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Hereditarily Hurewicz spaces and Arhangel'skiĭ sheaf amalgamations

Abstract. A classical theorem of Hurewicz characterizes spaces with the Hurewicz covering property as those having bounded continuous images in the Baire space. We give a similar characterization for spaces X which have the Hurewicz property hereditarily.

We proceed to consider the class of Arhangel'skiĭ α_1 spaces, for which every sheaf at a point can be amalgamated in a natural way. Let $C_p(X)$ denote the space of continuous real-valued functions on X with the topology of pointwise convergence. Our main result is that $C_p(X)$ is an α_1 space if, and only if, each *Borel* image of X in the Baire space is bounded. Using this characterization, we solve a variety of problems posed in the literature concerning spaces of continuous functions.

Keywords. Pointwise convergence, point-cofinite covers, α_1, eventual dominance, Hurewicz property, selection principles, QN sets, ideal convergence

Contents

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1. Introduction

We are mainly concerned with spaces X which are (homeomorphic to) sets of irrational numbers, and we recommend adopting this restriction for clarity. Our results (and proofs) apply to all topological spaces X in which each open set is a union of countably many clopen sets, and the spaces considered are assumed to have this property.[1]

Fix a topological space X. Let $\mathscr{A}, \mathscr{B}$ be families of covers of X. The space X may or may not have the following property [35].

$\mathsf{U}_{\text{fin}}(\mathscr{A}, \mathscr{B})$: Whenever $\mathcal{U}_1, \mathcal{U}_2, \ldots \in \mathscr{A}$ and none contains a finite subcover, there exist finite sets $\mathcal{F}_n \subseteq \mathcal{U}_n, n \in \mathbb{N}$, such that $\{\bigcup \mathcal{F}_n : n \in \mathbb{N}\} \in \mathscr{B}$.

Let O denote the collection of all countable open covers of X.[2] A cover $\mathcal{U}$ of X is *point-cofinite* if $\mathcal{U}$ is infinite and each $x \in X$ is a member of all but finitely many members of $\mathcal{U}$.[3] Let Γ denote the collection of all open point-cofinite covers of X. Motivated by studies of Menger [26], Hurewicz [19] introduced the *Hurewicz property* $\mathsf{U}_{\text{fin}}(\text{O}, \Gamma)$.

Hurewicz [19] essentially obtained the following combinatorial characterization of $\mathsf{U}_{\text{fin}}(\text{O}, \Gamma)$ (see Recław [29]). For $f, g \in \mathbb{N}^{\mathbb{N}}$, $f \leq^* g$ means $f(n) \leq g(n)$ for all but finitely many n. A subset Y of $\mathbb{N}^{\mathbb{N}}$ is *bounded* if there is $g \in \mathbb{N}^{\mathbb{N}}$ such that $f \leq^* g$ for all $f \in Y$.

Theorem 1 (Hurewicz). *X satisfies $\mathsf{U}_{\text{fin}}(\text{O}, \Gamma)$ if, and only if, every continuous image of X in $\mathbb{N}^{\mathbb{N}}$ is bounded.*

This characterization has found numerous applications—see [38, 24, 41] and references therein. We give a similar characterization for *hereditarily Hurewicz* spaces, that is, spaces X such that each subspace of X satisfies $\mathsf{U}_{\text{fin}}(\text{O}, \Gamma)$.

The property of being hereditarily Hurewicz was studied in, e.g., [15, 28, 27]. Rubin introduced a property of subsets of $\mathbb{R}$ such that the existence of a set with this property is equivalent to the possibility of a certain construction of boolean algebras [31]. Miller [27] proved that the *Rubin spaces* are exactly the hereditarily Hurewicz spaces.

The property of being hereditarily Hurewicz also manifests itself as follows: A set $X \subseteq \mathbb{R}$ is a *σ' space* [32] if for each F_σ set E, there is an F_σ set F such that $E \cap F = \emptyset$ and $X \subseteq E \cup F$. This property was effectively used in studies of generalized metric spaces [13]. Recently, Sakai proved that X is a σ' space if, and only if, X is hereditarily Hurewicz (Theorem 6 below).

There exist additional classes of hereditarily Hurewicz spaces in the literature. We describe some of them.

A topological space is *Fréchet* if each point in the closure of a subset of the space is a limit of a convergent sequence of points from that subset. The following concepts,

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[1] Every perfectly normal space (open sets are F_σ) with upper inductive dimension 0 (disjoint closed sets can be separated by a clopen set) has the required property. Thus, the spaces considered in the references also have the required property.

[2] If X is Lindelöf, we can consider arbitrary open covers of X.

[3] Traditionally, point-cofinite covers were called γ-*covers* [17].

due to Arhangel'skiĭ [1, 2], are important in determining when a product of Fréchet spaces is Fréchet. Let Y be a general topological space (not necessarily Lindelöf or zero-dimensional). A *sheaf* at a point $y \in Y$ is a family of sequences, each converging to y. To avoid trivialities, we consider only sequences of distinct elements. We say that a countable set A converges to y if some (equivalently, each) bijective enumeration of A converges to y. The space Y is an α_1 *space* if for each $y \in Y$, each countable sheaf $\{A_n : n \in \mathbb{N}\}$ at y can be amalgamated as follows: There are cofinite subsets $B_n \subseteq A_n$, $n \in \mathbb{N}$, such that the set $B = \bigcup_n B_n$ converges to y. The references dealing with α_1 spaces are too numerous to be listed here; see [40] and the references therein for a partial list.

Fix a space X. Denote by $C_p(X)$ the family of all continuous real-valued functions on X, viewed as a subspace of the Tychonoff product $\mathbb{R}^X$. A sequence of results by Bukovský–Recław–Repický [10], Recław [30], Sakai [33], and Bukovský–Haleš [9], culminated in the result that if $C_p(X)$ is an α_1 space, then X is hereditarily Hurewicz. Our main result is that if $C_p(X)$ is an α_1 space, then each Borel image of X in $\mathbb{N}^{\mathbb{N}}$ is bounded. It is easy to see that the converse implication also holds, and we obtain a powerful characterization of spaces X such that $C_p(X)$ is an α_1 space.

Historically, the realization that if $C_p(X)$ is an α_1 space then X is hereditarily Hurewicz goes through QN spaces [10]: Let Y be a metric space. A function $f : X \to Y$ is a *quasi-normal limit* of functions $f_n : X \to Y$ if there are positive reals ϵ_n, $n \in \mathbb{N}$, converging to 0 such that for each $x \in X$, $d(f_n(x), f(x)) < \epsilon_n$ for all but finitely many n. A topological space X is a *QN space* if whenever 0 is a pointwise limit of a sequence of continuous real-valued functions on X, then 0 is a quasi-normal limit of that sequence. QN spaces are studied in, e.g., [10, 30, 36, 28, 11, 33, 9]. In [33, 9] it was shown that X is a QN space if, and only if, $C_p(X)$ is an α_1 space. Thus, QN spaces are also characterized by having bounded Borel images in $\mathbb{N}^{\mathbb{N}}$.

We use our main theorem to show that quite a few additional properties studied in the literature are equivalent to having bounded Borel images in $\mathbb{N}^{\mathbb{N}}$, and consequently solve a variety of problems posed in the literature. To make the paper self-contained and accessible to a wide audience, we supply proofs for all needed results. Often, our proofs of known results are slightly simpler than those available in the literature.

2. A characterization of hereditarily Hurewicz spaces

Let $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ be the one-point compactification of $\mathbb{N}$, and endow $\overline{\mathbb{N}}^{\mathbb{N}}$ with the Tychonoff product topology. An element $f \in \overline{\mathbb{N}}^{\mathbb{N}}$ is *eventually finite* if there is m such that $f(n) < \infty$ for each $n \geq m$. Let $\mathbb{EF}$ be the subspace of $\overline{\mathbb{N}}^{\mathbb{N}}$ consisting of all eventually finite elements of $\overline{\mathbb{N}}^{\mathbb{N}}$. The relation $\leq^*$ extends to $\mathbb{EF}$ in the natural way.

Theorem 2. *X is hereditarily $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$ if, and only if, every continuous image of X in $\mathbb{EF}$ is bounded.*

Proof. ($\Rightarrow$) Assume that $\Psi : X \to \mathbb{EF}$ is continuous. For each n, define the following (G_δ) subset of $\Psi[X]$:

$$G_n = \{f \in \Psi[X] : (\forall m \geq n)\ f(m) < \infty\}.$$

Let $T^n : G_n \to \mathbb{N}^{\mathbb{N}}$ be the shift transformation defined by $T^n(f)(m) = f(m + n)$ for all m.

For each n, $X_n = \Psi^{-1}[G_n] \subseteq X$, and therefore X_n satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$. By Theorem 1, $T^n[\Psi[X_n]] = T^n[G_n]$ is a bounded subset of $\mathbb{N}^{\mathbb{N}}$. Thus, G_n is a bounded subset of $\mathbb{EF}$, and therefore so is $\Psi[X] = \bigcup_n G_n$.

($\Leftarrow$) First, note that Hurewicz's Theorem 1 and our assumption on X imply that X satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$.

Lemma 3. *If each G_δ subset of X satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$, then X is hereditarily $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$.*

Proof. Let $Y \subseteq X$. Assume that $\mathcal{U}_n$, $n \in \mathbb{N}$, are covers of Y by open subsets of X, which do not contain finite subcovers. Each $\mathcal{U}_n$ is an open cover of $G = \bigcap_n \bigcup \mathcal{U}_n \supseteq Y$, and has no finite subcover of G. As G is a G_δ subset of X, it satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$. Thus, there are finite $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\{\bigcup \mathcal{F}_n : n \in \mathbb{N}\}$ is a point-cofinite cover of G, and therefore of Y. $\square$

Assume that G is a G_δ subset of X.

Lemma 4 (Sakai [33]). *For each G_δ subset G of X, there is an open point-cofinite cover $\{U_n : n \in \mathbb{N}\}$ of X such that $G = \bigcap_n U_n$.*

Proof. We have $G^{\mathsf{c}} = \bigcup_n C_n$ with each C_n closed. If A is closed and B is open, then B is a union of countably many disjoint clopen sets, and therefore $A \cap B$ is a union of countably many disjoint closed sets. Thus, each of the disjoint sets $C_n \setminus (C_1 \cup \cdots \cup C_{n-1})$, $n \in \mathbb{N}$, is a union of countably many disjoint closed sets. Hence, $G^{\mathsf{c}} = \bigcup_n \tilde{C}_n$ where the sets $\tilde{C}_n$ are closed and disjoint, and therefore $G = \bigcap_n \tilde{C}_n^{\mathsf{c}}$, where $\{\tilde{C}_n^{\mathsf{c}} : n \in \mathbb{N}\}$ is an open point-cofinite cover of X. $\square$

So, let $\{U_n : n \in \mathbb{N}\}$ be an open point-cofinite cover of X such that $G = \bigcap_n U_n$. For each n, let $U_n = \bigcup_m C_m^n$, a union of disjoint clopen sets. Define $\Psi : X \to \mathbb{EF}$ by

$$\Psi(x)(n) = \begin{cases} m, & m \in \mathbb{N}, \ x \in C_m^n, \\ \infty, & x \notin U_n. \end{cases}$$

As $\{U_n : n \in \mathbb{N}\}$ is a point-cofinite cover of X, $\Psi(x)$ is eventually finite for each $x \in X$.

Ψ is continuous: A basic open set in $\mathbb{EF}$ has the form $\prod_n V_n$ such that there are finite $I_0, I_1 \subseteq \mathbb{N}$ and elements m_n, $n \in I_0 \cup I_1$, for which: For each $n \in I_0$, $V_n = \{m_n\}$, for each $n \in I_1$, $V_n = \{m_n, m_n + 1, \ldots\} \cup \{\infty\}$, and for each $n \notin I_0 \cup I_1$, $V_n = \mathbb{N} \cup \{\infty\}$. Now,

$$\Psi^{-1}\Big[\prod_{n \in \mathbb{N}} V_n\Big] = \bigcap_{n \in I_0} C_{m_n}^n \cap \bigcap_{n \in I_1} \Big(X \setminus \Big(\bigcup_{k < m_n} C_k^n\Big)\Big)$$

is open.

Thus, $\Psi[X]$ is bounded by some $g \in \mathbb{N}^{\mathbb{N}}$. Now,

$$G = \{x \in X : (\forall n) \ \Psi(x)(n) < \infty\} = \Psi^{-1}[\{f \in \mathbb{N}^{\mathbb{N}} : f \leq^* g\}].$$

The set $\{f \in \mathbb{N}^{\mathbb{N}} : f \leq^* g\}$ is an F_σ subset of $\mathbb{EF}$. Indeed, let $\{g_n : n \in \mathbb{N}\}$ enumerate all elements of $\mathbb{N}^{\mathbb{N}}$ which are eventually equal to g. Then $\{f \in \mathbb{N}^{\mathbb{N}} : f \leq^* g\} =$

$\bigcup_n \{f \in \mathbb{EF} : f \leq g_n\}$. Thus, G is an F_σ subset of X. As $\mathsf{U}_{\text{fin}}(O, \Gamma)$ is hereditary for closed subsets and preserved by countable unions, G satisfies $\mathsf{U}_{\text{fin}}(O, \Gamma)$. □

Recall that a topological space X is a *σ space* if each G_δ subset of X is an F_σ subset of X. The proof of Theorem 2 actually shows that $(2) \Rightarrow (3)$, $(2) \Rightarrow (1)$, $(3) \Rightarrow (4)$, and $(4) \Rightarrow (2)$ in the following theorem (and therefore establishes it).

Theorem 5. *The following are equivalent:*

(1) *X is hereditarily $\mathsf{U}_{\text{fin}}(O, \Gamma)$.*
(2) *Each G_δ subset of X satisfies $\mathsf{U}_{\text{fin}}(O, \Gamma)$.*
(3) *Every continuous image of X in $\mathbb{EF}$ is bounded.*
(4) *X satisfies $\mathsf{U}_{\text{fin}}(O, \Gamma)$ and is a σ space.* □

The implication $(1) \Rightarrow (4)$ in Theorem 5 was first proved by Fremlin and Miller [15]. The implication $(4) \Rightarrow (1)$ can be alternatively deduced from Theorem 3.12 of [11] and Corollary 10 of [8]. An additional equivalent formulation was discovered by Sakai. Recall the definition of σ' space from the introduction (page 354).

Theorem 6 (Sakai). *Let $X \subseteq \mathbb{R}$. Then X is a σ' space if, and only if, X is hereditarily $\mathsf{U}_{\text{fin}}(O, \Gamma)$.*

Proof. This follows from Theorem 5.7 of [22]: X satisfies $\mathsf{U}_{\text{fin}}(O, \Gamma)$ if, and only if, for each G_δ set $G \subseteq \mathbb{R}$ containing X, there is an F_σ set $F \subseteq \mathbb{R}$ such that $X \subseteq F \subseteq G$.

($\Rightarrow$) As being a σ' space is hereditary, it suffices to show that X satisfies $\mathsf{U}_{\text{fin}}(O, \Gamma)$. Indeed, for each G_δ set $G \subseteq \mathbb{R}$ containing X, let $E = \mathbb{R} \setminus G$, and take an F_σ set $F \subseteq \mathbb{R}$ disjoint from E such that $X \subseteq E \cup F$. Then $X \subseteq F \subseteq G$.

($\Leftarrow$) Let $E \subseteq \mathbb{R}$ be F_σ. As $X \setminus E$ satisfies $\mathsf{U}_{\text{fin}}(O, \Gamma)$ and is a subset of the G_δ set $\mathbb{R} \setminus E$, there is an F_σ set $F \subseteq \mathbb{R}$ such that $X \setminus E \subseteq F \subseteq \mathbb{R} \setminus E$. Then $E \cap F = \emptyset$ and $X \subseteq E \cup F$. □

To indicate the potential usefulness of Theorem 2, we use it to give slightly more direct proofs of two known theorems. Recall the definition of QN spaces from the introduction (page 355).

Theorem 7 (Recław [30]). *If X is a QN space, then X is hereditarily $\mathsf{U}_{\text{fin}}(O, \Gamma)$.*

Proof. Let $Y \subseteq \mathbb{EF}$ be a continuous image of X. Then Y is a QN space. By Theorem 2, it suffices to show that Y is bounded. For each n, and each $y \in Y$, define

$$f_n(y) = \frac{1}{\min y^{-1}(n)},$$

using the natural conventions that $\min \emptyset = \infty$ and $1/\infty = 0$. Then $\lim_n f_n(y) = 0$ for all $y \in Y$. As Y is a QN space, there are positive ϵ_n, $n \in \mathbb{N}$, dominating this convergence. For each k, let

$$Y_k = \{y \in Y : (\forall n \geq k) \ f_n(y) < \epsilon_n\}.$$

Then $Y = \bigcup_k Y_k$. We will show that each Y_k is bounded.

Fix k. Take an increasing $g \in \mathbb{N}^{\mathbb{N}}$ such that $g(1) = k$ and for each n, $\epsilon_m < 1/n$ for all $m \geq g(n)$. Then Y_k is bounded by g: Let $y \in Y_k$. Fix n such that $y(n) < \infty$. If $y(n) \leq k$, then $y(n) \leq g(1) \leq g(n)$. Otherwise, $y(n) > k$, and since $y \in Y_k$, $f_{y(n)}(y) < \epsilon_{y(n)}$. Thus,

$$\frac{1}{n} \leq \frac{1}{\min y^{-1}(y(n))} = f_{y(n)}(y) < \epsilon_{y(n)},$$

and therefore $y(n)$ cannot be greater than $g(n)$.   □

Theorem 8 (Recław [30]). *If X is a QN space, then X is a σ space.*

Proof. Apply Theorems 5 and 7.   □

The following sections give a deeper reason for the last two theorems.

3. Bounded Borel images

Our main goal in this section is to establish the equivalence in the following Theorem 9. The implication (2)⇒(1) in this theorem is Proposition 9 of Scheepers [36]. The implication (1)⇒(2) is the more difficult one, and will be proved in a sequence of related results.

Theorem 9. *The following are equivalent:*

(1) $C_p(X)$ *is an α_1 space.*
(2) *Each Borel image of X in $\mathbb{N}^{\mathbb{N}}$ is bounded.*

Proof. (2)⇒(1) Consider a sheaf $\{A_n : n \in \mathbb{N}\}$ at $f \in C_p(X)$. For each n, enumerate $A_n = \{f_m^n : m \in \mathbb{N}\}$ bijectively. Define a Borel function $\Psi : X \to \mathbb{N}^{\mathbb{N}}$ by

$$\Psi(x)(n) = \min\{k : (\forall m \geq k)\, |f_m^n(x) - f(x)| \leq 1/n\}.$$

Let $g \in \mathbb{N}^{\mathbb{N}}$ bound $\Psi[X]$, and take the amalgamation $B = \bigcup_n \{f_m^n : m \geq g(n)\}$. Then B converges to f.

(1)⇒(2) Assume that $C_p(X)$ is an α_1 space. Then the subspace $C_p(X, \{0, 1\})$ of $C_p(X)$, consisting of all continuous functions $f : X \to \{0, 1\}$, is an α_1 space.[4] Each element of $C_p(X, \{0, 1\})$ has the form χ_U, the characteristic function of a clopen set $U \subseteq X$. Immediately from the definition, a sequence χ_{U_n} of elements of $C_p(X, \{0, 1\})$ converges pointwise to the constant function 1 if, and only if, $\{U_n : n \in \mathbb{N}\}$ is a clopen point-cofinite cover of X. This gives the following, which is due to Bukovský–Haleš (cf. [9, Theorem 17]), and independently Sakai (cf. [33, Theorem 3.7]).

Lemma 10. *The following are equivalent:*

(1) $C_p(X, \{0, 1\})$ *is an α_1 space.*
(2) *For each family $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of pairwise disjoint clopen point-cofinite covers of X, there are cofinite $\mathcal{V}_n \subseteq \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\bigcup_n \mathcal{V}_n$ is a point-cofinite cover of X.*   □

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[4] In fact, by the methods of Gerlits–Nagy [18], the converse implication also holds. This fact will not be used in our proof.

A function f with domain X is a *discrete limit* of functions f_n, $n \in \mathbb{N}$, if for each $x \in X$, $f_n(x) = f(x)$ for all but finitely many n.

Each bijectively enumerated family $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of subsets of a set X induces a *Marczewski map* $\mathcal{U} : X \to P(\mathbb{N})$ defined by

$$\mathcal{U}(x) = \{n \in \mathbb{N} : x \in U_n\}$$

for each $x \in X$. The main step in our proof is the following.

Lemma 11. *Assume that $C_p(X, \{0, 1\})$ is an α_1 space, and $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ is a bijectively enumerated family of open subsets of X. Then the Marczewski map $\mathcal{U} : X \to P(\mathbb{N})$ is a discrete limit of continuous functions.*

Proof. First, consider the case where for each n, U_n is not clopen.

For each n, write U_n as a union $\bigcup_m C_m^n$ of nonempty disjoint clopen sets. We may assume that the partitions are disjoint: Inductively, for each $n = 2, 3, \ldots$, consider the elements C_m^n, $m \in \mathbb{N}$, of the nth partition. For each m, if C_m^n appears in the partition of U_k for some $k < n$, merge (in the nth partition) C_m^n with some other element of the nth partition. Continue in this manner until the nth partition is disjoint from all previous partitions.

Thus, the families $\mathcal{U}_n = \{(C_m^n)^{\mathrm{c}} : m \in \mathbb{N}\}$ are disjoint clopen point-cofinite covers of X. By Lemma 10, there are k_n, $n \in \mathbb{N}$, and subsets $\mathcal{V}_n = \{(C_m^n)^{\mathrm{c}} : m \geq k_n\} \subseteq \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\bigcup_n \mathcal{V}_n$ is a point-cofinite cover of X. In other words,

$$\mathcal{V} = \left\{ \bigcap_{m=k_n}^{\infty} (C_m^n)^{\mathrm{c}} : n \in \mathbb{N} \right\}$$

is a point-cofinite cover of X.

For each n, m, let

$$U_m^n = \bigcup_{i=1}^{\max\{m, k_n\}} C_i^n.$$

For each m, define $\Psi_m : X \to P(\mathbb{N})$ by

$$\Psi_m(x) = \{n : x \in U_m^n\}.$$

As each U_m^n is clopen, Ψ_m is continuous. It remains to prove that, viewed as a Marczewski map, $\mathcal{U}$ is a discrete limit of the maps Ψ_m, $m \in \mathbb{N}$.

Fix $x \in X$. Let N be such that $x \in \bigcap_{m=k_n}^{\infty} (C_m^n)^{\mathrm{c}}$ for all $n \geq N$. For each $n < N$ with $x \in U_n$, let m_n be such that $x \in U_{m_n}^n$. Set $M = \max\{m_n : n < N\}$.

Fix $m \geq M$. We show that $x \in U_m^n$ if, and only if, $x \in U_n$. One direction follows from $U_m^n \subseteq U_n$. To prove the other direction, assume that $x \in U_n$, and consider the two possible cases: If $n < N$, then $x \in U_m^n$ because $m \geq M \geq m_n$, and we are done. Thus, assume that $n \geq N$. Then $x \in \bigcap_{i=k_n}^{\infty} (C_i^n)^{\mathrm{c}}$. As $x \in U_n = \bigcup_m C_m^n$, it follows that $x \in \bigcup_{i=1}^{k_n-1} C_i^n \subseteq U_m^n$.

Thus, for each $x \in X$ there is M such that $\Psi_m(x) = \mathcal{U}(x)$ for all $m \geq M$. This completes the proof in the case that no U_n is clopen.

For the remaining case, let $I \subseteq \mathbb{N}$ be the set of all n such that U_n is not clopen. The previous case shows that $\mathcal{U}_I = \{U_n : n \in I\}$, viewed as a Marczewski function from X to $P(I)$, is a discrete limit of continuous functions $\Psi_m : X \to P(I)$.

For each m, define $\Phi_m : X \to P(\mathbb{N})$ by

$$\Phi_m(x) = \{n : (n \in I \text{ and } n \in \Psi_m(x)) \text{ or } (n \notin I \text{ and } x \in U_n)\}.$$

Then $\mathcal{U}$ is a discrete limit of the continuous functions Φ_m, $m \in \mathbb{N}$. □

As X satisfies item (2) of Lemma 10, it satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$: Refine each given cover to a clopen cover, turn it to a clopen point-cofinite cover by taking finite unions, and make the point-cofinite covers disjoint.

Assume that $\mathcal{U}$ is a countable family of open subsets of X. By Lemma 11, the Marczewski map $\mathcal{U} : X \to P(\mathbb{N})$ is a discrete limit of continuous functions Ψ_n.

Clearly, every discrete limit is a quasi-normal limit. The proof of [10, Theorem 4.8] actually establishes the following.

Lemma 12. *Assume that P is a property of topological spaces, which is preserved by taking closed subsets, continuous images and countable unions. If X has the property P and $\Psi : X \to Y$ is a quasi-normal limit of continuous functions into a metric space Y, then $\Psi[X]$ has the property P.*

Proof. Let Ψ_n, $n \in \mathbb{N}$, be continuous functions as in the premise of the lemma, and let ϵ_n, $n \in \mathbb{N}$, be as in the definition of quasi-normal convergence. For each k,

$$X_k = \{x \in X : (\forall n, m \geq k)\, d(\Psi_n(x),\, \Psi_m(x)) \leq \epsilon_n + \epsilon_m\}$$

is a closed subset of X, and the functions Ψ_n converge to Ψ uniformly on X_k. Thus, Ψ is continuous on X_k, and therefore $\Psi[X_k]$ has the property P.

Now, $X = \bigcup_k X_k$, and therefore $\Psi[X] = \bigcup_k \Psi[X_k]$ has the property P. □

It follows that for each countable family $\mathcal{U}$ of open subsets of X, $\mathcal{U}[X]$ satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$.

Let F, B denote the families of all countable closed and all countable Borel covers of X, respectively. Similarly, let F_Γ, B_Γ denote the families of all countable closed and all Borel point-cofinite covers of X. Following is a striking result of Bukovský, Recław, and Repický [10]. In their terminology, it tells that the family of closed subsets of X is weakly distributive if, and only if, so is the family of Borel subsets of X. In the language of selection principles, this result has the following compact form.

Lemma 13 (Bukovský–Recław–Repický [10]). $\mathsf{U}_{\mathrm{fin}}(\mathrm{F}, \mathrm{F}_\Gamma) = \mathsf{U}_{\mathrm{fin}}(\mathrm{B}, \mathrm{B}_\Gamma)$.

Proof. Assume that X satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{F}, \mathrm{F}_\Gamma)$. We first show that X is a σ space [10, Theorem 5.2].

Assume that $G = \bigcap_n U_n$ where for each n, $U_n \supseteq U_{n+1}$ are open subsets of X. Write, for each n,

$$U_n = \bigcup_{m \in \mathbb{N}} C_m^n,$$

where for each m, $C_m^n \subseteq C_{m+1}^n$ are closed subsets of X. We may assume that the closed

cover $\{C_m^n \cup (X \setminus U_n) : m \in \mathbb{N}\}$ of X has no finite subcover.[5] As X satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{F}, \mathrm{F}_\Gamma)$ and each given cover is monotone, there are $m_n, n \in \mathbb{N}$, such that $\{C_{m_n}^n \cup (X \setminus U_n) : n \in \mathbb{N}\}$ is a closed point-cofinite cover of X. For each k define

$$Z_k = \bigcap_{n=k}^{\infty} C_{m_n}^n.$$

Then each Z_k is a closed subset of X, and $G = \bigcup_k Z_k$ is F_σ. This shows that X is a σ space.

Now, assume that $\mathcal{U}_n \in \mathrm{B}$, $n \in \mathbb{N}$. Then for each n, each element of $\mathcal{U}_n$ is F_σ and can therefore be replaced by countably many closed sets. Applying $\mathsf{U}_{\mathrm{fin}}(\mathrm{F}, \mathrm{F}_\Gamma)$ to the thus modified covers, we obtain a cover in F_Γ. For each n, extend each of the finitely many chosen elements of the nth cover to an F_σ set from the original cover $\mathcal{U}_n$, to obtain an element of B_Γ chosen in accordance with the definition of $\mathsf{U}_{\mathrm{fin}}(\mathrm{B}, \mathrm{B}_\Gamma)$.[6] □

Lemma 14. *The following are equivalent:*

(1) *X satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{B}, \mathrm{B}_\Gamma)$.*
(2) *For each countable family $\mathcal{U}$ of open subsets of X, $\mathcal{U}[X]$ satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$.*
(3) *For each countable family $\mathcal{C}$ of closed subsets of X, $\mathcal{C}[X]$ satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$.*

Proof. (2)⇔(3) Use the auto-homeomorphism of $P(\mathbb{N})$ defined by mapping a set to its complement.

(1)⇒(2) The Marczewski map $\mathcal{U} : X \to P(\mathbb{N})$ is Borel. It is easy to see that $\mathsf{U}_{\mathrm{fin}}(\mathrm{B}, \mathrm{B}_\Gamma)$ is preserved by Borel images [39]. Thus, $\mathcal{U}[X]$ satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{B}, \mathrm{B}_\Gamma)$, and in particular $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$.

(3)⇒(1) By Lemma 13, it suffices to show that X satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{F}, \mathrm{F}_\Gamma)$. For each $\mathcal{C} = \{C_n : n \in \mathbb{N}\} \in \mathrm{F}$ which does not contain a finite subcover, $\{\bigcup_{m \leq n} C_m : n \in \mathbb{N}\} \in \mathrm{F}_\Gamma$. Thus, $\mathsf{U}_{\mathrm{fin}}(\mathrm{F}, \mathrm{F}_\Gamma) = \mathsf{U}_{\mathrm{fin}}(\mathrm{F}_\Gamma, \mathrm{F}_\Gamma)$,[7] and we prove the latter property.

Let $\mathcal{C}_n = \{C_m^n : m \in \mathbb{N}\}$, $n \in \mathbb{N}$, be bijectively enumerated closed point-cofinite covers of X which do not contain finite subcovers. We may assume that these covers are pairwise disjoint [35].

Let $\mathcal{C} = \bigcup_n \mathcal{C}_n$, and consider the Marczewski map $\mathcal{C} : X \to P(\mathbb{N} \times \mathbb{N})$ defined by

$$\mathcal{C}(x) = \{(n, m) : x \in C_m^n\}$$

for all $x \in X$. For each (n, m), $O_{(n,m)} = \{A \subseteq \mathbb{N} \times \mathbb{N} : (n, m) \in A\}$ is an open subset of $P(\mathbb{N} \times \mathbb{N})$, and for each n, $\mathcal{U}_n = \{O_{(n,m)} : m \in \mathbb{N}\}$ is an open cover of $\mathcal{C}[X]$ which does not contain a finite subcover. As $\mathcal{C}[X]$ satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$, there are $k_n, n \in \mathbb{N}$, such that $\{\bigcup_{m<k_n} O_{(n,m)} : n \in \mathbb{N}\}$ is a point-cofinite cover of $\mathcal{C}[X]$. Then $\{\bigcup_{m<k_n} C_m^n : n \in \mathbb{N}\}$ is a point-cofinite cover of X (it is infinite because X does not appear there as an element).[8] □

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[5] If there are infinitely many n for which there is some m_n with $C_{m_n}^n = U_n$, then $G = \bigcap_n C_{m_n}^n$ is closed and we are done. Otherwise, we can ignore finitely many n and assume that there are no n, m such that $C_m^n = U_n$ contains G.

[6] This argument, in more general form, appears in [11, Theorem 2.1].

[7] This statement holds in a more general form [22].

[8] The argument is standard: For each finite family of proper subsets of X, there is a finite subset of X not contained in any member of this family.

By Lemma 14, X satisfies $\mathsf{U}_{\mathrm{fin}}(B, B_\Gamma)$. It remains to observe the following. For the reader's convenience, we reproduce the proof of the implication needed in the present proof.

Lemma 15 (Bartoszyński–Scheepers [4]). *X satisfies $\mathsf{U}_{\mathrm{fin}}(B, B_\Gamma)$ if, and only if, each Borel image of X in $\mathbb{N}^{\mathbb{N}}$ is bounded.*

Proof. ($\Rightarrow$) Assume that $Y \subseteq \mathbb{N}^{\mathbb{N}}$ is a Borel image of X. Then Y satisfies $\mathsf{U}_{\mathrm{fin}}(B, B_\Gamma)$. By taking the image of Y under the continuous mapping $f(n) \mapsto f(1) + \cdots + f(n)$ defined on $\mathbb{N}^{\mathbb{N}}$, we may assume that all elements in Y are nondecreasing.

We first consider the uninteresting case: There is an infinite $I \subseteq \mathbb{N}$ such that for each $n \in I$, $F_n = \{f(n) : f \in Y\}$ is finite. For each n, let $m \in I$ be minimal such that $n \leq m$, and define $g(n) = \max F_m$. Then Y is bounded by g.

Thus, assume that there is N such that for each $n \geq N$, $\{f(n) : f \in Y\}$ is infinite. For all n, m, consider the open set $U_m^n = \{f \in Y : f(n) \leq m\}$. Then for each $n \geq N$, $\mathcal{U}_n = \{U_m^n : m \in \mathbb{N}\}$ is an open point-cofinite cover of Y. Apply $\mathsf{U}_{\mathrm{fin}}(B, B_\Gamma)$ to obtain for each $n \geq N$ a finite set $F_n \subseteq \mathbb{N}$ such that $\{\bigcup_{m \in F_n} U_m^n : n \in \mathbb{N}\}$ is a point-cofinite cover of Y. Define $g \in \mathbb{N}^{\mathbb{N}}$ by $g(n) = \max F_n$ for each $n \geq N$ (and arbitrary for $n < N$). Then Y is bounded by g.   □

This completes the proof of Theorem 9.   □

Remark 16. Let $A \subseteq^* B$ mean that $A \setminus B$ is finite. A *semifilter* is a family $\mathcal{F}$ of infinite subsets of $\mathbb{N}$ such that for each $A \in \mathcal{F}$ and each $B \subseteq \mathbb{N}$ with $A \subseteq^* B$, we have $B \in \mathcal{F}$. In [43] it is proved that if in item (2) of Lemma 14 we replace $\mathcal{U}[X]$ with the semifilter it generates, then we obtain a characterization of $\mathsf{U}_{\mathrm{fin}}(O, \Gamma)$. Lemma 14 shows that moving to the generated semifilter is essential to obtain this result, since $\mathsf{U}_{\mathrm{fin}}(B, B_\Gamma)$ is strictly stronger than $\mathsf{U}_{\mathrm{fin}}(O, \Gamma)$.

4. Applications

4.1. QN spaces

We begin with a straightforward proof of one implication in the following theorem (which answers in the affirmative Problem 2 of Scheepers [36]). Because of the importance of this result, we also supply a proof for the other implication.

Theorem 17 (Sakai [33], Bukovský–Haleš [9]). *X is a QN space if, and only if, $C_p(X)$ is an α_1 space.*

Proof. ($\Leftarrow$) This is Theorem 4 of [36]. Using Theorem 9 this becomes straightforward: Assume that $C_p(X)$ is an α_1 space. Given $f_n, n \in \mathbb{N}$, converging pointwise to 0, define a Borel function $\Psi : X \to \mathbb{N}^{\mathbb{N}}$ by

$$\Psi(x)(n) = \min\{k : (\forall m \geq k) \, |f_m(x)| < 1/n\}.$$

By Theorem 9, $\Psi[X]$ is bounded by some $g \in \mathbb{N}^{\mathbb{N}}$. For each $x \in X$ and all but finitely many n, $|f_m(x)| < 1/n$ for each $m \geq g(n)$. For each n and each m with $g(n) \leq m < g(n+1)$, take $\epsilon_m = 1/n$.

($\Rightarrow$) Assume that X is a QN space, and $\{A_n : n \in \mathbb{N}\}$ is a countable sheaf at $f \in C_p(X)$. We may assume that f is the constant zero function, and that the image of each member of each A_n is contained in the unit interval $[0, 1]$.

For each n, enumerate $A_n = \{f_m^n : m \in \mathbb{N}\}$ bijectively. For each m, define $g_m \in C_p(X)$ by

$$g_m(x) = \sup\{f_m^n(x)/n : n \in \mathbb{N}\}$$

for all $x \in X$. Then $\{g_m : m \in \mathbb{N}\}$ converges pointwise to the zero function. As X is a QN space, there are positive ϵ_m, $m \in \mathbb{N}$, converging to 0, such that X is the increasing union of the sets

$$X_n = \{x \in X : (\forall m \geq n)\, g_m(x) \leq \epsilon_m\}.$$

For each n, choose m_n such that $n\epsilon_m \leq 1/n$ for all $m \geq m_n$. We claim that the amalgamation $B = \bigcup_n \{f_m^n : m \geq m_n\}$ converges pointwise to the zero function. Indeed, fix $x \in X$ and a positive ϵ. Take N such that x belongs to X_N (and thus to all X_k with $k \geq N$) and such that $1/N \leq \epsilon$. For each $n \geq N$ and each $m \geq m_n$,

$$f_m^n(x) \leq n \cdot g_m(x) \leq n\epsilon_m \leq 1/n \leq \epsilon.$$

And for each $n < N$, there are only finitely many m such that $f_m^n(x) > \epsilon$. Thus, for all but finitely many $f \in B$, $f(x) \leq \epsilon$. □

A beautiful direct (but tricky) proof for ($\Leftarrow$) of Theorem 17 was recently discovered by Bukovský [7].

Theorems 9 and 17 solve in the affirmative Problem 22 from [9].

Corollary 18. *X is a QN space if, and only if, each Borel image of X in $\mathbb{N}^{\mathbb{N}}$ is bounded.*

□

Theorem 19 (Recław [30]). *The QN property is hereditary.*

Proof. The property of having bounded Borel images in $\mathbb{N}^{\mathbb{N}}$ is hereditary. □

Answering Question 5.8 of Shakhmatov [40] (attributed to Scheepers), Sakai [33] and independently Bukovský–Haleš [9] gave a characterization of the QN property in terms of covering properties of X. Their characterization uses the new *Kočinac α_1 selection principle* [25]. Theorem 15 and Corollary 18 give a new characterization in terms of the classical Hurewicz selection principle: $\mathsf{U}_{\mathrm{fin}}(\mathsf{B}, \mathsf{B}_\Gamma)$. This selection hypothesis can be stated in a more elegant manner. For families of covers $\mathscr{A}$, $\mathscr{B}$ of X, define

$\mathsf{S}_1(\mathscr{A}, \mathscr{B})$: Whenever $\mathcal{U}_1, \mathcal{U}_2, \ldots \in \mathscr{A}$, there exist elements $U_n \in \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\{U_n : n \in \mathbb{N}\} \in \mathscr{B}$.

Then $\mathsf{U}_{\mathrm{fin}}(\mathsf{B}, \mathsf{B}_\Gamma) = \mathsf{S}_1(\mathsf{B}_\Gamma, \mathsf{B}_\Gamma)$ [39]. By Lemma 13, also $\mathsf{U}_{\mathrm{fin}}(\mathsf{F}, \mathsf{F}_\Gamma) = \mathsf{S}_1(\mathsf{F}_\Gamma, \mathsf{F}_\Gamma)$. (This can also be proved directly.) We obtain the following new characterizations.

Corollary 20. *The following are equivalent:*

(1) *$C_p(X)$ is an α_1 space.*
(2) *X is a QN space.*
(3) *X satisfies $\mathsf{S}_1(\mathsf{F}_\Gamma, \mathsf{F}_\Gamma)$.*
(4) *X satisfies $\mathsf{S}_1(\mathsf{B}_\Gamma, \mathsf{B}_\Gamma)$.* □

4.2. Convergent sequences of Borel functions

Let $B_p(X)$ be the space of all *Borel* real-valued functions on X, with the topology of pointwise convergence. We obtain the surprising result that if $C_p(X)$ is an α_1 space, then so is $B_p(X)$. This is *not* provably the case for Arhangel'skiĭ's properties α_2, α_3, and α_4. A topological space Y is an α_2 *space* if it satisfies $S_1(\Gamma_y, \Gamma_y)$ for each $y \in Y$, where Γ_y is the family of all sequences converging to y. For the definitions of α_3 and α_4, see e.g. [36].

Corollary 21. *The following are equivalent:*

(1) *Each Borel image of X in $\mathbb{N}^{\mathbb{N}}$ is bounded.*
(2) *$C_p(X)$ is an α_1 space.*
(3) *$B_p(X)$ is an α_1 space.*
(4) *$B_p(X)$ is an α_2 space.*
(5) *$B_p(X)$ is an α_3 space.*
(6) *$B_p(X)$ is an α_4 space.*

Proof. (1)$\Rightarrow$(3) This is proved verbatim as the proof of (2)$\Rightarrow$(1) in Theorem 9.

(3)$\Rightarrow$(2) is evident.

(2)$\Rightarrow$(1) is due to the above mentioned result of Scheepers, and the equivalence of being a QN space and (1).

(4)$\Leftrightarrow$(5)$\Leftrightarrow$(6) is proved as in Gerlits–Nagy [18] or Scheepers [36] (in fact, the Borel case is easier).

(3)$\Rightarrow$(4) is evident.

(4)$\Rightarrow$(1) It suffices to show that X satisfies $S_1(B_\Gamma, B_\Gamma)$. Given $\mathcal{U}_n \in B_\Gamma$, $n \in \mathbb{N}$, we find that for each n, $A_n = \{\chi_U : U \in \mathcal{U}_n\} \subseteq B_p(X)$ converges pointwise to 0. Applying α_2, let $U_n \in \mathcal{U}_n$, $n \in \mathbb{N}$, be such that χ_{U_n} converges pointwise to 0. Then $\{U_n : n \in \mathbb{N}\}$ is a point-cofinite cover of X. $\qquad\square$

4.3. Almost continuous functions

A function $f : X \to Y$ is *almost continuous* [3] if for each nonempty $A \subseteq X$, the restriction of f to A has a point of continuity. $AC_p(X)$ is the space of all almost continuous real-valued functions on X, with the topology of pointwise convergence [5].

If X and Y are Tychonoff and $f : X \to Y$ is almost continuous, then for each $A \subseteq X$ the set of points of continuity of the restriction of f to A is open dense in A [5]. Each function with the latter property is Borel [42]. Thus, $C_p(X) \subseteq AC_p(X) \subseteq B_p(X)$.

Corollary 22. *$AC_p(X)$ is an α_1 space if, and only if, $C_p(X)$ is an α_1 space.*

4.4. wQN spaces and the Scheepers Conjecture

X is a *wQN space* [10] if each sequence of continuous real-valued functions on X converging pointwise to zero has a subsequence converging to zero quasi-normally.

Two fundamental problems concerning wQN spaces appear in the literature: In [36, p. 269], [9, Problem 23], and [6, Problems 10.3–10.4], we are asked whether, consistently,

every wQN space is a QN space. The *Scheepers Conjecture* [37] asserts that X is a wQN space if, and only if, X satisfies $S_1(\Gamma, \Gamma)$. It is still open whether the Scheepers Conjecture is provable. A striking result of Dow gives a positive answer to the first problem, and a consistently positive answer to the second.

Theorem 23 (Dow [14]). *In the Laver model, each α_2 space is an α_1 space.*

Let C_Γ denote the family of all clopen point-cofinite covers of X. Clearly, $S_1(\Gamma, \Gamma)$ implies $S_1(C_\Gamma, C_\Gamma)$.

Corollary 24. *In the Laver model:*

(1) $S_1(B_\Gamma, B_\Gamma) = S_1(C_\Gamma, C_\Gamma)$.
(2) *X is a wQN space if, and only if, X is a QN space.*
(3) *The Scheepers Conjecture holds.*

In particular, these assertions are (*simultaneously*) *consistent.*

Proof. (1) Using the correspondence described just before Lemma 10, we find that $C_p(X)$ is an α_2 space if, and only if, X satisfies $S_1(C_\Gamma, C_\Gamma)$. Thus, if X satisfies $S_1(C_\Gamma, C_\Gamma)$, then by Dow's Theorem 23, $C_p(X)$ is an α_1 space. By Theorem 9, X satisfies $S_1(B_\Gamma, B_\Gamma)$.

(2) Assume that X is a wQN space. Then $C_p(X)$ is an α_2 space [34]. By Dow's Theorem 23, $C_p(X)$ is an α_1 space. By Theorem 17, X is a QN space.

(3) $S_1(\Gamma, \Gamma)$ implies (in ZFC) being a wQN space [37]. Now, back in the Laver model, assume that X is a wQN space. By (2), X is a QN space. By Corollary 20, X satisfies $S_1(B_\Gamma, B_\Gamma)$, and in particular $S_1(\Gamma, \Gamma)$. □

Remark 25. In [33, 9] it is shown that X is a wQN space if, and only if, X satisfies $S_1(C_\Gamma, C_\Gamma)$. Using this, (2) and (3) follow immediately from Corollary 24(1).

4.5. $\overline{QN}$ spaces and M spaces

X is a $\overline{QN}$ *space* [11] if each real-valued function (not necessarily continuous) on X which is a pointwise limit of a sequence of continuous functions, is in fact a quasi-normal limit of those functions.

The following result is immediate from Theorem 9 and [11, Theorem 5.10(9)]. For completeness, we give a simple, direct proof.

Theorem 26. *The following are equivalent:*

(1) *X is a $\overline{QN}$ space.*
(2) *X is a QN space.*
(3) *Each sequence of Borel functions converging pointwise to 0, converges to 0 quasi-normally.*
(4) *Each sequence of Borel functions converging pointwise to any function converges quasi-normally to this function.*

Proof. (1)⇒(2) is immediate.

(2)⇒(3) Assume (2). By Theorem 9, each Borel image of X in $\mathbb{N}^\mathbb{N}$ is bounded. Thus, an argument verbatim as in our proof of (⟸) of Theorem 9 gives (3).

(3)⇒(4) The limit function f is also Borel, and $f_n - f$ converges to 0.

(4)⇒(1) is immediate. □

This shows that the first assumption in [11, Theorem 5.10(9)] is not needed. It also answers [11, Problem 6.11] in the affirmative. Based on [11, Theorem 6.9] and improving it, we also obtain the following solution of [11, Problem 6.10].

Corollary 27. *Every QN space is an M space.* □

The definition of M space is available in [11].

4.6. wQN_* spaces

X is a *wQN_* space* if each sequence of lower semi-continuous real-valued functions on X converging pointwise to zero has a subsequence converging to zero quasi-normally. In his talk at the *Third Workshop on Coverings, Selections, and Games in Topology* (Serbia, April 2007), Bukovský defined wQN_* spaces and described his recent investigations of this property and its upper semi-continuous variant. The main problem he posed was: Is every QN space a wQN_* space?

Theorem 28. *Every QN space is a wQN_* space.*

Proof. Every lower semi-continuous function is Borel. Use Theorem 26. □

Bukovský has later proved that the converse to Theorem 28 also holds [7], and therefore the notions coincide (with one another and with having bounded Borel images).

4.7. Bounded-ideal convergence spaces

The notion of ideal convergence originates in works of Steinhaus and Fast on statistical convergence, and was generalized by Bernstein, Katětov, and others (see [16] for an introduction). The following definitions are as in Jasinski–Recław [20]. Let D be a countable set, and $\mathcal{I} \subseteq P(D)$ be an ideal (i.e., $\mathcal{I}$ contains all singletons and is closed under taking subsets and finite unions). Let $\mathcal{I}^*$ denote the filter $\{D \setminus A : A \in \mathcal{I}\}$ dual to $\mathcal{I}$. A sequence $\{r_d\}_{d \in D}$ of real numbers $\mathcal{I}$-*converges* to 0 if $\{d \in D : |r_d| < \epsilon\} \in \mathcal{I}^*$ for each positive ϵ. A sequence $\{f_d\}_{d \in D}$ of continuous real-valued functions on X $\mathcal{I}$-converges to 0 if for each $x \in X$, the sequence $\{f_d(x)\}_{d \in D}$ of real numbers $\mathcal{I}$-converges to 0. The space X has the $\mathcal{I}$-*convergence property* if for each sequence $\{f_d\}_{d \in D}$ of continuous real-valued functions on X which $\mathcal{I}$-converges to 0, there is $A \in \mathcal{I}^*$ such that $\{f_d\}_{d \in A}$ converges pointwise to 0.

We will use the following.

Lemma 29. *In the definition of $\mathcal{I}$-convergence, it suffices to consider only sequences of distinct elements.*

Proof. Let $\{f_d\}_{d \in D}$ be given. Enumerate $D = \{d_n : n \in \mathbb{N}\}$ bijectively. For each n, as the functions $f_{d_n} + 1/m$, $m \in \mathbb{N}$, are all distinct, there is $m(d_n) \in \mathbb{N}$ such that $m(d_n) \geq n$ and $f_{d_n} + 1/m(d_n) \notin \{f_{d_1} + 1/m(d_1), \dots, f_{d_{n-1}} + 1/m(d_{n-1})\}$.

It is easy to see that $\{f_d\}_{d \in D}$ $\mathcal{I}$-converges to 0 if, and only if, $\{f_d + 1/m(d)\}_{d \in D}$ $\mathcal{I}$-converges to 0. □

We use these definitions for $D = \mathbb{N} \times \mathbb{N}$. For $h \in \mathbb{N}^{\mathbb{N}}$, define $A_h = \{(n, m) : m \leq h(n)\}$. The family $\{A_h : h \in \mathbb{N}^{\mathbb{N}}\}$ is closed under finite intersections, and generates the *bounded-ideal*

$$\mathcal{I}_b = \{B \subseteq \mathbb{N} \times \mathbb{N} : (\exists h \in \mathbb{N}^{\mathbb{N}}) \; B \subseteq A_h\}.$$

X has the *bounded-ideal convergence property* if it has the $\mathcal{I}_b$-convergence property.

The bounded-ideal, which is also the Fubini product $\emptyset \times \mathrm{Fin}$ of the trivial ideal and the ideal of finite sets, plays a central role in studies of ideal convergence. For each analytic P-ideal $\mathcal{I}$, if any $X \subseteq \mathbb{R}$ not having Lebesgue measure zero has the $\mathcal{I}$-convergence property, then $\mathcal{I}$ is isomorphic to $\mathcal{I}_b$ [21]. For additional uses of this ideal and its associated convergence, see [16].

Jasinski and Recław [20] proved that every Sierpiński set has the bounded-ideal convergence property, and that if X has the bounded-ideal convergence property, then X is a σ space. Both of these assertions follow at once from the following.

Theorem 30. *The following are equivalent:*

(1) *X has the bounded-ideal convergence property.*
(2) *$C_p(X)$ is an α_1 space.*
(3) *Each Borel image of X in $\mathbb{N}^{\mathbb{N}}$ is bounded.*

Proof. By Theorem 9, it suffices to show that (1)$\Leftrightarrow$(2).

(2)$\Rightarrow$(1) Assume that $\{f_{(n,m)}\}_{(n,m)\in\mathbb{N}\times\mathbb{N}}$ $\mathcal{I}_b$-converges to 0. By Lemma 29, we may assume that the elements $f_{(n,m)}$, $(n, m) \in \mathbb{N} \times \mathbb{N}$, are distinct.

For each $x \in X$ and each positive ϵ, $\{(n, m) : |f_{(n,m)}(x)| < \epsilon\} \in \mathcal{I}_b^*$, that is, there is $h \in \mathbb{N}^{\mathbb{N}}$ such that $\{(n, m) : |f_{(n,m)}(x)| < \epsilon\} \supseteq (\mathbb{N} \times \mathbb{N}) \setminus A_h$. Thus, $|f_{(n,m)}(x)| < \epsilon$ for all $n, m \in \mathbb{N}$ such that $h(n) < m$. It follows that for each n, $\{f_{(n,m)}\}_{m\in\mathbb{N}}$ converges pointwise to 0.

As $C_p(X)$ is an α_1 space, there is for each n a number $h(n) \in \mathbb{N}$ such that $\{f_{(n,m)} : n, m \in \mathbb{N}, \; m > h(n)\}$ converges pointwise to 0, and since the enumeration is bijective, the sequence $\{f_{(n,m)}\}_{(n,m)\in(\mathbb{N}\times\mathbb{N})\setminus A_h}$ also converges pointwise to 0. As $(\mathbb{N} \times \mathbb{N}) \setminus A_h \in \mathcal{I}_b^*$, this shows that X has the bounded-ideal convergence property.

(1)$\Rightarrow$(2) Assume that for each n the sequence $\{f_{(n,m)}\}_{m\in\mathbb{N}}$ converges pointwise to 0. For each $x \in X$, each positive ϵ, and each n, there is $h(n) \in \mathbb{N}$ such that $|f_{(n,m)}(x)| < \epsilon$ for all $m > h(n)$. Thus, $\{(n, m) : |f_{(n,m)}(x)| < \epsilon\} \supseteq (\mathbb{N} \times \mathbb{N}) \setminus A_h$, that is, $\{f_{(n,m)}\}_{(n,m)\in\mathbb{N}\times\mathbb{N}}$ $\mathcal{I}_b$-converges to 0.

The bounded-ideal convergence property implies that there is $h \in \mathbb{N}^{\mathbb{N}}$ such that $\{f_{(n,m)}\}_{(n,m)\in(\mathbb{N}\times\mathbb{N})\setminus A_h}$ converges pointwise to 0, and therefore so does the sheaf amalgamation $\{f_{(n,m)} : n, m \in \mathbb{N}, \; m > h(n)\}$ (which can be enumerated as a subsequence of $\{f_{(n,m)}\}_{(n,m)\in(\mathbb{N}\times\mathbb{N})\setminus A_h}$). $\square$

Corollary 31. *The following are equivalent:*

(1) *X has the bounded-ideal convergence property.*
(2) *For each sequence $\{f_d\}_{d\in\mathbb{N}\times\mathbb{N}}$ of Borel real-valued functions on X which $\mathcal{I}_b$-converges to 0, there is $A \in \mathcal{I}_b^*$ such that $\{f_d\}_{d\in A}$ converges pointwise to 0.*

(3) *For each sequence $\{f_d\}_{d\in\mathbb{N}\times\mathbb{N}}$ of Borel real-valued functions on X which $\mathcal{I}_b$-converges to a Borel function f, there is $A \in \mathcal{I}_b^*$ such that $\{f_d\}_{d\in A}$ converges pointwise to f.*

(4) *For each sequence $\{f_d\}_{d\in\mathbb{N}\times\mathbb{N}}$ of Borel real-valued functions on X which $\mathcal{I}_b$-converges to a function f, there is $A \in \mathcal{I}_b^*$ such that $\{f_d\}_{d\in A}$ converges pointwise to f.*

Proof. (1)$\Rightarrow$(2) Replace "continuous" by "Borel" in the proof of Theorem 30 and use Theorem 21.

(2)$\Rightarrow$(3) $B_p(X)$ is a topological group, and in particular homogeneous.

(3)$\Rightarrow$(4) The assumption in (4) implies, in particular, that f is a pointwise limit of $\{f_{(1,m)}\}_{m\in\mathbb{N}}$. Thus, f is Borel. $\qquad\square$

4.8. Bounded Baire-class α images

Continuous functions and Borel functions are the extremal notions in the *Baire hierarchy* of functions: A real-valued function f is of *Baire class* 0 if it is continuous. For $0 < \alpha \le \aleph_1$, f is of *Baire class* α if f is the pointwise limit of a sequence of functions, each of Baire class smaller than α. The function f is Borel if, and only if, f is of Baire class $\aleph_1$ (see [23]). A natural question in light of our study is: Which spaces X have the property that each Baire-class α image of X in $\mathbb{N}^{\mathbb{N}}$ is bounded?

Theorem 32. *For each $\alpha > 0$, the following are equivalent:*

(1) *Each Baire-class α image of X in $\mathbb{N}^{\mathbb{N}}$ is bounded.*
(2) *Each Borel image of X in $\mathbb{N}^{\mathbb{N}}$ is bounded.*

Proof. Assume that each Baire-class 1 image of X in $\mathbb{N}^{\mathbb{N}}$ is bounded. Baire-class 1 functions are exactly the F_σ-measurable functions.

One way to proceed is to use Lemma 14, since for each bijectively enumerated family $\mathcal{C} = \{C_n : n \in \mathbb{N}\}$ of closed sets, the corresponding Marczewski function is F_σ-measurable (and by the proof of Lemma 14, we may assume that for each $x \in X$, $\mathcal{C}(x)$ is infinite).

However, there is a more direct proof. By Lemma 13, it suffices to prove that X satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{F}, \mathrm{F}_\Gamma)$. Assume that $\mathcal{U}_n = \{C_m^n : m \in \mathbb{N}\}$, $n \in \mathbb{N}$, are closed covers of X not containing a finite subcover. Define $\Psi : X \to \mathbb{N}^{\mathbb{N}}$ by

$$\Psi(x)(n) = \min\{m : x \in C_m^n\}$$

for all $n \in \mathbb{N}$. Each basic open subset of $\mathbb{N}^{\mathbb{N}}$ is an intersection of finitely many sets of the form $O_m^n = \{f \in \mathbb{N}^{\mathbb{N}} : f(n) = m\}$. As $\Psi^{-1}[O_m^n] = C_m^n \setminus \bigcup_{k<m} C_k^n$ is an F_σ set for all n and m, Ψ is F_σ-measurable. Thus, $\Psi[X]$ is bounded by some $g \in \mathbb{N}^{\mathbb{N}}$. Then $\{\bigcup_{m\le g(n)} C_m^n : n \in \mathbb{N}\}$ is a point-cofinite cover of X. $\qquad\square$

5. Closing the circle: Continuous bounded images again

The proof of Theorem 9 gives us the following analogue of Theorem 2. Say that a set $Y \subseteq \overline{\mathbb{N}}^{\mathbb{N}}$ is *bounded* if there is $g \in \mathbb{N}^{\mathbb{N}}$ such that for each $f \in Y$ and all but finitely many n, $f(n) < \infty$ implies $f(n) \leq g(n)$. This generalizes the standard notions of boundedness in $\mathbb{N}^{\mathbb{N}}$ or $\mathbb{EF}$.

Theorem 33. *The following are equivalent:*

(1) *Each Borel image of X in $\mathbb{N}^{\mathbb{N}}$ is bounded.*
(2) *Each continuous image of X in $\overline{\mathbb{N}}^{\mathbb{N}}$ is bounded.*

Proof. (1)$\Rightarrow$(2) Assume that $\Psi : X \to \overline{\mathbb{N}}^{\mathbb{N}}$ is continuous. Define $d : \overline{\mathbb{N}}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ by $d(x)(n) = x(n)$ if $x(n) < \infty$, and $d(x)(n) = 1$ if $x(n) = \infty$. Then $d \circ \Psi : X \to \mathbb{N}^{\mathbb{N}}$ is Borel, and therefore $d[\Psi[X]]$ is a bounded subset of $\mathbb{N}^{\mathbb{N}}$. Thus, $\Psi[X]$ is a bounded subset of $\overline{\mathbb{N}}^{\mathbb{N}}$.

(2)$\Rightarrow$(1) Assume that each continuous image of X in $\overline{\mathbb{N}}^{\mathbb{N}}$ is bounded. We first prove that for each bijectively enumerated family $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of open sets, $\mathcal{U}$ is a discrete limit of continuous functions. The proof is similar to the proof of Lemma 11. As shown at the end of the proof of Lemma 11, we may assume that no U_n is clopen.

For each n, write U_n as a union $\bigcup_m C_m^n$ of disjoint clopen sets. Define $\Psi : X \to \overline{\mathbb{N}}^{\mathbb{N}}$ by

$$\Psi(x)(n) = \begin{cases} m, & x \in U_m^n, \\ \infty, & x \notin U_n. \end{cases}$$

Then Ψ is continuous. Let $g \in \mathbb{N}^{\mathbb{N}}$ bound $\Psi[X]$. For each n, m, let

$$U_m^n = \bigcup_{i=1}^{\max\{m, g(n)\}} C_i^n.$$

For each m, define a continuous function $\Psi_m : X \to P(\mathbb{N})$ by

$$\Psi_m(x) = \{n : x \in U_m^n\}.$$

We claim that $\mathcal{U}$ is a discrete limit of the maps Ψ_m, $m \in \mathbb{N}$.

Fix $x \in X$. Let N be such that for all $n \geq N$, $\Psi(x)(n) < \infty$ implies $\Psi(x)(n) \leq g(n)$. For each $n < N$ with $x \in U_n$, let m_n be such that $x \in U_{m_n}^n$. Set $M = \max\{m_n : n < N\}$.

Fix $m \geq M$. We show that $n \in \Psi_m(x)$ if, and only if, $x \in U_n$. One direction follows from $U_m^n \subseteq U_n$. To prove the other direction, assume that $x \in U_n$, and consider the two possible cases: If $n < N$, then $x \in U_m^n$ because $m \geq M \geq m_n$, and we are done. Thus, assume that $n \geq N$. Then $\Psi(x)(n) \leq g(n)$, and therefore $x \in \bigcup_{i=1}^{g(n)} C_i^n \subseteq U_m^n$.

Thus, for each $x \in X$ there is M such that for all $m \geq M$, $\Psi_m(x) = \mathcal{U}(x)$.

Now, each continuous image of X in $\mathbb{N}^{\mathbb{N}}$ is bounded because $\mathbb{N}^{\mathbb{N}}$ is a subspace of $\overline{\mathbb{N}}^{\mathbb{N}}$. By Hurewicz's Theorem 1, X satisfies $\mathsf{U}_{\mathrm{fin}}(\mathsf{O}, \Gamma)$, and by Lemma 12, so does $\mathcal{U}[X]$. By Lemma 14, each Borel image of X in $\mathbb{N}^{\mathbb{N}}$ is bounded. $\qquad\square$

We therefore obtain the aesthetically pleasing result, that the chain of properties

$$\mathsf{U}_{\mathrm{fin}}(\mathrm{B}, \mathrm{B}_\Gamma) \quad \Longrightarrow \quad \text{hereditarily-}\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma) \quad \Longrightarrow \quad \mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$$

is obtained by requiring bounded continuous images in the chain of subspaces

$$\overline{\mathbb{N}}^{\mathbb{N}} \quad \supseteq \quad \mathbb{E}\mathbb{F} \quad \supseteq \quad \mathbb{N}^{\mathbb{N}},$$

respectively.

Note. After the present paper was accepted for publication, Bukovský and Šupina [12] devised an alternative, more analytic and less combinatorial, proof of our main Theorem 9.

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