Applied Probability Models for CS (2019-2020)
Exercise 1 Solution

1. (a) The probability to answer a question correctly is p=0.85. Assuming that the questions are independent, this is a binomial experiment with n=20. The expected number of correct answers is thus $E[X] = np = 20 \cdot 0.85 = 17$, and the expected grade is $E[5 \cdot X] = 5 \cdot E[X] = 5 \cdot 17 = 85$.

   (b) The probability that a random student got 90 is:

$$P(90) = P(W) \cdot P(90|W) + P(E) \cdot P(90|E) = 0.7 \cdot P(90|W) + 0.3 \cdot P(90|E)$$

   Using the formula for binomial experiments, $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$:

$$P(90) = 0.7 \cdot \binom{20}{18} \cdot 0.85^{18} \cdot 0.15^2 + 0.3 \cdot \binom{20}{18} \cdot 0.95^{18} \cdot 0.05^2$$

$$= 0.160537 + 0.0566 = 0.2171$$

   (c) Using Bayes rule:

$$P(W|90) = \frac{P(W) \cdot P(90|W)}{P(90)} = \frac{0.7 \cdot \binom{20}{18} \cdot 0.85^{18} \cdot 0.15^2}{0.2171} = 0.7395$$

2. (a) The probability that you will be the first to get a frosted cupcake is the joint probability that every one of the 19 students before you got a non-frosted cupcake, and that you got a frosted one. The probability of each student to get a frosted / non-frosted cupcake is the number of frosted / non-frosted cupcakes out of all the cupcakes left in the box, and the overall probability is:

$$\frac{25}{40} \cdot \frac{24}{39} \cdot \ldots \cdot \frac{7}{22} \cdot \frac{15}{21} = \frac{25! \cdot 20! \cdot 15}{6! \cdot 40!} = 9.636 \cdot 10^{-7}$$

   (b) The probability that all the frosted cupcakes were taken already is the joint probability that 15 of the 19 students before you got frosted cupcakes. Then, the probability that you and the other 4 before you got non-frosted cupcakes is 1. The overall probability is:

$$\binom{19}{15} \cdot \frac{15! \cdot 25!}{40!} = 9.636 \cdot 10^{-8}$$

   (c) Two students received frosted cupcakes so far, so the probability to get a frosted cupcake is $p = \frac{13}{21} = 0.619$.

      i. If you don't accept the deal, it doesn't matter which cupcake your friend gets. According to the expectation formula, your expected reward is:

$$p \cdot r_f + (1-p) \cdot r_n = 0.619 \cdot r_f + 0.381 \cdot r_n$$

      ii. If you accept the deal, there are three different outcomes: (1) you get a non-frosted cupcake, (2) both you and your friend get frosted cupcakes, and (3) you get a frosted cupcake and your friend doesn't, and you switch cupcakes. The probability of (1) is $p_1 = 0.381$.
      The probability of (2) is the joint probability of you getting a frosted cupcake (0.619) and that only 11 more students after you and before your friend got a frosted cupcake:

$$p_2 = 0.619 \cdot \frac{\binom{19}{11} \cdot 1}{\binom{20}{12}} = 0.619 \cdot \frac{12}{20} = 0.3714$$

The probability of (3) is the joint probability of you getting a frosted cupcake (0.619) and that 12 more students after you and before your friend got a frosted cupcake (and also equals $1 - p_1 - p_2$):

$$p_3 = 0.619 \cdot \frac{\binom{19}{12} \cdot 1}{\binom{20}{12}} = 0.619 \cdot \frac{8}{20} = 0.248$$

The expected reward is:

$$E = p_1 \cdot r_n + p_2 \cdot r_f + p_3 \cdot (r_n + 2) = 0.381 \cdot r_n + 0.371 \cdot r_f + 0.248 \cdot (r_n + 2)$$

$$E = 0.629 \cdot r_n + 0.371 \cdot r_f + 0.496$$

iii. The deal doesn't change the probabilities of you and your friend getting frosted and non-frosted cupcakes, so you should decide according to the rewards, and accept the deal if

$$E(reward_with_deal) > E(reward)$$

$$0.629 \cdot r_n + 0.371 \cdot r_f + 0.496 > 0.619 \cdot r_f + 0.381 \cdot r_n$$

$$r_f < r_n + 2$$

3. (a)

$$E[X+Y] = \sum_{x\in X}\sum_{y\in Y}(x+y)\cdot P(x,y) = \sum_{x\in X}\sum_{y\in Y}(x\cdot P(x,y) + y\cdot P(x,y)) =$$

$$= \sum_{x\in X}\sum_{y\in Y} x\cdot P(x,y) + \sum_{y\in Y}\sum_{x\in X} y\cdot P(x,y) = \sum_{x\in X} x \sum_{y\in Y} P(x,y) + \sum_{y\in Y} y \sum_{x\in X} P(x,y) =$$

Using the definition of marginal probability ($P(x) = \sum_{y\in Y} P(x,y)$):

$$= \sum_{x\in X} x\cdot P(x) + \sum_{y\in Y} y\cdot P(y) = E[X] + E[Y]$$

(b) The expected grade of a student who used Wikipedia was computed in 1 (a): $E[X] = 85$. Alice's probability to get a frosted cupcake is $p = \frac{15}{40} = \frac{3}{8}$, and her expected reward from eating a cupcake is $E[Y] = \frac{3}{8}\cdot 2 + \frac{5}{8}\cdot 1 = \frac{11}{8}$. Her overall expected reward is therefore $E[X+Y] = E[X] + E[Y] = 86.375$.

4. (a) Using the expectation formula:

$$E[X\cdot Y] = \sum_{x\in X}\sum_{y\in Y} x\cdot y\cdot P(x,y)$$

X and Y are independent, so $P(x,y) = P(x)\cdot P(y)$, and:

$$E[X\cdot Y] = \sum_{x\in X}\sum_{y\in Y} x\cdot y\cdot P(x)\cdot P(y) = \sum_{x\in X} x\cdot P(x)\cdot \sum_{y\in Y} y\cdot P(y) = E[X]\cdot E[Y]$$

(b) From set theory:

$$E\cup F = (E\setminus F)\cup(F\setminus E)\cup(E\cap F)$$

With probability:

$$P(E\cup F) = P(E\setminus F) + P(F\setminus E) + P(EF)$$

$$= (P(E) - P(EF)) + (P(F) - P(EF) + P(EF)$$

$$= P(E) + P(F) - P(EF)$$

(c) i. $P(E,F) = P(\{1\}) = \frac{1}{4} = \frac{1}{2}\cdot\frac{1}{2} = P(E)P(F)$ and similarly for E, G and G, F.

ii. They are not mutually independent. For example, $P(E,F,G) = P(\{1\}) = \frac{1}{4} \neq \frac{1}{8} = P(E)P(F)P(G)$.