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# ADDITIVITY PROPERTIES OF TOPOLOGICAL DIAGONALIZATIONS

TOMEK BARTOSZYNSKI†, SAHARON SHELAH‡, AND BOAZ TSABAN*

**Abstract.** We answer a question of Just, Miller, Scheepers and Szeptycki whether certain diagonalization properties for sequences of open covers are provably closed under taking finite or countable unions.

**§1. Introduction.** In [7] Just, Miller, Scheepers and Szeptycki studied a unified framework for topological diagonalizations and asked about the additivity of the corresponding families of sets. In this paper we answer their question. Some of the properties considered in [7] were studied earlier by Hurewicz ($\mathsf{U}_{fin}(\Gamma, \Gamma)$), Menger ($\mathsf{U}_{fin}(\Gamma, \mathscr{O})$), Rothberger ($\mathsf{S}_1(\mathscr{O}, \mathscr{O})$, traditionally known as the $C''$ property), Gerlits and Nagy ($\mathsf{S}_1(\Omega, \Gamma)$, traditionally known as the $\gamma$- property), and others.

We have tried to be as concise as possible in this paper. An extension of this paper containing the basic definitions, complete proofs, extended results and consequences is available online [1].

**§2. Preliminaries.** By a *set of reals* we mean a subset of $\mathbb{R} \setminus \mathbb{Q}$. Recall that each separable, zero-dimensional metric space is homeomorphic to a set of reals. Let $X$ be a set of reals. A countable open cover $\mathscr{U}$ of $X$ is said to be

1. an *$\omega$-cover* if $X$ is not in $\mathscr{U}$ and for each finite subset $F$ of $X$, there is a set $U \in \mathscr{U}$ such that $F \subseteq U$;
2. a *$\gamma$-cover* if it is infinite and for each $x$ in $X$ the set $\{U \in \mathscr{U} : x \notin U\}$ is finite.

Let $\mathscr{O}$, $\Omega$, and $\Gamma$ denote the collections of all countable open covers, $\omega$-covers, and $\gamma$-covers of $X$, respectively. Let $\mathscr{A}$ and $\mathscr{B}$ be any of these three classes. We consider the following three properties which $X$ may or may not have.

$\mathsf{S}_1(\mathscr{A}, \mathscr{B})$: For each sequence $\langle \mathscr{U}_n : n \in \omega \rangle$ of elements of $\mathscr{A}$, there exist elements $U_n \in \mathscr{U}_n$, $n \in \omega$, such that $\{U_n : n \in \omega\}$ is a member of $\mathscr{B}$.

$\mathsf{S}_{fin}(\mathscr{A}, \mathscr{B})$: For each sequence $\langle \mathscr{U}_n : n \in \omega \rangle$ of elements of $\mathscr{A}$, there exist finite sets $\mathscr{V}_n \subseteq \mathscr{U}_n$, $n \in \omega$, such that $\bigcup_{n \in \omega} \mathscr{V}_n$ is an element of $\mathscr{B}$.

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$\mathsf{U}_{\mathit{fin}}(\mathscr{A},\mathscr{B})$: For each sequence $\langle \mathscr{U}_n : n \in \omega\rangle$ of elements of $\mathscr{A}$ which do not contain a finite subcover, there exist finite sets $\mathscr{V}_n \subseteq \mathscr{U}_n$ such that $\{\bigcup \mathscr{V}_n : n \in \omega\}$ is a member of $\mathscr{B}$.

Many equivalences hold among these properties, and the surviving ones appear in the following diagram (where an arrow denotes implication), to which no arrow can be added except perhaps from $\mathsf{U}_{\mathit{fin}}(\Gamma,\Gamma)$ or $\mathsf{U}_{\mathit{fin}}(\Gamma,\Omega)$ to $\mathsf{S}_{\mathit{fin}}(\Gamma,\Omega)$ [7].

$$\begin{tikzcd}
 & & \mathsf{U}_{\mathit{fin}}(\Gamma,\Gamma) \arrow[r] & \mathsf{U}_{\mathit{fin}}(\Gamma,\Omega) \arrow[r] & \mathsf{U}_{\mathit{fin}}(\Gamma,\mathscr{O}) \\
 & & \mathsf{S}_{\mathit{fin}}(\Gamma,\Omega) \arrow[ur] & & \\
\mathsf{S}_1(\Gamma,\Gamma) \arrow[r] \arrow[uurr] & \mathsf{S}_1(\Gamma,\Omega) \arrow[rr] \arrow[ur] & & \mathsf{S}_1(\Gamma,\mathscr{O}) \arrow[uur] & \\
 & & \mathsf{S}_{\mathit{fin}}(\Omega,\Omega) \arrow[uu] & & \\
\mathsf{S}_1(\Omega,\Gamma) \arrow[r] \arrow[uu] & \mathsf{S}_1(\Omega,\Omega) \arrow[rr] \arrow[uu] \arrow[ur] & & \mathsf{S}_1(\mathscr{O},\mathscr{O}) \arrow[uu] &
\end{tikzcd}$$

THEOREM 2.1 (folklore). *$\mathsf{S}_1(\mathscr{O},\mathscr{O})$, $\mathsf{S}_1(\Gamma,\mathscr{O})$ and $\mathsf{U}_{\mathit{fin}}(\Gamma,\mathscr{O})$ are countably additive.*

PROOF. Given $X = \bigcup_{n\in\omega} X_n$, where each $X_n$ has the appropriate selection property, let $\langle \mathscr{U}_n : n \in \omega\rangle$ be a sequence of covers. Partition $\omega$ into infinite sets $\langle A_n : n \in \omega\rangle$ and apply the selection principle to $X_i$ and the covers $\langle \mathscr{U}_n : n \in A_i\rangle$. Afterwards take the union of the selected covers. ⊣

DEFINITION 2.2. Let $\omega^{\uparrow\omega} = \{f \in \omega^\omega : f \text{ is non-decreasing}\}$, and for $f, g \in \omega^{\uparrow\omega}$ let $f \leq^\star g$ mean that $f(n) \leq g(n)$ for all but finitely many $n$. A family $F \subseteq \omega^{\uparrow\omega}$ is

1. *dominating* if for every $g \in \omega^{\uparrow\omega}$ there is $f \in F$ such that $g \leq^\star f$,
2. *finitely dominating* if every $g \in \omega^{\uparrow\omega}$ there are $f_1, f_2, \dots, f_k \in F$ such that $g \leq^\star \max\{f_1,\dots,f_k\}$,
3. *unbounded* if for every $g \in \omega^{\uparrow\omega}$ there is $f \in F$ such that $f \not\leq^\star g$.

THEOREM 2.3 ([6, 8, 12]). *For a set of reals $X$:*

1. *$X$ satisfies $\mathsf{U}_{\mathit{fin}}(\Gamma,\Gamma)$ iff for every continuous mapping $X \ni x \rightsquigarrow f_x \in \omega^{\uparrow\omega}$, $\{f_x : x \in X\}$ is bounded,*
2. *$X$ satisfies $\mathsf{U}_{\mathit{fin}}(\Gamma,\mathscr{O})$ iff for every continuous mapping $X \ni x \rightsquigarrow f_x \in \omega^{\uparrow\omega}$, $\{f_x : x \in X\}$ is not dominating,*
3. *$X$ satisfies $\mathsf{U}_{\mathit{fin}}(\Gamma,\Omega)$ iff for every continuous mapping $X \ni x \rightsquigarrow f_x \in \omega^{\uparrow\omega}$, $\{f_x : x \in X\}$ is not finitely dominating.*

For completeness, we sketch a proof for (1). The proofs for (2) and (3) are similar.

PROOF. ($\rightarrow$) Suppose that $X$ satisfies $\mathsf{U}_{\mathit{fin}}(\Gamma,\Gamma)$ and $x \rightsquigarrow f_x$ is a continuous mapping. Then $\{f_x : x \in X\}$ satisfies $\mathsf{U}_{\mathit{fin}}(\Gamma,\Gamma)$. Define $U^n_k = \{x : f_x(n) \leq k\}$ for $n, k \in \omega$, and $\mathscr{U}_n = \{U^n_k : k \in \omega\}$ for $n \in \omega$. Assume that for each $n$, $\mathscr{U}_n$ does not contain a finite subcover of $\{f_x : x \in X\}$ (we leave the other case to the reader). Apply $\mathsf{U}_{fin}(\Gamma, \Gamma)$ to get a $\gamma$-cover which in turn will give us a function which bounds $\{f_x : x \in X\}$.

($\leftarrow$) It suffices to show $\mathsf{U}_{fin}(\mathscr{O}, \Gamma)$ (since $\mathsf{U}_{fin}(\mathscr{O}, \Gamma)$ implies $\mathsf{U}_{fin}(\Gamma, \Gamma)$). Suppose that $\langle \mathscr{U}_n : n \in \omega \rangle$ is a sequence of open covers of $X$. Since $X$ is zero-dimensional, by passing to finer covers we can assume that $\mathscr{U}_n = \{U^n_k : k \in \omega\}$, where the sets $U^n_k$ are clopen. Define a continuous mapping $x \rightsquigarrow f_x$ as $f_x(n+1) = f_x(n) + \min\{k : x \in U^{n+1}_k\}$. If $g \in \omega^\omega$ bounds $\{f_x : x \in X\}$, then $\{\bigcup_{j \leq g(n)} U^n_j : n \in \omega\}$ is a $\gamma$-cover of $X$. ⊣

An immediate consequence of Theorem 2.3 is that $\mathsf{U}_{fin}(\Gamma, \Gamma)$ is countably additive. But not all properties we consider are provably additive: In [5] it was proved that, assuming the Continuum Hypothesis, $\mathsf{S}_1(\Omega, \Gamma)$ is not finitely additive. In Problem 5 of [7] it was asked which of the remaining properties is countably, or at least finitely, additive. In [10] it was proved that $\mathsf{S}_1(\Gamma, \Gamma)$ is countably additive. We will show that assuming (a small portion of) the Continuum Hypothesis, none of the remaining properties is finitely additive.

## §3. Negative results.

The following theorem is a generalization of the constructions of [7] and [11].

THEOREM 3.1. *Assume that $2^\omega$ is not the union of $< 2^{\aleph_0}$ meager sets. There exist sets of reals $X_1, X_2 \in \mathsf{S}_1(\Omega, \Omega)$ such that $X_1 \cup X_2 \notin \mathsf{U}_{fin}(\Gamma, \Omega)$.*

PROOF. For simplicity we will work in $\mathbb{Z}^\omega$, where $\mathbb{Z}$ denotes the set of integers. We will construct sets $X_1, X_2 \in \mathbb{Z}^\omega$ such that $X_1 + X_2 = \{x_1 + x_2 : x_1 \in X_1, x_2 \in X_2\} = \mathbb{Z}^\omega$. Since $2 \cdot \max(x_1, x_2) \geq x_1 + x_2$ it follows that $X_1 \cup X_2$ is 2-dominating. Thus by Theorem 2.3(3), $X_1 \cup X_2 \notin \mathsf{U}_{fin}(\Gamma, \Omega)$.

Let $\{f_\alpha : \alpha < 2^{\aleph_0}\}$ enumerate $\mathbb{Z}^\omega$, $\{\langle \mathscr{U}^\alpha_n : n \in \omega \rangle : \alpha < 2^{\aleph_0}\}$ enumerate all countable sequences of countable families of open sets, and let $Q = \{q \in \mathbb{Z}^\omega : \forall^\infty n\ q(n) = 0\}$.

We construct $X_1 = \{x^1_\beta : \beta < 2^{\aleph_0}\} \cup Q$ and $X_2 = \{x^2_\beta : \beta < 2^{\aleph_0}\} \cup Q$ by induction on $\alpha < 2^{\aleph_0}$. Let $X^i_\alpha = \{x^i_\beta : \beta < \alpha\}$ for $i = 1, 2$ be given. We will describe how to choose $x^1_\alpha$ and $x^2_\alpha$.

LEMMA 3.2 ([7]). *Assume that $2^\omega$ is not the union of $< 2^{\aleph_0}$ meager sets. Suppose that $Y \subseteq \mathbb{Z}^\omega$ has size $< 2^{\aleph_0}$. Then $Y$ satisfies $\mathsf{S}_1(\Omega, \Omega)$.*

We give a proof as a hint to the proof of a forthcoming assertion.

PROOF. Suppose that $\langle \mathscr{U}_n : n \in \omega \rangle = \langle U^n_k : n, k \in \omega \rangle$ is a sequence of $\omega$-covers of $Y$. For each $F \in [Y]^{<\omega}$ let $f_F(n) = \min\{k : F \subseteq U^n_k\}$. The set $H_F = \{x \in \mathbb{Z}^\omega : \forall^\infty n\ x(n) \neq f_F(n)\}$ is meager in $\mathbb{Z}^\omega$. Thus any $z \notin \bigcup_{F \in [Y]^{<\omega}} H_F$ gives the desired selector. ⊣

For $i = 1, 2$, say that $\alpha$ is *$i$-good* if for each $n$ $\mathscr{U}^\alpha_n$ is an $\omega$-cover of $X^i_\alpha$. Assume that $\alpha$ is $i$-good. Apply Lemma 3.2 and choose a selector $U^{\alpha,i}_n \in \mathscr{U}^{\alpha,i}_n$ such that $\{U^{\alpha,i}_n : n \in \omega\}$ is an $\omega$-cover of $X^i_\alpha$. We make the *inductive hypothesis* that for each $i$-good $\beta < \alpha$, $\{U^{\beta,i}_n : n \in \omega\}$ is an $\omega$-cover of $X^i_\alpha$. For each finite $F \subseteq X^i_\alpha$, and each $i$-good $\beta \leq \alpha$, define $G^{i,\beta}_F = \bigcup\{U^{\beta,i}_n : F \subseteq U^{\beta,i}_n\}$. Observe that $G^{i,\beta}_F$ is open dense in $\mathbb{Z}^\omega$ since $Q \subseteq X^i_\alpha$.

LEMMA 3.3. *Assume that $2^\omega$ is not the union of $< 2^{\aleph_0}$ meager sets. Let $\{U_\gamma : \gamma < \lambda < 2^{\aleph_0}\}$ be a family of open dense subsets of $\mathbb{Z}^\omega$. Then for every $f \in \mathbb{Z}^\omega$ there are $x_1, x_2 \in \bigcap_{\gamma<\lambda} U_\gamma$ such that $x_1 + x_2 = f$.*

PROOF. Consider $f - Y = \{f - x : x \in Y\}$, and let $x_1 \in \bigcap_{\gamma<\lambda} U_\gamma \cap \bigcap_{\gamma<\lambda}(f - U_\gamma)$. It follows that $x_1 + x_2 = f$ for some $x_2 \in \bigcap_{\gamma<\lambda} U_\gamma$. ⊣

Apply Lemma 3.3 to find $x^1_\alpha, x^2_\alpha \in \bigcap\{G^{i,\beta}_F : i = 1, 2,\ F \in [X^i_\alpha]^{<\omega},\ i\text{-good } \beta \leq \alpha\}$, such that $x^1_\alpha + x^2_\alpha = f_\alpha$. The induction hypothesis remains true after the construction step.

We have that $X_1 + X_2 = \mathbb{Z}^\omega$, so it remains to check that $X_i \in \mathsf{S}_1(\Omega, \Omega)$ for $i = 1, 2$. Fix $i$. Suppose that $\langle \mathscr{U}_n : n \in \omega\rangle$ is a sequence of $\omega$-covers of $X_i$, and let $\alpha$ be such that $\langle \mathscr{U}_n : n \in \omega\rangle = \langle \mathscr{U}^\alpha_n : n \in \omega\rangle$. Clearly, $\langle \mathscr{U}^\alpha_n : n \in \omega\rangle$ is an $\omega$-cover of $X^i_\alpha$ so we have to show that the selector $\{U^{\alpha,i}_n : n \in \omega\}$ chosen at the step $\alpha$ is an $\omega$-cover of $X_i$. Take any $F \in [X_i]^{<\omega}$ and write it as $F = F_0 \cup F'$, where $F_0 = F \cap X^i_\alpha$ and $F' = F \setminus F_0 = \{x^i_{\beta_1}, x^i_{\beta_2}, \dots, x^i_{\beta_\ell}\}$, where $\alpha \leq \beta_1 < \beta_2 < \cdots < \beta_\ell$. Note that $x^i_{\beta_1} \in G^{i,\alpha}_{F_0}$ and for $j > 1$, $x^i_{\beta_{j+1}} \in G^{i,\alpha}_{F_0 \cup \{x^1_{\beta_1}, \dots, x^1_{\beta_j}\}}$, which finishes the proof. ⊣

Let $\Omega_{\mathrm{Borel}}$ be the collection of all countable Borel $\omega$-covers of $X$. A modification of the above proof gives us the following stronger result, which settles the additivity question in the case of Borel covers.

THEOREM 3.4. *Assume that $2^\omega$ is not the union of $< 2^{\aleph_0}$ meager sets. There exist sets $X_1, X_2 \in \mathsf{S}_1(\Omega_{\mathrm{Borel}}, \Omega_{\mathrm{Borel}})$ such that $X_1 \cup X_2 \notin \mathsf{U}_{fin}(\Gamma, \Omega)$.*

PROOF. We will need the following definition [11]: A cover $\mathscr{U} = \{U_n : n \in \omega\} \in \Omega_{\mathrm{Borel}}$ is called *$\omega$-fat* if for every $F \in [X]^{<\omega}$ and finitely many nonempty open sets $O_1, \dots, O_k$, there exists $U \in \mathscr{U}$ such that $F \subseteq U$ and none of the sets $U \cap O_1, \dots, U \cap O_k$ is meager. Let $\Omega^{\mathrm{fat}}_{\mathrm{Borel}}$ be the collection of all countable $\omega$-fat Borel covers of $X$. We will use some simple properties of these covers (the proofs are easy – see [1]).

LEMMA 3.5. *Assume that $\mathscr{U}$ is a countable collection of Borel sets. Then $\cup\mathscr{U}$ is comeager if, and only if, for each nonempty basic open set $O$ there exists $U \in \mathscr{U}$ such that $U \cap O$ is not meager.*

COROLLARY 3.6. Assume that $\mathscr{U}$ is an $\omega$-fat cover of some set $X$. Then:

1. For each finite $F \subseteq X$ and finite family $\mathscr{F}$ of nonempty *basic* open sets, the set
$$\cup\{U \in \mathscr{U} : F \subseteq U \text{ and for each } O \in \mathscr{F},\ U \cap O \notin \mathscr{M}\}$$
is comeager.
2. For each element $x$ in the intersection of all sets of this form, $\mathscr{U}$ is an $\omega$-fat cover of $X \cup \{x\}$.

A modification of the proof of Lemma 3.2 gives the following.

LEMMA 3.7. *Assume that $2^\omega$ is not the union of $< 2^{\aleph_0}$ meager sets. If $Y \subseteq \mathbb{Z}^\omega$ has size $< 2^{\aleph_0}$, then $Y$ satisfies $\mathsf{S}_1\big(\Omega^{\mathrm{fat}}_{\mathrm{Borel}}, \Omega^{\mathrm{fat}}_{\mathrm{Borel}}\big)$.*

The following lemma justifies our focusing on $\omega$-fat covers.

LEMMA 3.8. *Assume that $X$ is a set of reals such that for each nonempty basic open set $O$, $X \cap O$ is not meager. Then every countable Borel $\omega$-cover $\mathscr{U}$ of $X$ is an $\omega$-fat cover of $X$.*

Let $\mathbb{Z}^\omega = \{f_\alpha : \alpha < 2^{\aleph_0}\}$, and $\{G_\alpha : \alpha < 2^{\aleph_0}\}$ be all dense $G_\delta$ subsets of $\mathbb{Z}^\omega$. Let $\{O_n : n \in \omega\}$ and $\{\mathscr{F}_m : m \in \omega\}$ be all nonempty basic open sets and all finite families of nonempty basic open sets, respectively, in $\mathbb{Z}^\omega$. Let $\{\langle \mathscr{U}_n^\alpha : n \in \omega\rangle : \alpha < 2^{\aleph_0}\}$ be all sequences of countable families of Borel sets.

We construct, by induction on $\alpha < 2^{\aleph_0}$, sets $X_i = \{x_\beta^i : \beta < 2^{\aleph_0}\}$ $(i = 1, 2)$ which have the property needed in Lemma 3.8. At stage $\alpha \geq 0$ set $X_\alpha^i = \{x_\beta^i : \beta < \alpha\}$ and consider the sequence $\langle \mathscr{U}_n^\alpha : n \in \omega\rangle$. Say that $\alpha$ is $i$-good if for each $n$ $\mathscr{U}_n^\alpha$ is an $\omega$-fat cover of $X_\alpha^i$. In this case, by Lemma 3.7 there exist elements $U_n^{\alpha,i} \in \mathscr{U}_n^\alpha$ such that $\langle U_n^{\alpha,i} : n \in \omega\rangle$ is an $\omega$-fat cover of $X_\alpha^i$. We make the inductive hypothesis that for each $i$-good $\beta < \alpha$, $\langle U_n^{\beta,i} : n \in \omega\rangle$ is an $\omega$-fat cover of $X_\alpha^i$. For each finite $F \subseteq X_\alpha^i$, $i$-good $\beta \leq \alpha$, and $m$ define

$$G_{F,m}^{i,\beta} = \cup\{U_n^{\beta,i} : F \subseteq U_n^{\beta,i} \text{ and for each } O \in \mathscr{F}_m,\ U_n^{\beta,i} \cap O \notin \mathscr{M}\}.$$

By the inductive hypothesis, $G_{F,m}^{i,\beta}$ is comeager.

Set

$$Y_\alpha = \bigcap_{\beta<\alpha} G_\beta \cap \bigcap\{G_{F,m}^{i,\beta} : i < 2,\ i\text{-good } \beta \leq \alpha,\ m \in \omega,\ \text{Finite } F \subseteq X_\alpha^i\}.$$

Let $k = \alpha \bmod \omega$. Use Lemma 3.3 to pick $x_\alpha^0, x_\alpha^1 \in O_k \cap Y_\alpha$ such that $x_\alpha^0 + x_\alpha^1 =^* f_\alpha$. By Corollary 3.6(2), the inductive hypothesis is preserved.

Thus each $X_i$ satisfies $\mathsf{S}_1(\Omega_{\text{Borel}}^{\text{fat}}, \Omega_{\text{Borel}}^{\text{fat}})$ and its intersection with each nonempty basic open set has size $2^{\aleph_0}$. By Lemma 3.8, $\Omega_{\text{Borel}}^{\text{fat}} = \Omega_{\text{Borel}}$ for $X_i$. Finally, $X_0 + X_1$ is dominating, so $X_0 \cup X_1$ is 2-dominating. $\dashv$

## §4. Consistency results.

THEOREM 4.1 (folklore). *It is consistent that the properties* $\mathsf{S}_1(\Omega, \Gamma)$, $\mathsf{S}_1(\Omega, \Omega)$, *and* $\mathsf{S}_1(\mathscr{O}, \mathscr{O})$ *are countably additive.*

PROOF. It is well known that the Borel Conjecture implies that $\mathsf{S}_1(\mathscr{O}, \mathscr{O}) = [2^\omega]^{\leq\aleph_0}$. Thus $\mathsf{S}_1(\Omega, \Gamma) = \mathsf{S}_1(\Omega, \Omega) = [2^\omega]^{\leq\aleph_0}$. $\dashv$

We do not know if any of the properties $\mathsf{S}_{fin}(\Omega, \Omega)$, $\mathsf{S}_1(\Gamma, \Omega)$, and $\mathsf{S}_{fin}(\Gamma, \Omega)$ is consistently closed under taking finite unions, however $\mathsf{U}_{fin}(\Gamma, \Omega)$ is.

DEFINITION 4.2.

1. For any finite-to-one function $f \in \omega^\omega$ and an ultrafilter $\mathscr{U}$ on $\omega$ let $f(\mathscr{U})$ be the ultrafilter $\{X \subseteq \omega : f^{-1}(X) \in \mathscr{U}\}$.
2. Two ultrafilters $\mathscr{U}$ and $\mathscr{V}$ on $\omega$ are *nearly coherent* if there is a finite-to-one function $f \in \omega^\omega$ such that $f(\mathscr{U}) = f(\mathscr{V})$.
3. Let ***NCF*** stand for the statement: any two non-principal ultrafilters $\mathscr{U}$ and $\mathscr{V}$ on $\omega$ are nearly coherent.
4. Let $\mathfrak{D}_{\text{fin}}$ be the family of subsets of $\omega^{\uparrow\omega}$ that are not finitely dominating.

THEOREM 4.3. ***NCF*** *iff* $\mathfrak{D}_{\text{fin}}$ *is closed under finite unions.*

PROOF. $(\leftarrow)$ As this was also proved by Blass [4, Proposition 4.11], we omit our proof (see [1]).

$(\rightarrow)$ Note that the relation $Y \in \mathfrak{D}_{\text{fin}}$ is witnessed by a filter and a function, that is there exists a function $g \in \omega^{\uparrow\omega}$ such that the family $\{X_f^g : f \in Y\}$ is a filter base, where $X_f^g = \{n : f(n) \leq g(n)\}$, and can therefore be extended to an ultrafilter.

Suppose that $Y_1, Y_2 \in \mathfrak{D}_{\text{fin}}$ and let $r \in \omega^{\uparrow\omega}$ and ultrafilters $\mathscr{U}_1, \mathscr{U}_2$ witness that. By ***NCF*** there exists $h \in \omega^\omega$ such that $h(\mathscr{U}_1) = h(\mathscr{U}_2)$. Without loss of generality we can assume (see [2]) that $h \in \omega^{\uparrow\omega}$. Let $I_n = h^{-1}(\{n\})$ for $n \in \omega$ and let $g \in \omega^{\uparrow\omega}$ be any function such that $g(\min(I_n)) \geq r(\max(I_n))$, $n \in \omega$. Suppose that $F_1 \in [Y_1]^{<\omega}$ and $F_2 \in [Y_2]^{<\omega}$. We will show that $g$ is not dominated by $\max(F_1, F_2)$. By the choice of $r$, $X^r_{\max(F_1)} \in \mathscr{U}_1$ and $X^r_{\max(F_2)} \in \mathscr{U}_2$. Since $h(\mathscr{U}_1) = h(\mathscr{U}_2)$ it follows that the set $B = \{n \in \omega : I_n \cap X^r_{\max(F_1)} \neq \emptyset \text{ and } I_n \cap X^r_{\max(F_2)} \neq \emptyset\}$ is infinite. For every $n \in B$ and $i = 1, 2$ let $k^n_i \in I_n \cap X^r_{\max(F_i)}$. It follows that for $i = 1, 2$, $g\big(\min(I_n)\big) \geq r\big(\max(I_n)\big) \geq r(k^n_i) \geq \max(F_i)(k^n_i) \geq \max(F_i)\big(\min(I_n)\big)$. ⊣

THEOREM 4.4. *It is consistent that $\mathsf{U}_{fin}(\Gamma, \Omega)$ is countably additive.*

PROOF. By Theorem 2.3(3), if $\mathfrak{D}_{\text{fin}}$ is countably additive so is $\mathsf{U}_{fin}(\Gamma, \Omega)$. By [3] it is known that ***NCF*** is consistent. It is easy to see that if $\mathfrak{D}_{\text{fin}}$ is finitely additive, then it is countably additive. Together with Theorem 4.3 this finishes the proof. ⊣

*Remark* 4.5. 1. Whereas the results in this paper settle all additivity problems for the classical types of covers (namely, general open covers, $\omega$-covers, and $\gamma$- covers), there remain many open problems when $\tau$-covers are considered – see [1].
2. We have recently found out that in [9], Scheepers used the Continuum Hypothesis to construct two sets satisfying $\mathsf{S}_1(\Omega, \Omega)$ such that their union does not satisfy $\mathsf{S}_{fin}(\Omega, \Omega)$. This is extended by our Proposition 3.1, which is extended further by Theorem 3.4. Moreover, the Continuum Hypothesis is stronger than our assumption that the real line is not the union of less than continuum many meager sets.

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DEPARTMENT OF MATHEMATICS
BOISE STATE UNIVERSITY
BOISE, IDAHO 83725, USA
*E-mail*: tomek@math.boisestate.edu
*URL*: math.boisestate.edu/~tomek

INSTITUTE OF MATHEMATICS
HEBREW UNIVERSITY OF JERUSALEM
GIVAT RAM, 91904 JERUSALEM, ISRAEL
and
MATHEMATICS DEPARTMENT
RUTGERS UNIVERSITY
NEW BRUNSWICK, NJ 08903, USA
*E-mail*: shelah@math.huji.ac.il
*URL*: math.rutgers.edu/~shelah

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE,
BAR-ILAN UNIVERSITY
RAMAT-GAN 52900, ISRAEL
*E-mail*: tsaban@macs.biu.ac.il
*URL*: www.cs.biu.ac.il/~tsaban