A NOTE ON THE METHOD OF NYIKOS OF DETECTING SPACES X WITH NON-STRATIFIABLE $C_k(X)$

BOAZ TSABAN AND LYUBOMYR ZDOMSKYY

Abstract. In this note we use the ideas of P. Nyikos to establish some properties of zero-dimensional metrizable spaces X such that $C_k(X)$ is stratifiable. In particular, we prove that $C_k(\mathcal{F})$ is not stratifiable for every ultrafilter $\mathcal{F}$.

In what follows we consider only zero-dimensional metrizable spaces. The main result of this note is the following

Theorem 1. *Let X be a dense Baire subspace of a Polish P such that $C_k(X)$ is stratifiable. Then X is comeager in P.*

The proof of Theorem 1 is based on the subsequent two facts. The first of them is Lemma 29 of Gartside and Reznichenko [1], and the second was established by Nyikos [3].

For a topological space X we shall denote by $\mathcal{O}_*(X)$ and $\mathcal{K}_*(X)$ the families of all nonempty clopen and compact subspaces of X respectively.

Lemma 1. *For a space X the following conditions are equivalent:*

1. *$C_k(X)$ is stratifiable;*
2. *there exist maps $\phi : \mathcal{K}_*(X) \to \mathcal{K}_*(X)$ and $\Phi : \mathcal{O}_*(X) \to \mathcal{K}_*(X)$ such that $\Phi(U) \subset U$ for $U \in \mathcal{O}_*(X)$ and if $V \cap K \neq \emptyset$ then $\phi(K) \cap \Phi(V) \neq \emptyset$ for $V \in \mathcal{O}_*(X)$ and $K \in \mathcal{K}_*(X)$.*

Lemma 2. *Let X be topological space and $\phi : \mathcal{K}_*(X) \to \mathcal{K}_*(X)$ be a map. Assume that there exists a sequence $(W_n)_{n\in\mathbb{N}}$ of clopen subsets of X and a descending sequence $(\mathcal{K}_n)_{n\in\mathbb{N}}$ of collections of compact sets such that $\bigcup_{n\in\mathbb{N}} W_n$ is clopen, and such that $W_n \cap \phi(K) \neq \emptyset$ for all $K \in \mathcal{K}_n$ but $W_n \cap K \neq \emptyset$ for some $K \in \mathcal{K}_i$ for all $i < n$. Then there is no $\Phi : \mathcal{O}_*(X) \to \mathcal{K}_*(X)$ such that the pair (ϕ, Φ) fulfills the requirements of Lemma 1.*

In the proof of Theorem 1 we shall also use the characterization of Baire spaces by means of the Choquet game. We recall from [2] that

Date: January 6, 2008.
Supported by the Koshland Center for Basic Research.

the Choquet game on a topological space X is played by two players, say I and II, as follows: the first player starts the game by choosing an open nonempty subset U_0 of X and the second one responds by a nonempty open subset V_0 of U_0. Then the first player chooses an open nonempty $U_1 \subset V_0$, and the second player responds with a nonempty open $V_1 \subset U_1$, and so on. At the end of the game they construct sequences $(U_n)_{n\in\mathbb{N}}$ and $(V_n)_{n\in\mathbb{N}}$ of open nonempty subsets of X such that

$$U_0 \supset V_0 \supset U_1 \supset V_1 \supset \cdots \supset U_n \supset V_n \supset \cdots .$$

The first player wins, if $\bigcap_{n\in\mathbb{N}} U_n = \emptyset$. It is known [2] that a space X is Baire if and only if the first player has no winning strategy in the Choquet game on it.

In what follows we shall denote by $\mathcal{O}_{**}(X)$ the family of all subspaces of X with nonempty interior.

Proof of Theorem 1. Assume, contrary to our claim, that $Y = P \setminus X$ is not meager, and hence there exists a clopen subset U of P such that $\overline{Y \cap U} = U$ and $Y \cap U$ is a Baire space. Since $C_k(Z)$ is stratifiable for every closed subspace Z of X (see [1, Proposition 27]), there is no loss of generality to assume that $X = U$.

Let $\mathcal{B}$ be a countable clopen base of the the topology of P and $\phi : \mathcal{K}_*(X) \to \mathcal{K}_*(X)$. We shall define two maps $\Upsilon_i : \mathcal{O}(P)_{**} \to \mathcal{B}$ such that

(*i*) $\Upsilon_i(U) \subset U$ for every U and $i \in \{0,1\}$; and
(*ii*) The set $\{x \in \Upsilon_1(U) : \phi\{x\} \cap \Upsilon_0(U) = \emptyset\}$ is a dense Baire subspace of $\Upsilon_1(U)$.

Let us fix $U \in \mathcal{O}_{**}(P)$. Since $\phi\{x\}$ is nowhere dense in $Int(U)$ for all $x \in U$, we can write U as the union $U = \bigcup_{B\in\mathcal{B}, B\subset U} U_B$, where $U_B = \{x \in U : \phi\{x\} \cap B = \emptyset\}$. Then one of the U_B's is nonmeager, and hence there are disjoint $B_0, B_1 \in \mathcal{B}$ such that $U_{B_0} \cap B_1$ is a dense Baire subspace of B_1. Now it suffices to set $\Upsilon_i(U) = B_i$, where $i \in \{0,1\}$.

Next, we shall define a strategy $\mathcal{T}$ of the fist player in the Choquet game on Y as follows: $\mathcal{T}(\emptyset) = \Upsilon_1(P) \cap Y$ and $\mathcal{T}(V_0, \dots, V_n) = \Upsilon_1(\overline{V_n}) \cap Y$ for all $(V_0, \dots, V_n) \in \mathcal{O}_*(Y)^{<\mathbb{N}}$ [overhead bars denote closures in P]. Since Y is a Baire space, $\mathcal{T}$ is not winning, which yields a sequence $(V_n)_{n\in\mathbb{N}}$ of open subsets of Y such that $V_n \subset \mathcal{T}(V_0, \dots, V_{n-1})$ for all $n \in \mathbb{N}$ and $\bigcap_{n\in\mathbb{N}} V_n = \{y_*\}$ for some $y_* \in Y$.

It suffices to note that $W_n = \Upsilon_0(\overline{V_n}) \cap X$ and $K_n = \{\{x\} : x \in \Upsilon_1(\overline{V_n}) \cap X\}$ fulfill the requirements of Lemma 2, which together with Lemma 1 contradicts the stratifiability of $C_k(X)$. □

Corollary 1. *For every ultrafilter $\mathcal{F}$ on $\mathbb{N}$ considered with the topology inherited from the Cantor space $\{0,1\}^{\mathbb{N}}$, the space $C_k(\mathcal{F})$ is not stratifiable.*

In light of [1, Proposition 27(1)] (asserting that if $C_k(X)$ is stratifiable and Y is a closed subspace of X, then $C_k(Y)$ is stratifiable), the subsequent statement is a self-strengthening of Theorem 1.

Corollary 2. *Let X be a subspace of the Baire space $[\mathbb{N}]^{\aleph_0}$. If $C_k(X)$ is stratifiable, then for every closed subspace F of $[\mathbb{N}]^{\aleph_0}$ the intersection $X \cap F$ has the Baire property in F.*

Remark. 1. In the proof of Theorem 1 we used the decreasing families of singletons.

2. It was proven in [3] that if $C_k(X)$ is stratifiable for a σ-compact X, then X is Polish. If ϕ is upper semicontinuous (or, at least, so is the restriction of ϕ to the family of singletons,) by the same methods as in the proof of Theorem 1 one can show that if X is dense in P and there exists Φ such that (ϕ, Φ) are such as in Lemma 1, then X is comeager. Consequently, if the stratifiability of $C_k(X)$ can be characterized in the same way as in Lemma 1 with $\phi|\{\{x\} : x \in X\}$ being upper semicontinuous, then $X \cap F$ is comeager in every closed subset F of a Polish space $P \supset X$ such that $\overline{X \cap F} = F$.

But even if this is true, it is not enough to prove that X is Polish provided $C_k(X)$ is stratifiable. For example, let Y be a perfecrly meager subspace of $[\mathbb{N}]^{\aleph_0}$ (e.g., a $\mathfrak{b}$-scale). Then the above results give no idea how to approach the question whether $C_k([\mathbb{N}]^{\aleph_0} \setminus Y)$ is stratifiable. □

After we finished writing this note, we learned that E. Reznichenko has obtained the same result earlier. Later, Reznichenko improved this to get the following complete result: For a separable metrizable X, $C_k(X)$ is separable if, and only if, X is Polish [4].

References

[1] P. Gartside and E. Reznichenko, *Near metric properties of function spaces*, Fundamenta Mathematicae **164** (2000), 97–114.

[2] A. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics **156**, Springer-Verlag, New York: 1995.

[3] P. Nyikos, *Non-stratifiability of $C_k(X)$ for a class of separable metrizable X*, Topology and its Applications, to appear. `http://www.math.sc.edu/ nyikos/strat.pdf`

[4] E. Reznivhenko, *Stratifiability of $C_k(X)$ for a class of separable metrizable X*, Topology and its Applications, to appear.

Boaz Tsaban, Department of Mathematics, Weizmann Institute of Science, Rehovot 76100, Israel
E-mail address: boaz.tsaban@weizmann.ac.il
URL: http://www.cs.biu.ac.il/~tsaban

Lyubomyr Zdomskyy, Department of Mechanics and Mathematics, Ivan Franko Lviv National University, Universytetska 1, Lviv 79000, Ukraine; and Department of Mathematics, Weizmann Institute of Science, Rehovot 76100, Israel
E-mail address: lzdomsky@rambler.ru