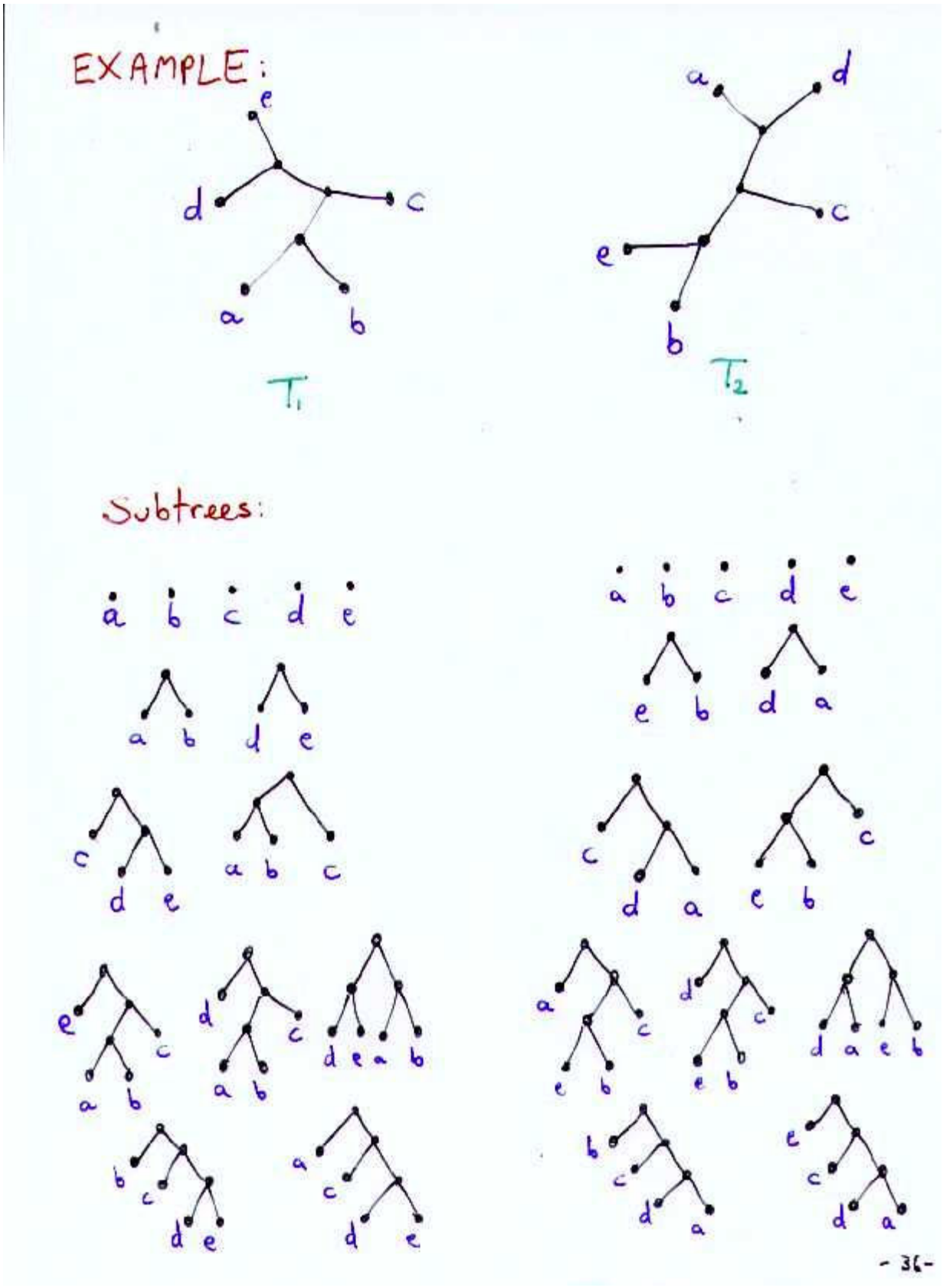

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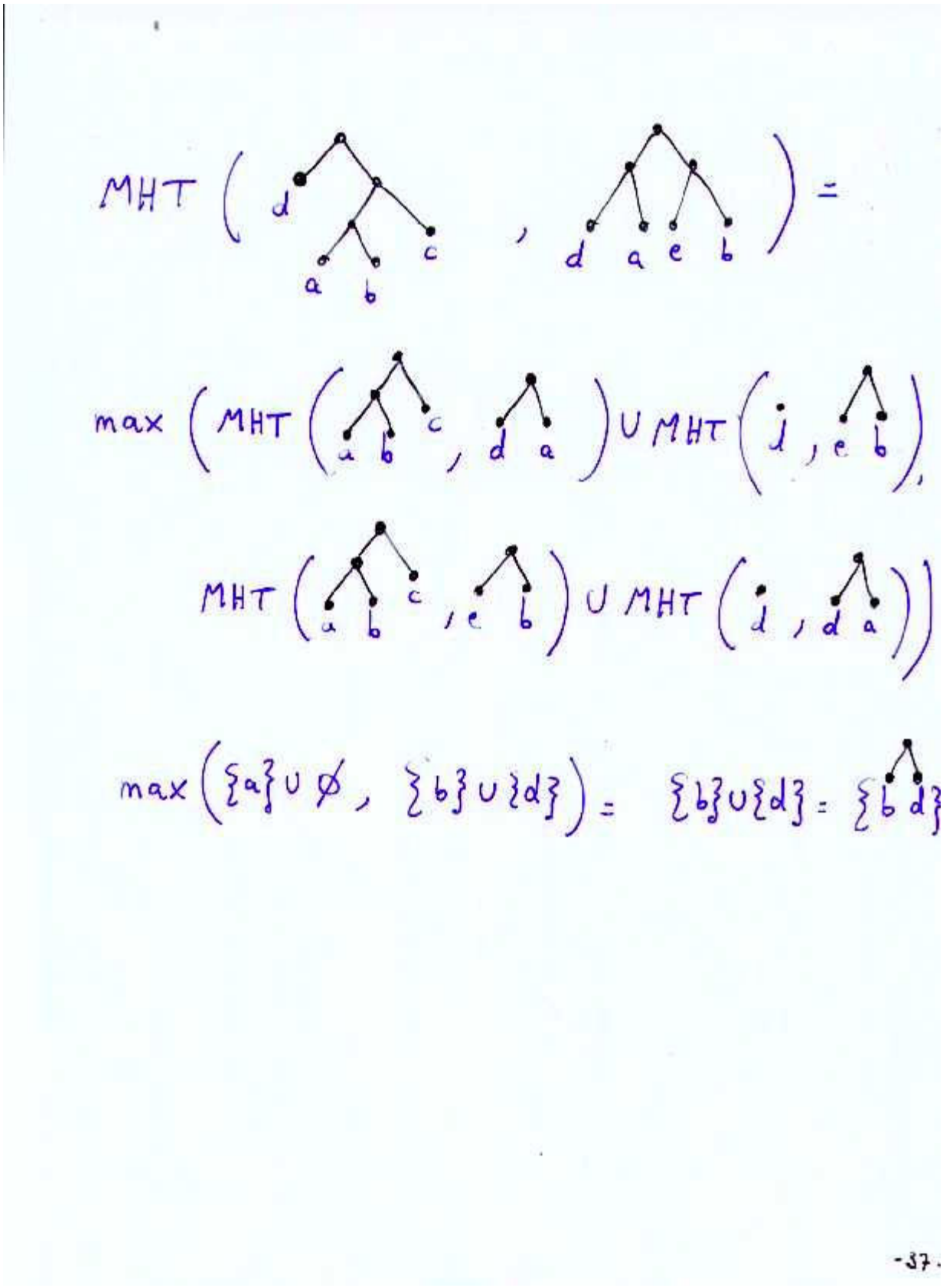

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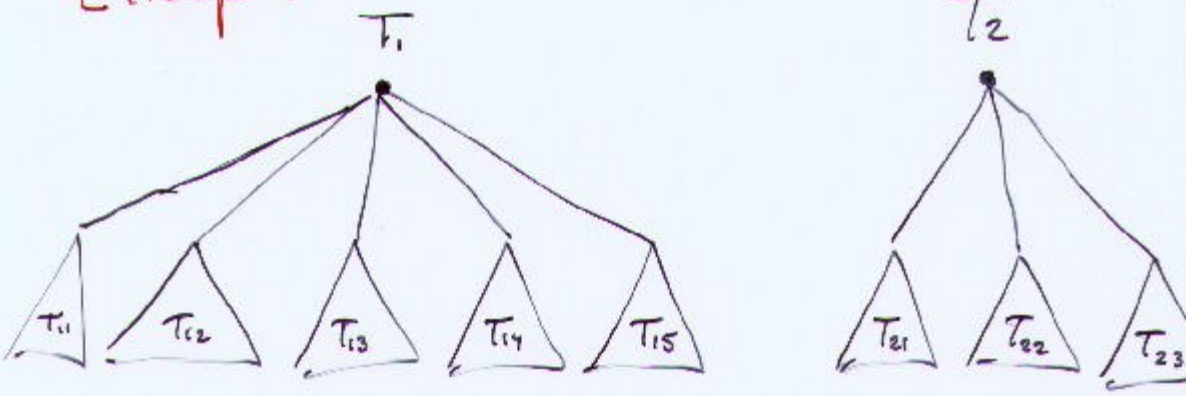
What happens when degree of tree > 3 ?

- All stages of dynamic programming remain Same.

Except:



We have: $MHT(T_{1i}, T_{2j})$ $i=1,\ldots,5$ $j=1,\ldots,3$

We want: The maximum combination.

How do we do it in polynomial time?

— Weighted Matching.

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WEIGHTED BIPARTITE MATCHING

Input: Weighted bipartite graph $G=(V,E,w)$
$w: E \to \mathbb{Z}^+$

Output: Matching $M \subseteq E'$ such that

$\sum_{e \in M} w(e)$ is maximum.

(We showed the special case with unit weights, $w(e)=1 \quad \forall e \in E$).

Hopcroft & Karp (1973)

Showed $O(n^{5/2})$ algorithm for maximum weighted bipartite matching.

(SIAM J. Computing, 1973, 225-231).

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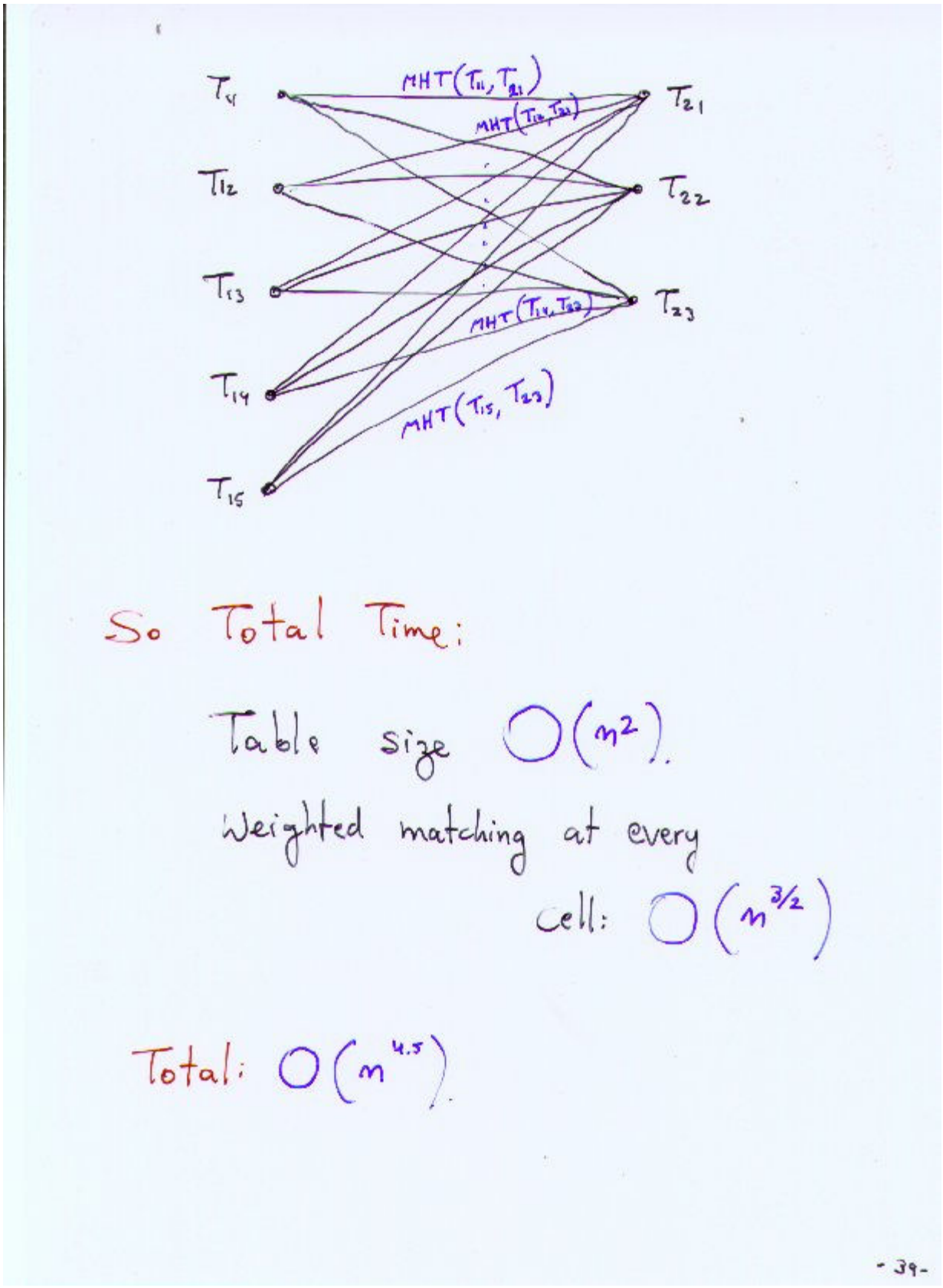

So Total Time:

Table size $O(n^2)$.

Weighted matching at every cell: $O(n^{3/2})$

Total: $O(n^{4.5})$.

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NP-Completeness of 3-MHT and 3-MWT (unbounded degree).

Decision Problem: Seek agreement subtree of given size rather than maximum.

3-Dimensional Matching

INPUT: $M \subseteq W \times X \times Y$

where W, X, Y are disjoint q element sets.

DECIDE: If $\exists M' \subseteq M$ of size q where no 2 elements of M' agree on any coordinate.

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EXAMPLE:

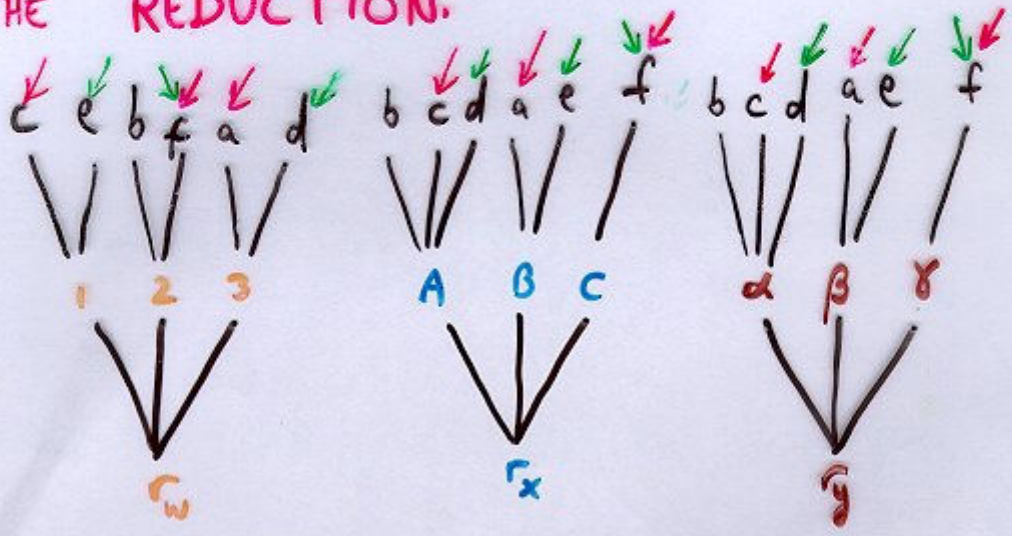
$W=\{1,2,3\}$ $X=\{A,B,C\}$

$Y=\{\alpha,\beta,\gamma\}$

$M=\{a:\langle 3\ B\beta\rangle, b:\langle 2\ A\alpha\rangle, c:\langle 1\ A\alpha\rangle$
$d:\langle 3\ A\alpha\rangle, e:\langle 1\ B\beta\rangle, f:\langle 2\ C\gamma\rangle\}$

$M'=\{a:\langle 3\ B\beta\rangle, c:\langle 1\ A\alpha\rangle, f:\langle 2\ C\gamma\rangle\}$

THE REDUCTION:



$\langle w,x,y\rangle$ is child of w in r_W, x in r_X and y in r_Y.

(For MHT - "anchor" roots)

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APPROXIMATION

Approximate set of leaves NOT in MHT.

(approximate the complement)

Theorem: (Bandelt & Dress 86)

Two leaf labeled trees are homeomorphic

iff

all generated 4-leaf subtrees are homeomorphic

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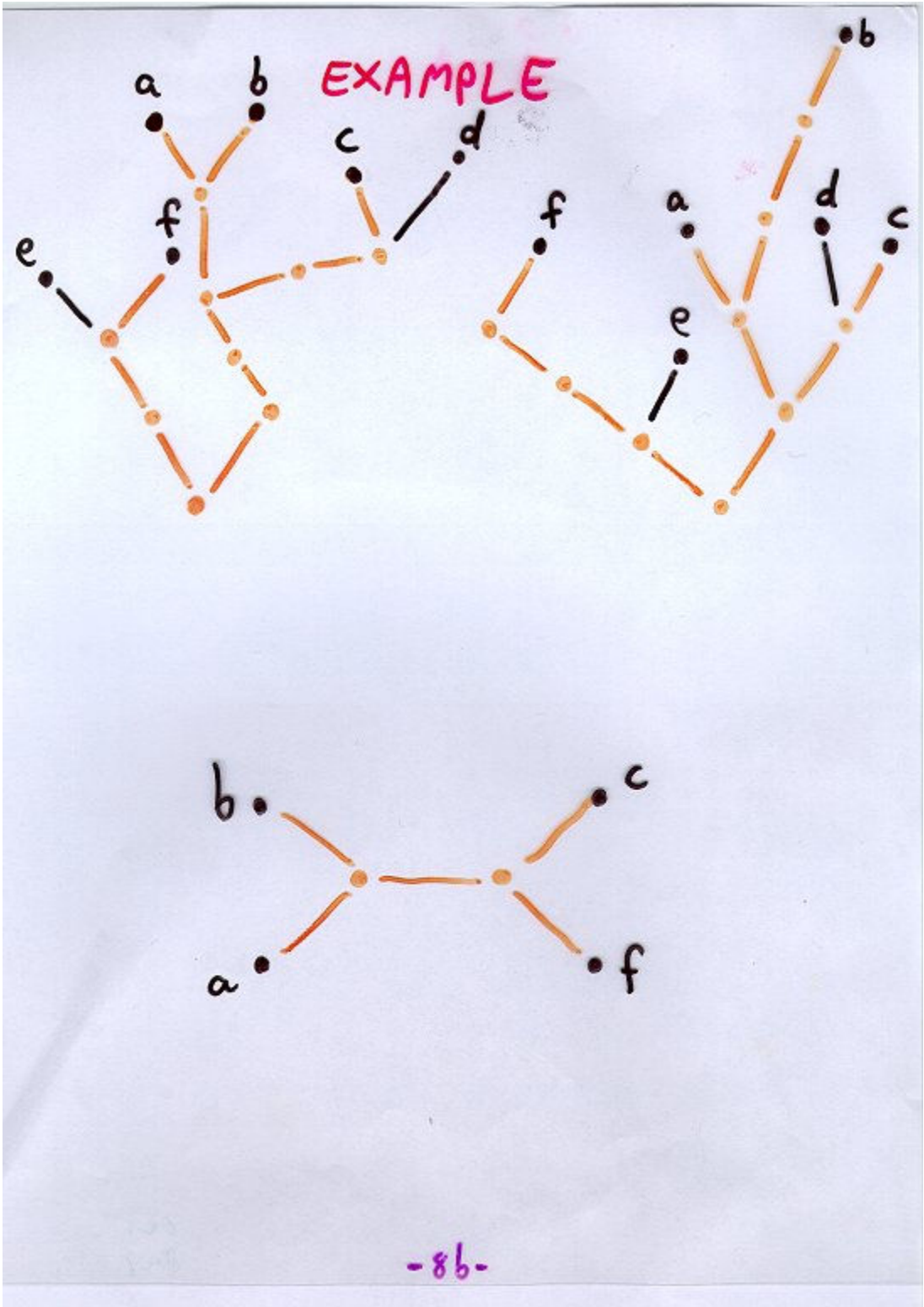

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IDEA: Generalize Vertex Cover

S_4: All 4-element subsets of S that do not generate a homeomorphic subtree in all trees.

$S' \subseteq S$ is a cover of S_4 if

$$\forall \{a,b,c,d\} \in S_4$$
$$\exists\ e \in S'$$
$$e \in \{a,b,c,d\}$$

S' is minimum cover if its size is smallest

From Thm: S' min cover $\Rightarrow$ homeomorphic restriction of $S-S'$ is MHT.

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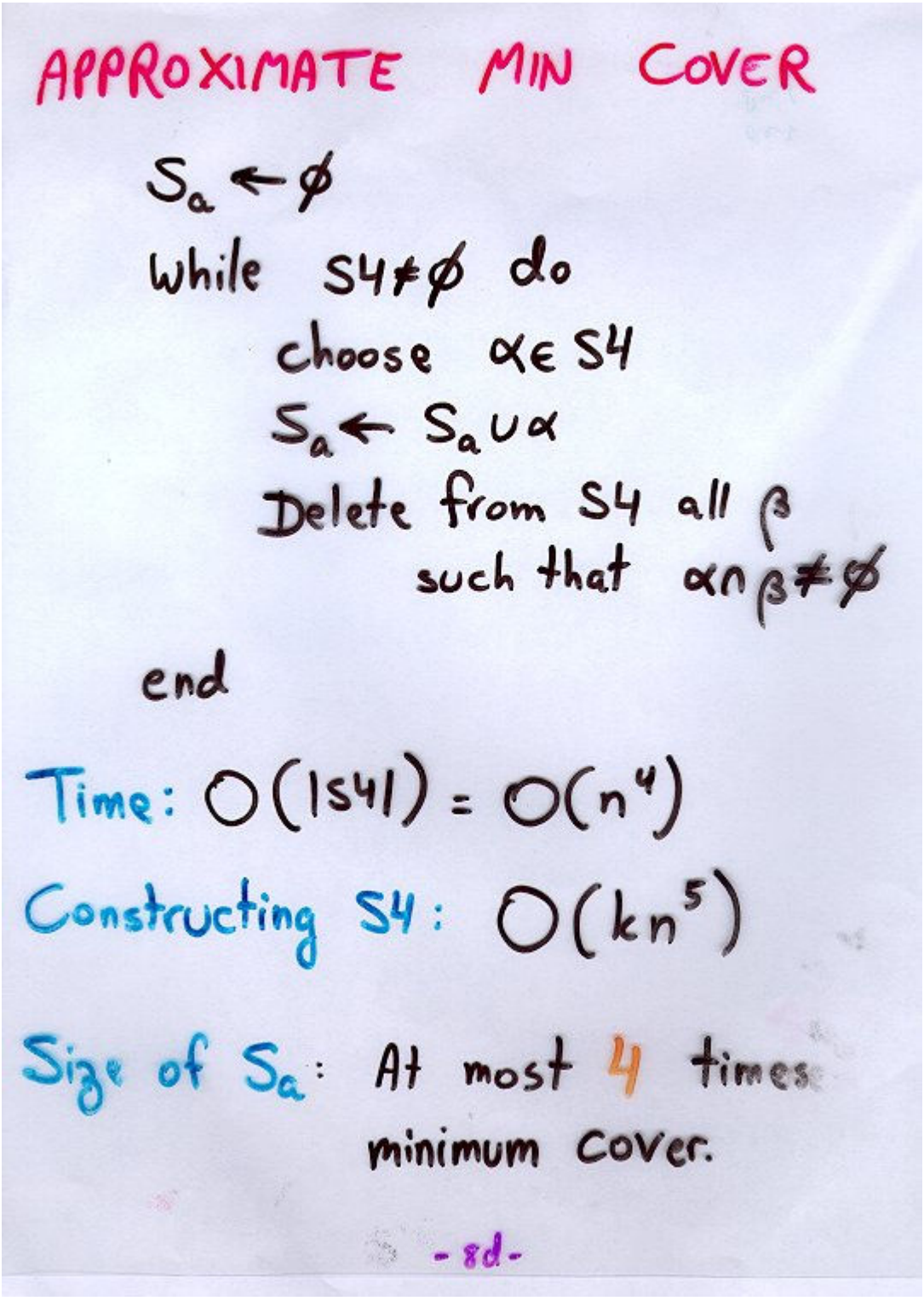

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