Applied Probability Models for CS (2019-2020)
Exercise 1

Submission Deadline: November 17, 2019

1. A history teacher gives a 20 question homework assignment. 70% of the students use Wikipedia to do their homework, while the other 30% prefer printed encyclopedias. Each student uses only one source to answer all the questions. Wikipedia's accuracy rate is 85% compared to 95% among printed encyclopedias. Each question is worth 5 points if answered correctly, and zero otherwise.
   (a) What is the expected grade of a student who uses Wikipedia?
   (b) What is the probability that a randomly chosen student got 90?
   (c) Bob got 90 in the assignment. What is the probability that he used Wikipedia?

2. The teacher was amazed by the success his students achieved in the homework assignment, and decided to reward the class by bringing homemade cupcakes. He baked 40 cupcakes, one for each student, but the vanilla frosting was only enough for 15 of them. To make it fair, he put the cupcakes in a box and had each student pick a cupcake without knowing whether it is frosted. You are the 20th student to get a cupcake.
   (a) What is the probability that you will be the first to get a frosted cupcake?
   (b) What is the probability that all the frosted cupcakes were taken already?
   (c) Two students received frosted cupcakes so far. Your rewards from eating a non-frosted and a frosted cupcake are r_n and r_f respectively ($r_n < r_f$). Your friend who is last in the line makes an offer that if you get a frosted cupcake and she doesn't, she will pay you \$2 to switch cupcakes with her.
      i. What is your expected reward if you don't accept the deal?
      ii. What is your expected reward if you accept the deal?
      iii. For which value of r_f should you accept the deal? (write an expression that depends on r_n).

3. Alice used Wikipedia to do her homework, and was the first to get a cupcake. Denote Alice's assignment grade by X, and the reward from eating a cupcake by Y (1 for a non-frosted cupcake and 2 for a frosted cupcake).
   (a) Prove that for any random variables X and Y, E[X + Y] = E[X] + E[Y].
   (b) Compute Alice's expected reward (assuming that rewards are additive).

4. (a) Given that X and Y are independent random variables, prove that E[X · Y] = E[X] · E[Y], where E stands for expectation.
   (b) For events E and F, prove: $P(E \cup F) = P(E) + P(F) - P(EF)$
   (c) A jar contains four numbered balls: $\{1, 2, 3, 4\}$. Let $E = \{1, 2\}, F = \{1, 3\}, G = \{1, 4\}$ be events specifying possible outcomes of picking up one ball randomly.
      i. Show that each pair of events is independent.
      ii. A finite set of events $\{E_1, ..., E_n\}$ is said to be mutually independent iff $P(E_{i_1}, ..., E_{i_k}) = \pi_{j=1}^{k} P(E_{i_j})$ for any subset of events $\{E_{i_1}, ..., E_{i_k}\} \subseteq \{E_1, ..., E_n\}$. Are $\{E, F, G\}$ mutually independent? Prove or show a counter-example.