ROTATED PATTERN MATCHING

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ROTATION

We need a model.

Landau & Vishkin 1994

developed a model for discretizing the digitization process.

Fredriksson & Ukkonen 1998

developed a similar

Geometric Model

for discrete rotations.

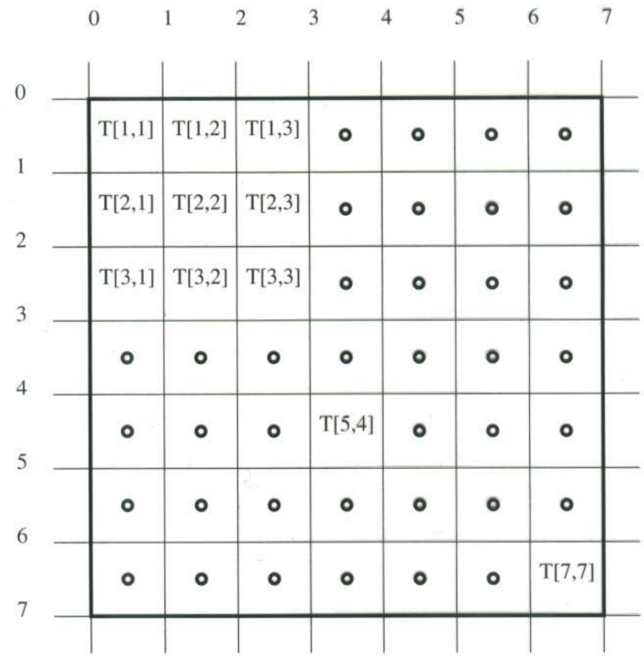


Fig. 1. The text grid and pixel centers of a 7×7 text.

r-6

PROPOSED SOLUTION:

Construct all possible rotated patterns.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

5a

	1,2	3	
5	6	7	4,8
13,9	10	11	12
	14	15,16	

5b

5	1	2	3
9	6	7	4
13	10	11	8
14	15	16	12

5c

		1			
	5		2	3	
	9	6	7		4
13		10	11	8	
	14	15		12	
			16		

5d

5	2	3	3
5	6	7	8
9	10	11	12
14	14	15	12

5e

	1	2	
9	6	7	4
13	10	11	8
	15	16	

5f

Fig. 5. An example of some possible 2-dimensional arrays that represent one pattern. Fig 5a - the original pattern. Figures 5b-d are computed in the "pattern over the text" model. Fig 5b - a representation of the pattern rotated by 19^0. Fig 5c - Pattern rotated by 21^0. Fig 5d - Pattern rotated by 26^0. Figures 5e-f are computed in the "pattern under the text" model. Fig 5e - Pattern rotated by 17^0. Fig 5f - Pattern rotated by 26^0.

r-c

Every rotated pattern can be found in the text using FFT in time:

$$O\left(n^2 \log m\right)$$

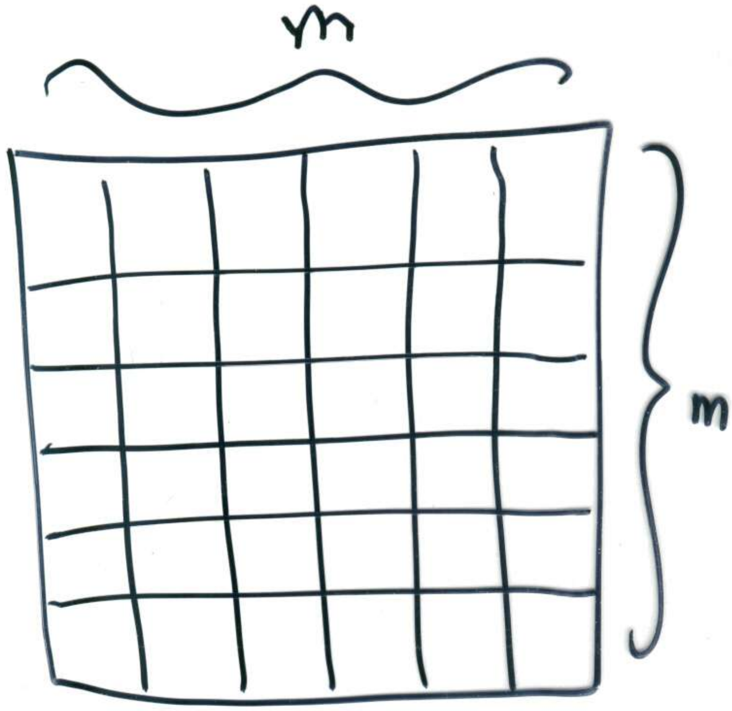
If there are N rotated patterns the total time is:

$$O\left(N\, n^2 \log m\right)$$

WHAT IS N ?

r-d

UPPER BOUND



m^2 pixels

Each pixel center crosses at most $4m$ grid lines.

$\Rightarrow O(m^3)$ different rotated patterns.

Could many points cross a gridline together?

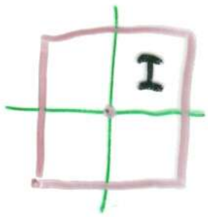
We will show:

LOWER BOUND: $\Omega(m^3)$

Restriction:

Set P =

1) Points in Quadrant I



2) Points (x, y) where x and y are co-prime.

$(\gcd(x, y) = 1)$

WE WILL SHOW: $\forall X_1, X_2 \in P$

It is impossible that X_1 and X_2 cross grid line at same angle.

How does it help?

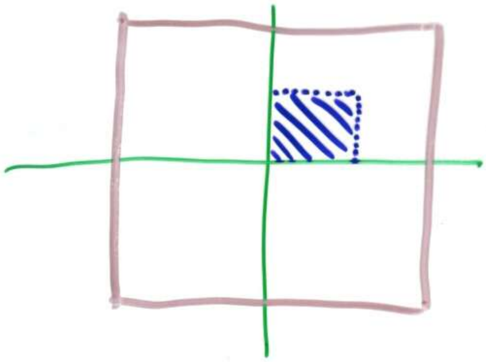
$$\|P\| = \frac{6m^2}{\pi^2} + o(m \log m)$$

(Geometry Thm).

Consider:

$$P_{\leq \frac{m}{4}} = \left\{ (x,y) \mid (x,y) \in P, \; 0 \leq x, y \leq \frac{m}{4} \right\}$$

Schematically: $P \cap$ shaded area



In shaded area: $\frac{m^2}{16}$ points.

So in $P - P_{\leq \frac{m}{4}}$, at least

$$\frac{6m^2}{\pi^2} - \frac{m^2}{16}$$ points, i.e.

$$\frac{96 - \pi^2}{16\pi^2} m^2 = \Theta(m^2)$$

points.

Each of the $\Theta(m^2)$ points in $P - P_{\leq \frac{m}{4}}$ crosses the grid $\Omega(m)$ times, and no two of them cross together.

Conclude: There are $\Omega(m^3)$ different rotated patterns.

Left to show:

Lemma: $\forall X_1, X_2 \in P$

It is impossible for them to cross a grid line at same angle.

Proof:

We discuss the case where

X_1 crosses horizontal grid line to Y_1

and

X_2 crosses horizontal grid line to Y_2.

(Other cases, both crossing vertical or one vertical & one horizontal, are similar.)

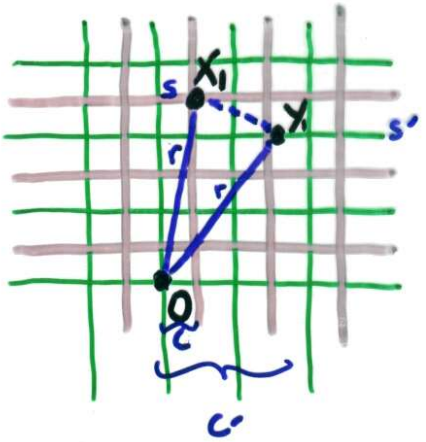
Let $X_1 = (c, s)$ $\quad Y_1 = (c', s')$.

c, s are odd. $\quad c = 2k_1 + 1$

$\quad s = 2k_2 + 1$

r-6-

s' is even $\quad s' = 2\ell_1$



$$c^2 + s^2 = r^2 = (2k_1+1)^2 + (2k_2+1)^2$$

$$c'^2 + s'^2 = r^2$$

$$c'^2 + 4\ell_1^2 = (2k_1+1)^2 + (2k_2+1)^2$$

$$c'^2 = (2k_1+1)^2 + (2k_2+1)^2 - 4\ell_1^2 =$$

$$= 4(k_1^2 + k_1 + k_2^2 + k_2 - \ell_1^2) + 2$$

So c'^2 is even. $c'^2 = 2\ell_2$

$$2\ell_2 = 4(k_1^2 + k_1 + k_2^2 + k_2 - \ell_1^2) + 2$$

$$\ell_2 = 2(k_1^2 + k_1 + k_2^2 + k_2 - \ell_1^2) + 1$$

Conclude:

$c' = \sqrt{2 l_2}$ where l_2 is odd

$s' = 2 l_1$

We can say even more:

$c' = n\sqrt{2m}$

where $n \in \mathbb{Z}^+$

and m is a square-free odd number

(does not have a square factor)

i.e. c' is an irrational number.

OUR SITUATION

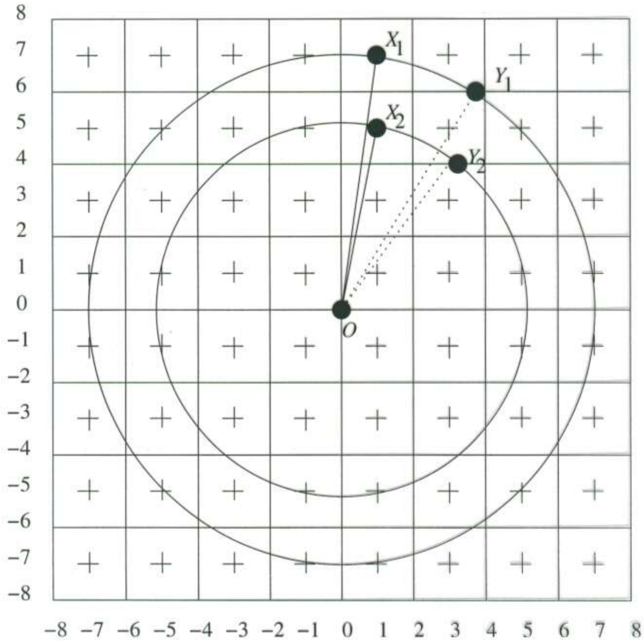


Fig. 6. Points X_1 and X_2 each have coprime integer coordinates. Orbits of them under rotation around point O cross an horizontal line at points Y_1 and Y_2 respectively. Then, $\measuredangle X_1OX_2 \neq \measuredangle Y_1OY_2$ by Claim 3.

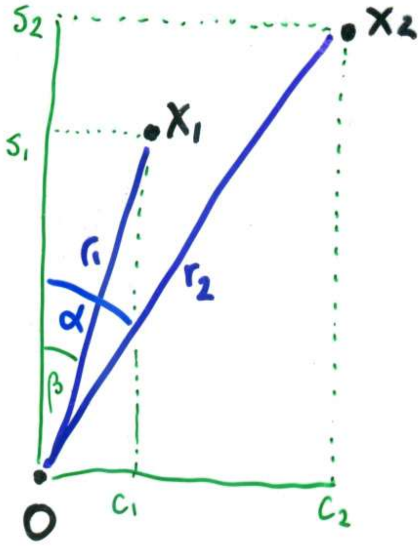
CLAIM: $\measuredangle X_1 O Y_1 \neq \measuredangle X_2 O Y_2$.

Proof: We show $\measuredangle X_1 O X_2 \neq \measuredangle Y_1 O Y_2$.

$X_1 : (c_1, s_1)$ $\qquad$ $Y_1 : (c_1', s_1')$

$X_2 : (c_2, s_2)$ $\qquad$ $Y_2 : (c_2', s_2')$

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$



$$\sin \measuredangle X_1 O X_2 = \frac{c_2}{r_2}\frac{s_1}{r_1} - \frac{s_2 c_1}{r_2 r_1} = \frac{s_1 c_2 - s_2 c_1}{r_1 r_2}$$

similarly: $\quad \sin \measuredangle Y_1 O Y_2 = \dfrac{s_1' c_2' - s_2' c_1'}{r_1 r_2}$

If $\measuredangle X_1 O X_2 = \measuredangle Y_1 O Y_2$ then

$\sin \measuredangle X_1 O X_2 = \sin \measuredangle Y_1 O Y_2$ i.e.

$$\underset{\mathbb{Z}}{\underset{\cap}{s_1 c_2 - s_2 c_1}} = s_1' c_2' - s_2' c_1'$$

What do we know?

$s_1', s_2' \in 2\mathbb{Z}$

$c_1' = n_1 \sqrt{2m_1}$

$c_2' = n_2 \sqrt{2m_2}$

where c_1', c_2' irrational.

We have:

$a\sqrt{2m_1} + b\sqrt{2m_2} \in \mathbb{Z}$, where $a, b \in \mathbb{Z}$

When can this happen?

$$a\sqrt{2m_1} + b\sqrt{2m_2} \in \mathbb{Z}$$

Options:

1) $m_1 \neq m_2$

Can not happen since $a, b \in \mathbb{Z}$ and $\sqrt{2m_1}$, $\sqrt{2m_2}$ are linearly independent in $\mathbb{Z}[\sqrt{2m_1}, \sqrt{2m_2}]$.

2) $m_1 = m_2$

$$a\sqrt{2m_1} + b\sqrt{2m_1} \in \mathbb{Z}$$

$$\sqrt{2m_1}\,(a+b) \in \mathbb{Z} \quad \text{iff} \quad a = -b$$

$$\text{iff}$$

$$a\sqrt{2m_1} + b\sqrt{2m_1} = 0$$

$$\parallel$$

$$s_1 c_2 - s_2 c_1$$

When can

$$s_1 c_2 = s_2 c_1 \quad ?$$

$$\frac{s_1}{c_1} = \frac{s_2}{c_2}$$

Since (c, s) are relatively prime,

this can only happen if

$s_1 = s_2$ and $c_1 = c_s$

i.e. $X_1 = X_2$.

WHAT IS LEFT?

Faster rotation.

Approximate rotation.

Approximate scaling.

Integration.

We have come a long way
but have a lot longer way
to go...