Algorithms II 89-322-01, 89-322-02, FINAL EXAM
MOED A

Instructor: *Prof. Amihood Amir*
Length of Exam: 2 hours
Time: August 14, 2008, 08:30
NO OUTSIDE MATERIAL ALLOWED!!!

1. What is the exact number of edges in a *complete* (no nodes of degree 2) unrooted binary tree with n leaves? Prove your claim.

   Answer:

   The number is $2n-3$. It can be proven by induction.
   Base case: For a tree with 2 leaves there is one edge.
   Induction hypothesis: Assume every tree with n leaves, where $n \geq 2$ has $2n-3$ edges. You are given a tree with $n+1$ leaves. Let a and b be two leaves such that the path between them is longest. Let c be the father of a, and let d be the other son of c (c can not be a leaf since we are considering the case where $n+1 > 2$). d must also be a leaf since otherwise the path between b and any leaf descendant from d is longer than the path between a and b and that path had maximum length.

   Now remove a and d and their edges from the tree. c becomes a leaf, and we now have a tree with n leaves. By induction it has $2n-3$ edges. Adding a and d and their 2 edges gives $2n-3+2 = 2(n+1)-3$. ∎

   Error Codes and Penalties:

   1. Assumption without proof that every binary tree with at least four nodes has an edge between two internal nodes. (-4)
   2. Proof on rooted tree. (-5)
   3. In induction step for rooted tree, the root was removed and then it was assumed that both sides have same size. (-8)
   4. Definition of complete tree, even though given in class and in the question, was wrong. (-17)
   5. Wrong induction direction. Started from a k node tree and then added to one of the leaves two sons, rather than starting from a k+1 node tree and showing how induction can be used. (-12)
   6. Wrong number given. (-8)
   7. Speaking about a "root" in an unrooted tree. (-12)
   8. Proof lacking of the assumption that there exist two leaves that are brothers. (-4)
   9. Base of induction erroneous or inexistant. (-9)
   10. Wrong direction of proof. The reduction should be from an unrooted tree to a rooted tree. (-12)

11. Confusion between rooted and unrooted trees in the proof. (-8)
12. Conclusion of the number of edges from the number of nodes without proof. (-8)
13. A recurrence relation is given as a result, rather than a closed form formula. (-10)
14. Induction step faulty, not proving right number. (-17)
15. Used internal nodes in proof, but did not specify or prove how many such nodes exist. (-10)
16. What is a “top edge” in an unrooted tree? (-4)
17. Proved number of nodes rather than the requested number of edges. (-5)
18. Problem in induction step, how do you choose $\frac{k}{2}$ parents? (-10)
19. Wrong handling of tree. (-7)
20. Induction not properly defined. (-5)
21. Node of degree 2 used in the induction, when none such exists in tree. (-7)

2. (a) Write an *integer program* to solve the **complement** of the MHT problem.
   (b) Perform a *relaxation* on that IP to create a *linear program* that approximates the solution of the complement of the MHT problem.
   (c) What approximation factor do you get? Why?

Answer:

(a) Recall that by the Bandelt and Dress theorem, the result is the minimum number of leaves that occur in all quadruples of leaves that are not homeomorphic. We therefore assign a variable x_i to every symbol x_i, These variables get integer value from $\{0, 1\}$. Assume there are n symbols.
The objective function is $\min \sum_{i=1}^{n} x_i$.
Subject to the constraints:
$x_i \in \{0, 1\}$, for $i = 1, ..., n$.
For every quadruple $< x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4} >$ that is not homeomorphic to all trees, write:

$$\sum_{j=1}^{4} x_{i_j} \geq 1.$$

(b) The relaxation is choosing all symbols whose value is at least $\frac{1}{4}$.

(c) The approximation ratio is 4. Clearly, every quadruple will have at least one element we choose, so our answer is correct, albeit not minimal. However, in the worst case we may choose all four elements rather than just one, thus the sum of the elements we choose may be up to 4 times larger than the optimum.

Error Codes and Penaties:

1. Gave an IP for the MHT problem. Not able to relax, since the problem not approximable. (-16)
2. Objective function OK, but no constraints. (-10)
3. Relaxation not given or the concept not understood. (-8)
4. No approximation. (-8)
5. Incorrect relaxation and approximation. (-8)
6. Gave IP and then LP relaxation for VC problem rather than $\overline{MHT}$. (-25)
7. Problem with objective function. (-8)
8. No approximation given, or wrong approximation and no justification. (-8)
9. Errors in the constraints. (-5)
10. Errors in choosing variables at relaxation stage, or choice not explicit. (-4)
11. Correct approximation ratio but no justification. (-4)

3. The *Bowling Shoes Problem* is the following: Mr. Kadouri decides to take up bowling. However, he does not know if he is going to like it. Renting a pair of shoes at his local bowling alley costs k NIS, while buying a pair of bowling shoes at his local shoe store costs n NIS. Mr. Kadouri decides upon the following strategy. He will try four times and rent the shoes. Assuming he likes bowling so much that he goes four times, he will then buy a pair of shoes.

   Is Mr. Kadouri's algorithm competitive? What is its competitive ratio?

   Answer:

   We need to consider 3 cases:

   (a) $\frac{n}{k} < 4$: In this case, assuming he bowls more than $\frac{n}{k}$ times, the optimum cost is n but he pays $n + 4k$. Assuming $4k = \alpha n$ then the competitive ratio is $(1 + \alpha)$.

   (b) $\frac{n}{k} = 4$: The competitive ratio is 2.

   (c) $\frac{n}{k} > 4$: For the worst (4 times) case the optimum costs $4k$ and he paid $n+4k$. Assuming $n = 4k\alpha$, the competitive ratio is $(1 + \alpha)$.

   Error Codes and Penalties:

   1. Competitive ratio not provided. (-11)
   2. Competitive ratio not proved. (-11)
   3. Erroneously claims not competitive. (-11)
   4. Does not cover one of the cases. (-11)
   5. Error in calculating competitive ratio. (-3)
   6. Given competitive ratio not tight. (-5)
   7. Shows minimal knowledge in on-line algorithms. (-23)
   8. Different cases not discussed. (-15)
   9. Right idea, confused proof. (-13)
   10. Assumes that if a given ratio (say 2) is contradicted, then the algorithm is not competitive. (-10)
   11. Concludes that if the optimum is 0 and the algorithm gives a non-zero value, the competitive ratio is 1. (-20)

GOOD LUCK