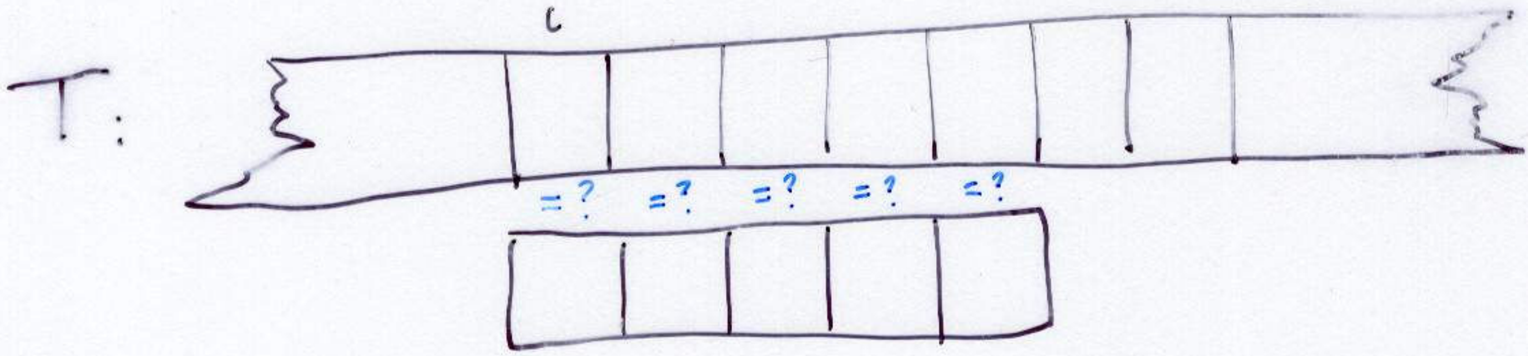
PATTERN MATCHING

Text: $T = T[1]\,T[2] \cdots T[n]$

Pattern: $P = P[1]\,P[2] \cdots P[m]$

Find all occurrences of P in T.

Idea:



Naive Algorithm

```
For i=1 to n-m+1 do

    match ← 1

    For j=1 to m do

        If P[j] ≠ T[i+j-1] then
                                match ← 0

    End

    If match=1 then output
                     "match at location" i

End
```

Time: $O(nm)$

EXAMPLE:

T: ABCDA

P: ABCDE

Does it make sense to try:

T: ABCDA

P: ABCDE

T: ABCDA

P: ABCDE

T: ABCDA

P: ABCDE

?

– Obviously Not!

KMP

ANOTHER EXAMPLE:

```
T:   ABCABCABCA
               ≠
P:   ABCABCABCD
```

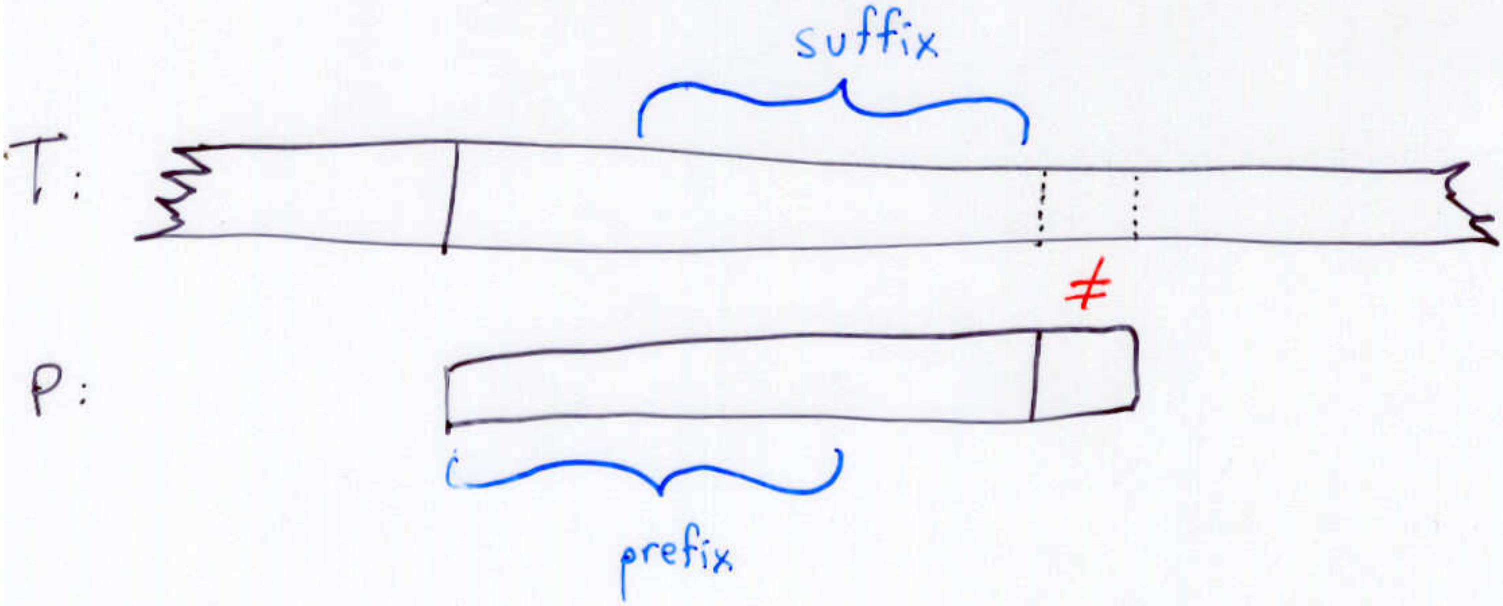
Where is the next place to try?

```
T:   ABCABCABCA
P:      ABCABCABCD
```

And the next?

```
T:   ABCABCABCA
P:         ABCABCABCD
```

What is the rule?



The longest proper prefix of the pattern that is a suffix of the text.

Problem: This needs to be calculated for every text location many times. Can it be done fast enough?

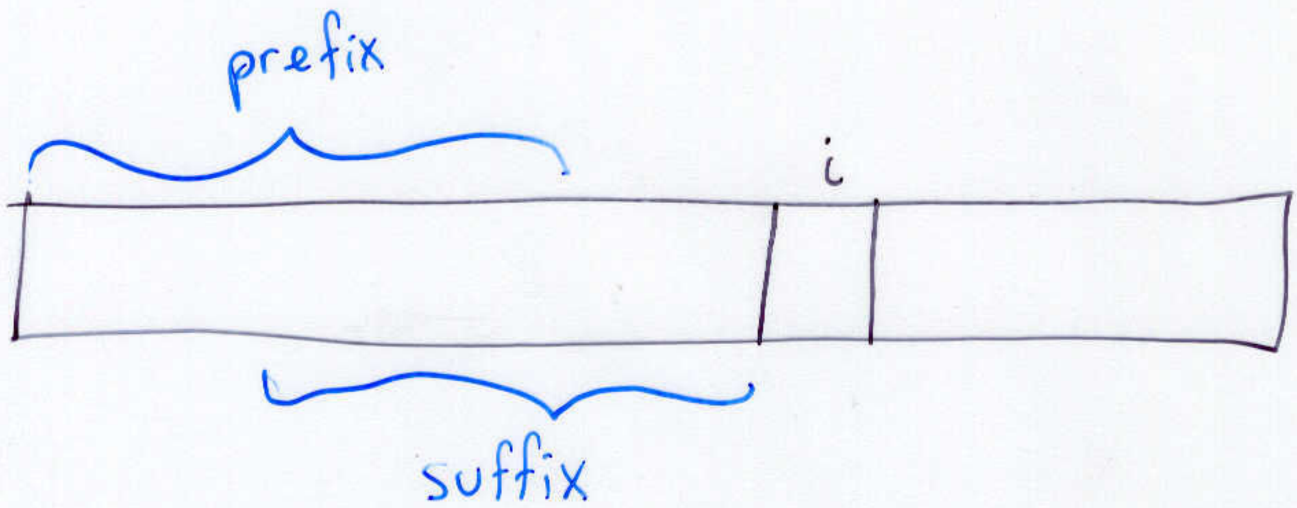
ANSWER: Build a table and get this longest proper prefix in constant time.

SIZE OF TABLE: $O(m)$

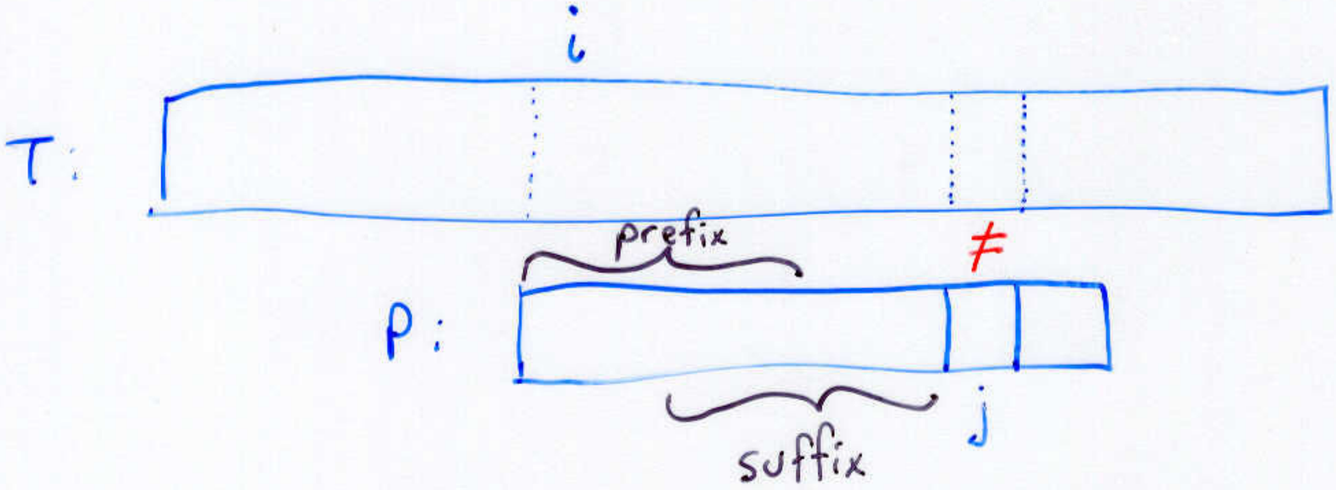
WHY? **Transitivity of "="**.

For every pattern location i

Give largest proper pattern prefix that is also a pattern suffix.



When situation is:



We have from table the largest prefix that is also a pattern suffix.

But

$$P[1] = T[i]$$
$$P[2] = T[i+1]$$
$$\vdots$$
$$P[j-1] = T[i+j-2]$$

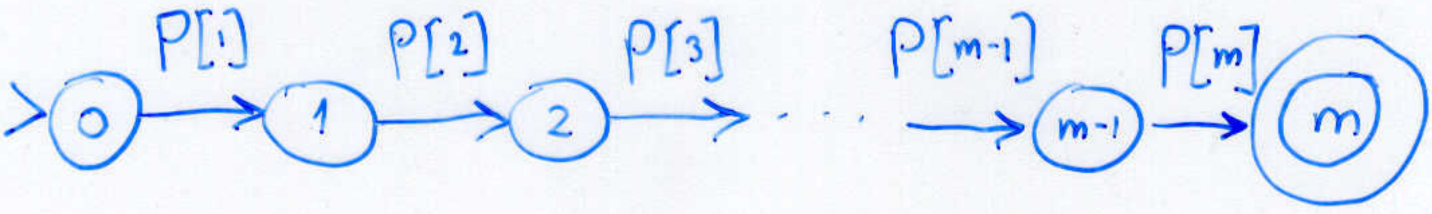
So by transitivity, the largest proper prefix of P that is a suffix of P ending at P[j-1] is also the largest prefix of P that is a suffix of T ending at T[i+j-2].

AUTOMATON IDEA

Knuth-Morris-Pratt (1972)

Construct automaton whose forward arrows are success links and whose back arrows are failure links.

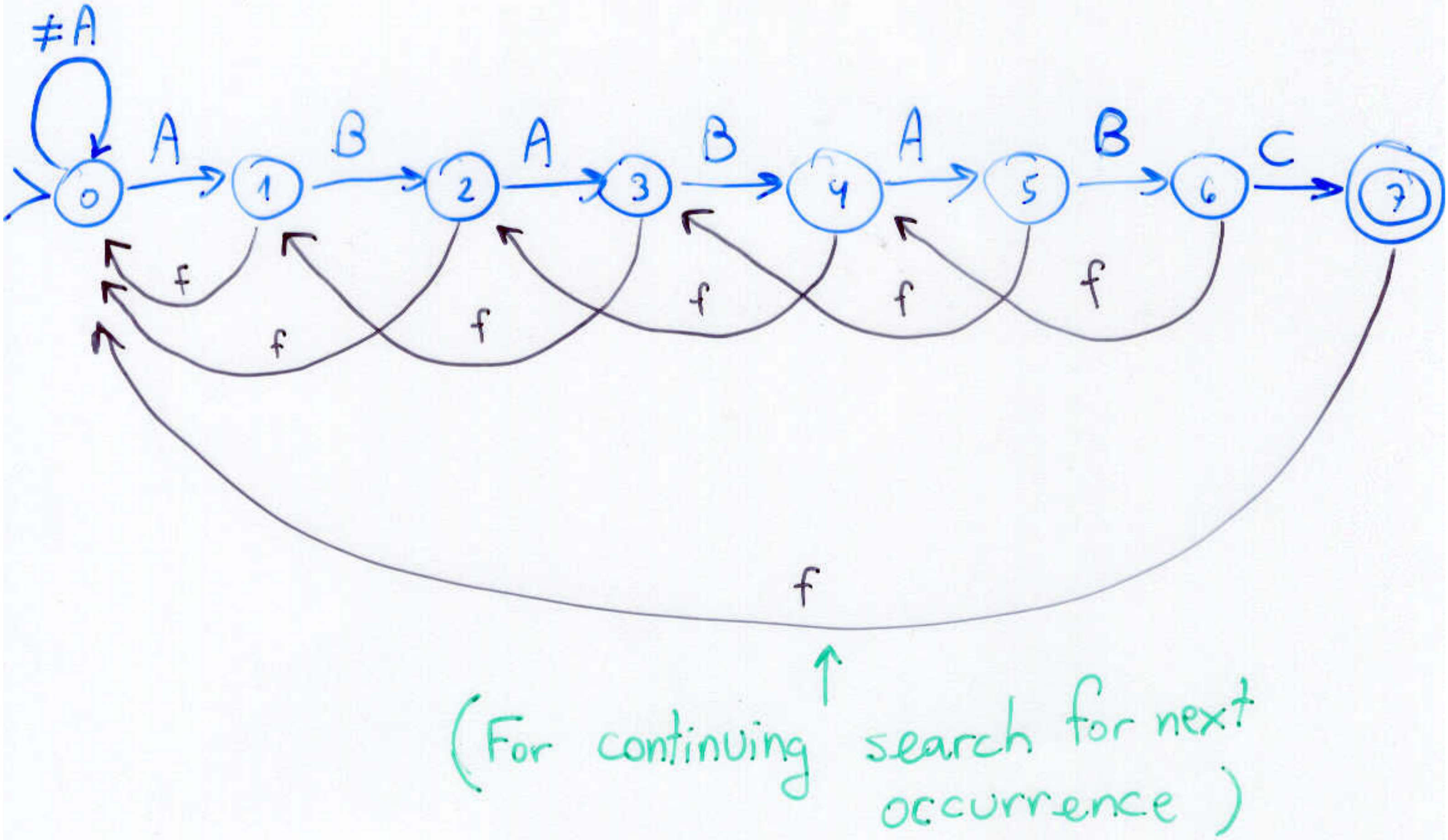
The Success Links: (For automaton of pattern P[1] ... P[m])



The Failure Links:

The failure link from node i points to node j, where j is the length of the longest proper prefix of P that is a suffix of P ending at $P[i]$.

EXAMPLE: ABABABC



Algorithm: Run on text with automaton.
For success link, move to next text location.
For failure, move on link but stay on text location.

EXAMPLE RUN:

T: ABABCCAABABABABC

Time:

Every move on success link was also a move forward in text. So how many times have we moved on success links?

$O(n)$

Since we never move back!

How many times do we move on failure links?

- Could be up to m in a row until start node is reached.

Then forward again.

But in fail links we do not advance on text.

Does this mean that the time is $O(nm)$?

KEY POINT: For every failure link that we follow back, we had to go forward with a success link!

So total # of fail links followed is not more than total # of success link followed, i.e. $O(n)$.

FORMALIZE

Define a counter F that is initialized to 0.

Increment F by 1: When a success link is followed.

Decrement F by 1: When a failure link is followed.

Claim: At any point in the algorithm run, if the automaton is at node k then $F \geq k$.

PROOF: By induction on the number of moves i of the ~~automaton~~ algorithm.

Base case: $i=0$

We are at node 0, $F=0$.

Induction Hypothesis: After move i, if the automaton is in node k then $F \geq k$.

Prove for move $i+1$.

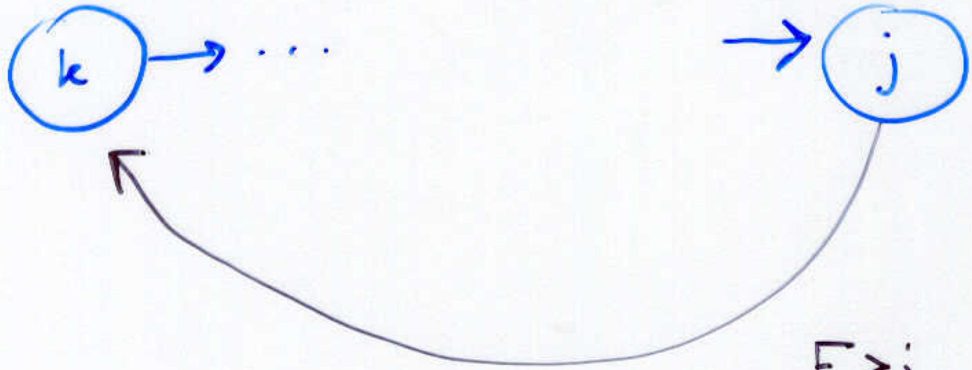
Cases: 1) Success link followed in move $i+1$

$F \geq k-1$
(ind hyp)

$F \leftarrow F+1$

$F \geq k$

2) Failure link followed in move $i+1$



$F \geq j$
(ind hyp)

$j > k$

$F \leftarrow F-1$ (fail link followed)

$\Downarrow$

$F \geq k$ ■

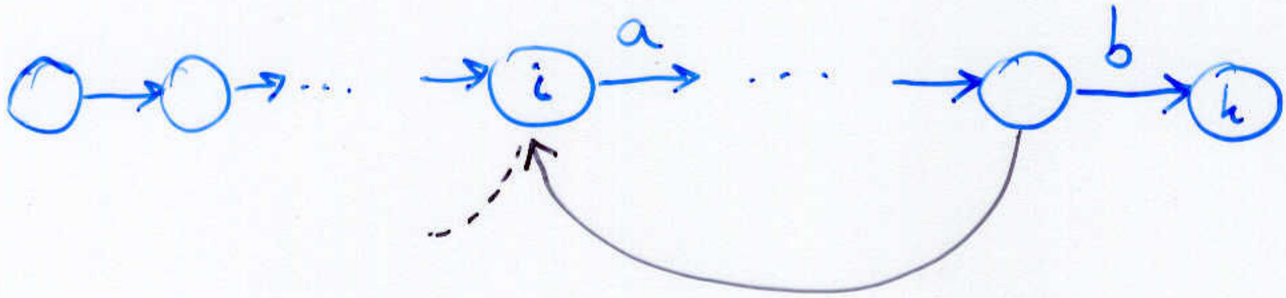
Conclude: F is incremented at most n times and is always non-negative. Therefore it is decremented (fail links) at most n times.

Total Time Text Scan for KMP Algorithm

$O(n)$.

How do we construct fail links?

Same method as search:



If $a = b$ then failure link of k points to $i+1$.

Otherwise follow fail link from i and repeat...

Time: $O(m)$.

Why?

Similar argument to text scan (can move back only if first moved forward).

Conclude: Total KMP time:

$$O(n+m) = O(n).$$