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ADVANCES IN Mathematics

Advances in Mathematics 232 (2013) 311–326

www.elsevier.com/locate/aim

Pointwise convergence of partial functions: The Gerlits–Nagy Problem

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Received 31 January 2011; accepted 21 September 2012

Communicated by the Managing Editors of AIM

Abstract

For a set $X \subseteq \mathbb{R}$, let $B(X) \subseteq \mathbb{R}^X$ denote the space of Borel real-valued functions on X, with the topology inherited from the Tychonoff product $\mathbb{R}^X$. Assume that for each countable $A \subseteq B(X)$, each f in the closure of A is in the closure of A under pointwise limits of sequences of partial functions. We show that in this case, $B(X)$ is countably Fréchet–Urysohn, that is, each point in the closure of a countable set is a limit of a sequence of elements of that set. This solves a problem of Arnold Miller. The continuous version of this problem is equivalent to a notorious open problem of Gerlits and Nagy. Answering a question of Salvador Hernańdez, we show that the same result holds for the space of all Baire class 1 functions on X.

We conjecture that, in the general context, the answer to the continuous version of this problem is negative, but we identify a nontrivial context where the problem has a positive solution.

The proofs establish new local-to-global correspondences, and use methods of infinite-combinatorial topology, including a new fusion result of Francis Jordan.

Keywords: Gerlits–Nagy Problem; Fréchet–Urysohn spaces; Sequential spaces; Sequential closure; Pointwise convergence; Covering properties; Borel covers; Baire class 1 functions; Selection principles

1. Introduction and basic results

Let $X \subseteq \mathbb{R}$. $\mathrm{C}(X)$ is the family of all continuous real-valued functions on X. We consider $\mathrm{C}(X)$ with the topology inherited from the Tychonoff product $\mathbb{R}^X$. A basis of the topology is

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0001-8708/$ - see front matter
doi:10.1016/j.aim.2012.09.017

given by the sets

$$[f; x_1, \dots, x_k; \epsilon] := \{g \in \mathrm{C}(X) : (\forall i = 1, \dots, k)\ |g(x_i) - f(x_i)| < \epsilon\},$$

where $f \in \mathrm{C}(X)$, $k \in \mathbb{N}$, $x_1, \dots, x_k \in X$, and ϵ is a positive real number. This is the *topology of pointwise convergence*, where a sequence (more generally, a net) f_n converges to f if and only if for each $x \in X$, the sequence of real numbers $f_n(x)$ converges to $f(x)$.

By definition, the (topological) closure $\overline{A}$ of a set $A \subseteq \mathrm{C}(X)$ is the set of all $f \in \mathrm{C}(X)$ such that, for all $k \in \mathbb{N}$, $x_1, \dots, x_k \in X$, and positive ϵ, there is an element $g \in A$ such that $|g(x_i) - f(x_i)| < \epsilon$ for $i = 1, \dots, k$. (Equivalently, there is a net in A converging pointwise to f.) $\mathrm{C}(X)$ is metrizable only when X is countable, and thus it makes sense to ask, when X is not countable, when do limits of sequences determine the closure of sets.

For a topological space Y and $A \subseteq Y$, the *closure of A under limits of sequences* is the smallest set $C \subseteq Y$ containing A, such that for each convergent (in Y) sequence of elements of C, the limit of this sequence is also in C. The closure of A under limits of sequences is contained in the topological closure $\overline{A}$ of A in Y.

Gerlits [6], and independently Pytkeev [17], proved that if limits determine the closure in $\mathrm{C}(X)$, then indeed it suffices to take limits once.

Theorem 1.1 (*Gerlits, Pytkeev*). *Let X be a Tychonoff space. Assume that, for each $A \subseteq \mathrm{C}(X)$, each $f \in \overline{A}$ (closure in $\mathrm{C}(X)$) belongs to the closure (in $\mathrm{C}(X)$) of A under limits of sequences. Then, for each $A \subseteq \mathrm{C}(X)$, each $f \in \overline{A}$ is a limit of a sequence of elements of A.*

The properties of $\mathrm{C}(X)$ in the premise and in the conclusion of Theorem 1.1 are often named *sequential* and *Fréchet–Urysohn*, respectively.

Consider now *partial* functions $f : X \to \mathbb{R}$, that is, functions whose domain is a (not necessarily proper) subset of X.

Definition 1.2. Let $f_1, f_2, \dots : X \to \mathbb{R}$ be partial functions. The *partial limit* function $f = \lim_n f_n$ is the partial real-valued function on X, with $\mathrm{dom}(f)$ being the set of all x such that $f_n(x)$ is eventually defined and converges, defined by $f(x) = \lim_n f_n(x)$ for each $x \in \mathrm{dom}(f)$.

Thus, for $f_1, f_2, \dots \in \mathrm{C}(X)$, the ordinary limit $\lim_n f_n$ exists in $\mathrm{C}(X)$ if and only if the domain of the partial limit function $f = \lim_n f_n$ is X, and f is continuous. The partial limit of a sequence of partial functions always exist, though it may be the empty function.

Definition 1.3. For a set A of partial functions $f : X \to \mathbb{R}$, the *closure of A under partial limits of sequences*, $\mathrm{partlims}(A)$, is the smallest set C of partial functions $f : X \to \mathbb{R}$, such that $A \subseteq C$ and for each sequence in C, the partial limit of this sequence is also in C.

Thus, the closure, in $\mathrm{C}(X)$, of a set $A \subseteq \mathrm{C}(X)$ under limits of sequences is a subset of $\mathrm{C}(X) \cap \mathrm{partlims}(A)$.

Lemma 1.4. *For each $A \subseteq \mathrm{C}(X)$, $\mathrm{C}(X) \cap \mathrm{partlims}(A)$ is contained in $\overline{A}$, the closure of A in $\mathrm{C}(X)$.*

Proof. The definition of basic open sets in $\mathrm{C}(X)$ (or $\mathbb{R}^X$) may be extended to partial functions, by letting $[f; x_1, \dots, x_k; \epsilon]$ be the set of all partial $g : X \to \mathbb{R}$ such that $x_1, \dots, x_k \in \mathrm{dom}(g)$ and $|g(x_i) - f(x_i)| < \epsilon$, for all $i = 1, \dots, k$.

Assume that $f \notin \overline{A}$. Take $x_1, \dots, x_k \in X$ and $\epsilon > 0$, such that $A \cap [f; x_1, \dots, x_k; \epsilon] = \emptyset$. Then $A \subseteq [f; x_1, \dots, x_k; \epsilon]^{\mathrm{c}}$, and $[f; x_1, \dots, x_k; \epsilon]^{\mathrm{c}}$ is closed under limits of partial functions:

Assume and $g = \lim_n g_n \in [f; x_1, \ldots, x_k; \epsilon]$. Then $x_1, \ldots, x_k \in \mathrm{dom}(g)$, and $|g(x_i) - f(x_i)| < \epsilon$, and therefore the same holds for g_n, for all but finitely many n. In particular, it cannot be the case that $g_1, g_2, \ldots \in [f; x_1, \ldots, x_k; \epsilon]^{\mathsf{c}}$.

It follows that f is not in the closure of A under partial limits of sequences. □

In 1982, Gerlits and Nagy published their seminal paper [7]. This paper has generated over 200 subsequent papers and a rich theory. Among the problems posed in [7], only one remains open. On its surface, the *Gerlits–Nagy Problem* is a combinatorial one, and we defer its combinatorial formulation to Section 4, where we prove that the Gerlits–Nagy Problem is equivalent to the following fundamental problem, dealing with pointwise convergence of real-valued functions.

Problem 1.5 (*Gerlits–Nagy [7]*)**.** Assume that, for each $A \subseteq \mathrm{C}(X)$, each $f \in \overline{A}$ belongs to the closure of A under partial limits of sequences. Does it follow that, for each $A \subseteq \mathrm{C}(X)$, each $f \in \overline{A}$ is a limit of a sequence of elements of A?

In the *Second Workshop on Coverings, Selections, and Games in Topology* (Lecce, Italy, 2005), Arnold Miller delivered a plenary lecture, where he posed the variant of the Gerlits–Nagy Problem, dealing with *Borel* rather than continuous functions [14].

Let $\mathrm{B}(X) \subseteq \mathbb{R}^X$ be the family of all Borel real-valued functions on X. One may consider the questions discussed above also for $\mathrm{B}(X)$, with the following reservation: Here, one must restrict attention to *countable* $A \subseteq \mathrm{B}(X)$, as we now show.

Each of the properties mentioned in the above discussion implies that $\mathrm{C}(X)$ is *countably tight*, that is, each point in the closure of a set is in the closure of a countable subset of that set. The standard proof would be by transfinite induction on the countable ordinals, but we adopt here an argument given in [2].

Proposition 1.6. *Let X be a topological space. Assume that, for each $A \subseteq \mathrm{C}(X)$, each $f \in \overline{A}$ belongs to the closure of A under partial limits of sequences. Then $\mathrm{C}(X)$ is countably tight.*

Proof. Let $A \subseteq \mathrm{C}(X)$. By Lemma 1.4, $\mathrm{partlims}(A) \cap \mathrm{C}(X) \subseteq \overline{A}$. Thus, it suffices to show that for each $f \in \overline{A}$, there is a countable $D \subseteq A$ such that $f \in \mathrm{partlims}(D)$.

Let $B = \bigcup\{\mathrm{partlims}(D) : D \subseteq A \text{ is countable}\}$. Then B is closed under partial limits of sequences: Let $f_1, f_2, \ldots \in B$. Then there are countable $D_1, D_2, \ldots \subseteq A$, such that $f_n \in \mathrm{partlims}(D_n)$ for all n. Let $D = \bigcup_n D_n$. Then $f_1, f_2, \ldots \in \mathrm{partlims}(D)$, and therefore $\lim_n f_n \in \mathrm{partlims}(D) \subseteq B$.

Thus, $\mathrm{partlims}(A) \subseteq B$, as required. □

By a classical result of Arhangel'skiĭ, $\mathrm{C}(X)$ is countably tight for all $X \subseteq \mathbb{R}$ (indeed, for all topological spaces X such that all finite powers of X are Lindelöf). However, $\mathrm{B}(X)$ is not countably tight, unless X is countable (in which case, $\mathbb{R}^X$, and thus $\mathrm{B}(X)$, is metrizable).

We denote by $\mathbf{1}$ the constant function identically equal to 1 on X.

Proposition 1.7. *Let X be an uncountable space, where each singleton is Borel. Then $\mathrm{B}(X)$ is not countably tight.*

Proof. Take $A = \{\chi_F : F \subseteq X \text{ finite}\} \subseteq \mathrm{B}(X)$, where χ_F denotes the characteristic function of F. Then the constant function $\mathbf{1}$ is in $\overline{A}$. Let $D = \{\chi_{F_n} : n \in \mathbb{N}\} \subseteq X$. Take $a \in X \setminus \bigcup_n F_n$. Then $\chi_{F_n}(a) = 0$ for all n, and thus $\mathbf{1} \notin \overline{D}$. □

Problem 1.8 (*Miller 2005 [14]*). Assume that, for each countable $A \subseteq \mathrm{B}(X)$, each $f \in \overline{A}$ belongs to the closure of A under partial limits of sequences. Does it follow that, for each countable $A \subseteq \mathrm{B}(X)$, each $f \in \overline{A}$ is a limit of a sequence of elements of A?

Our main result (Section 2) is a solution, in the affirmative, of Miller's problem. At the end of the second author's talk in the conference *Functional Analysis in Valencia 2010*, Salvador Hernańdez asked what is the solution to Miller's problem when considering Baire class 1 functions (i.e., functions which are pointwise limits of sequences of continuous functions). We solve Hernańdez's problem in Section 3. Finally, we establish several results concerning the original Gerlits–Nagy Problem, and pose some related problems.

2. Borel functions (Miller's problems)

We solve Miller's Problem 1.8 in the affirmative. Indeed, we do so not only for sets $X \subseteq \mathbb{R}$, but for all topological spaces X.

Theorem 2.1. *Let X be a topological space. Assume that, for each countable $A \subseteq \mathrm{B}(X)$, each $f \in \overline{A}$ belongs to the closure of A under partial limits of sequences. Then for each countable $A \subseteq \mathrm{B}(X)$, each $f \in \overline{A}$ is a limit of a sequence of elements of A.*

The proof is divided naturally into four steps. For brevity, we make the following convention, that will hold throughout the paper.

Convention 2.2. Let X be a topological space. We say that $\mathcal{U}$ is a *cover* of X if $X = \bigcup \mathcal{U}$, but $X \notin \mathcal{U}$. By *Borel cover* of X we always mean a *countable* family $\mathcal{U}$ of Borel subsets of X, such that the union of all members of $\mathcal{U}$ is X.

Step 1: Local to global

We deduce from the given local property of $\mathrm{B}(X)$, a global property of X.

Definition 2.3 (*Gerlits–Nagy [7]*). A cover $\mathcal{U}$ of X is an *ω-cover* of X if each finite $F \subseteq X$ is contained in a member of $\mathcal{U}$.

For sets $B_1, B_2, \ldots$, let

$$\mathrm{liminf}_n B_n = \bigcup_m \bigcap_{n \geq m} B_n,$$

that is, the set of all x which belong to B_n for all but finitely many n. Let $\mathrm{LI}(\mathcal{U})$ be the closure of $\mathcal{U}$ under the operator liminf.

A basic property of $\mathrm{liminf}_n B_n$ is that it does not depend on the first few sets B_n.

Lemma 2.4. *Let X be a topological space. Assume that, for each countable $A \subseteq \mathrm{B}(X)$, each $f \in \overline{A}$ belongs to* $\mathrm{partlims}(A)$. *Then for each Borel ω-cover $\mathcal{U}$ of X, $X \in \mathrm{LI}(\mathcal{U})$.*

Proof. Let $\mathcal{U}$ be a Borel ω-cover of X. Take $A = \{\chi_U : U \in \mathcal{U}\}$. Then $A \subseteq \mathrm{B}(X)$ is countable, and $\mathbf{1} \in \overline{A}$. Thus, $\mathbf{1} \in \mathrm{partlims}(A)$.

As each $f \in A$ is $\{0, 1\}$-valued, and limits of convergent sequences of 0's and 1's must be either 0 or 1, each f in $\mathrm{partlims}(A)$ is $\{0, 1\}$-valued. Let C be the set of all partial $\{0, 1\}$-valued functions f on X, such that $f^{-1}(1) \in \mathrm{LI}(\mathcal{U})$. Then $A \subseteq C$, and C is closed under partial limits of sequences. Indeed, let $f_1, f_2, \ldots \in C$, and $f = \lim_n f_n$. As $\lim_n f_n(x) = f(x)$ and the functions f_n are $\{0, 1\}$-valued, $f^{-1}(1) = \liminf_n f_n^{-1}(1) \in \mathrm{LI}(\mathcal{U})$.

Therefore, $\mathrm{partlims}(A)$ is contained in C, and in particular $\mathbf{1} \in C$, that is, there is $B \in \mathrm{LI}(\mathcal{U})$ such that $X = \mathbf{1}^{-1}(1) \subseteq B$. Thus, $X = B \in \mathrm{LI}(\mathcal{U})$. □

Step 2: A selective property

Definition 2.5. For a family $\mathcal{F}$ of subsets of X, let

$$\mathcal{F}_{\downarrow} = \{B \subseteq X : (\exists A \in \mathcal{F})\ B \subseteq A\},$$

the closure of $\mathcal{F}$ under taking subsets.

For a family $\mathcal{F}$ of sets, $\bigcup \mathcal{F}$ (without running index) denotes the union of all members of $\mathcal{F}$. We say that a family of sets $\mathcal{V}$ *refines* another family $\mathcal{U}$ if each $V \in \mathcal{V}$ is contained in some $U \in \mathcal{U}$. The following result may be obtained by following arguments of Gerlits and Nagy [7] and arguments of Nowik et al. [15], proved for open covers (under certain hypotheses on the space X). We provide a different, direct proof, which makes no assumption on X.

Proposition 2.6. *Let X be a topological space. Assume that for each Borel ω-cover $\mathcal{U}$ of X, $X \in \mathrm{LI}(\mathcal{U})$. Then for each sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of Borel covers of X, there are finite sets $\mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \ldots,$ such that for each $x \in X$, $x \in \bigcup \mathcal{F}_n$ for all but finitely many n.*

Proof. By moving to refinements, we may assume that for each n, the elements of $\mathcal{U}_n$ are pairwise disjoint, and $\mathcal{U}_{n+1}$ refines $\mathcal{U}_n$.[1] This way, if there are infinitely many n such that $\mathcal{U}_n$ contains a finite subcover $\mathcal{F}_n$ of X, then this is true for all n and the required assertion follows immediately. Thus, we may assume that for each $n, \mathcal{U}_n$ does not contain a finite subcover of X.

Let

$$\mathcal{B} = \left\{\liminf_n \bigcup \mathcal{F}_n : (\forall n)\mathcal{F}_n \text{ is a finite subset of } \mathcal{U}_n\right\}.$$

We must prove that $X \in \mathcal{B}$.

$\mathrm{LI}(\mathcal{B}_{\downarrow}) = \mathcal{B}_{\downarrow}$: For each k, assume that $B_k \subseteq \liminf_n \bigcup \mathcal{F}_n^k$, with each $\mathcal{F}_n^k$ a finite subset of $\mathcal{U}_n$. Take $\mathcal{F}_n = \mathcal{F}_n^1 \cup \mathcal{F}_n^2 \cup \cdots \cup \mathcal{F}_n^n$ for each n. Then

$$\liminf_k B_k \subseteq \liminf_k \liminf_n \bigcup \mathcal{F}_n^k \subseteq \liminf_n \bigcup \mathcal{F}_n \in \mathcal{B},$$

and thus $\liminf_n B_n \in \mathcal{B}_{\downarrow}$.

Thus, $\mathrm{LI}(\mathcal{B}) \subseteq \mathcal{B}_{\downarrow}$, and therefore if $X \in \mathrm{LI}(\mathcal{B})$ then $X \in \mathcal{B}$. $\mathcal{B}$ is an ω-cover of X and its elements are Borel, but $\mathcal{B}$ is in general *not* countable, and thus we cannot apply the premise of the lemma. To overcome this problem, we use a trick similar to one in [7]: Define

$$\mathcal{A} = \bigcup_{n\in\mathbb{N}} \left\{\bigcup \mathcal{F} : \mathcal{F} \subseteq \mathcal{U}_n, |\mathcal{F}| = n\right\}.$$

$\mathcal{A}$ is a Borel ω-cover of X, and therefore by the premise of the lemma, $X \in \mathrm{LI}(\mathcal{A}) \subseteq \mathrm{LI}(\mathcal{A}_{\downarrow} \cup \mathcal{B}_{\downarrow})$. As $X \notin \mathcal{A}$, it remains to show that $\mathrm{LI}(\mathcal{A}_{\downarrow} \cup \mathcal{B}_{\downarrow}) = \mathcal{A}_{\downarrow} \cup \mathcal{B}_{\downarrow}$.

Let $B_1, B_2, \ldots \in \mathcal{A}_{\downarrow} \cup \mathcal{B}_{\downarrow}$. As $\liminf_n B_n \subseteq \liminf_n B_{m_n}$ for each increasing sequence m_n, and $\mathcal{A}_{\downarrow} \cup \mathcal{B}_{\downarrow}$ is closed downwards, we may move to subsequences at our convenience.

[1] Given a Borel cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$, the Borel cover $\{U_n \setminus (U_1 \cup \cdots \cup U_{n-1}) : n \in \mathbb{N}\}$ refines $\mathcal{U}$, and its elements are pairwise disjoint. Given two Borel covers $\mathcal{U}, \mathcal{V}$ whose elements are pairwise disjoint, the Borel cover $\{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$ refines $\mathcal{U}$ and $\mathcal{V}$, and in particular its elements are pairwise disjoint.

If $B_n \in \mathcal{B}_\downarrow$ for infinitely many n, then by moving to a subsequence we may assume that $B_n \in \mathcal{B}_\downarrow$ for all n, and therefore $\liminf_n B_n \in \mathrm{LI}(\mathcal{B}_\downarrow) = \mathcal{B}_\downarrow \subseteq \mathcal{A}_\downarrow \cup \mathcal{B}_\downarrow$. In the remaining case, by moving to a subsequence, we may assume that $B_n \in \mathcal{A}_\downarrow$ for all n.

Consider first the case where, after moving to an appropriate subsequence of $B_1, B_2, \dots$, there is an increasing sequence k_n such that $B_n \subseteq \bigcup \mathcal{F}_{k_n}$, $\mathcal{F}_{k_n} \subseteq \mathcal{U}_{k_n}$ with $|\mathcal{F}_{k_n}| = k_n$, for all n. As the covers $\mathcal{U}_n$ are getting finer with n, for each $i \notin \{k_n : n \in \mathbb{N}\}$ there is a finite $\mathcal{F}_i \subseteq \mathcal{U}_i$ such that $\bigcup \mathcal{F}_i$ contains $\bigcup \mathcal{F}_{k_n}$ for the first n with $i < k_n$. Then

$$\liminf_n B_n \subseteq \liminf_n \bigcup \mathcal{F}_n \in \mathcal{B},$$

as required.

Finally, there remains the case where, after moving to an appropriate subsequence of B_1, $B_2, \dots$, there is k such that for each n, there is $\mathcal{F}_n \subseteq \mathcal{U}_k$ with $|\mathcal{F}_n| = k$, such that $B_n \subseteq \bigcup \mathcal{F}_n$. Let $B = \liminf B_n$. We will show that $B \in \mathcal{A}_\downarrow$. We may assume that $B \neq \emptyset$. Take $x_1 \in B$, and $U_1 \in \mathcal{U}_k$ such that $x_1 \in U_1$. If $B \subseteq U_1$, then $B \in \mathcal{A}_\downarrow$. Otherwise, take $x_2 \in B \setminus U_1$, and $U_2 \in \mathcal{U}_k$ such that $x_2 \in U_2$. Continue in the same manner until it is impossible to proceed, but not more than k steps, to have $x_1, \dots, x_i \in B$, where $i \leq k$, and distinct (and therefore disjoint) $U_1, \dots, U_i \in \mathcal{U}_k$. If $i < k$, then $B \subseteq U_1 \cup \dots \cup U_i$, a union of less than k elements of $\mathcal{U}_k$, and thus $B \in \mathcal{A}_\downarrow$. Otherwise $i = k$, and for all but finitely many n, $x_1, \dots, x_k \in B_n \subseteq \bigcup \mathcal{F}_n$, and as the elements of $\mathcal{U}_k$ are pairwise disjoint, $\mathcal{F}_n = \{U_1, \dots, U_k\}$ for all but finitely many n. Consequently, $B \subseteq \liminf_n \bigcup \mathcal{F}_n = U_1 \cup \dots \cup U_k \in \mathcal{A}$, and therefore $B \in \mathcal{A}_\downarrow$. □

Step 3: A stronger selective property

The selective property in the following theorem is stronger ([20], or Lemma 4.1) than the one introduced in the previous step. In its original formulation [14], Miller's Problem 1.8 asks whether the following theorem is true.

Theorem 2.7. *Assume that for each Borel ω-cover $\mathcal{U}$ of X, $X \in \mathrm{LI}(\mathcal{U})$. Then in fact, for each Borel ω-cover $\mathcal{U}$ of X, there are $U_1, U_2, \dots \in \mathcal{U}$ such that $X = \liminf_n U_n$.*

Proof. Let

$$\mathcal{B} = \{\liminf_n U_n : U_1, U_2, \dots \in \mathcal{U}\}_\downarrow.$$

It suffices to show that $\mathrm{LI}(\mathcal{B}) = \mathcal{B}$. Let $B_1, B_2, \dots \in \mathcal{B}$, and $B = \liminf_n B_n$. For each n, take $U_1^n, U_2^n, \dots \in \mathcal{U}$ such that $B_n \subseteq \liminf_m U_m^n$. Then for each n, the sets $V_m^n = \bigcap_{k \geq m} U_k^n$ are increasing to B_n, and therefore the sets $V_m^n \cup (X \setminus B_n)$ are increasing to X.

Applying Proposition 2.6 to the covers $\mathcal{U}_n = \{V_m^n \cup (X \setminus B_n) : m \in \mathbb{N}\}$, there are m_n such that $X = \liminf_n V_{m_n}^n \cup (X \setminus B_n)$ (since the covers are increasing, it suffices to pick one element from each cover). As $\liminf_n B_n = B$, we have that

$$B \subseteq (\liminf_n V_{m_n}^n \cup (X \setminus B_n)) \cap B \subseteq \liminf_n V_{m_n}^n \subseteq \liminf_n U_{m_n}^n,$$

and therefore $B \in \mathcal{B}$. □

Step 4: Global to local

The following lemma and its proof are, in the open/continuous case, due to Gerlits and Nagy [7]. Their argument also applies to the Borel case.

Lemma 2.8. *Assume that for each Borel ω-cover $\mathcal{U}$ of X, there are $U_1, U_2, \ldots \in \mathcal{U}$ such that $X = \liminf_n U_n$. Then for each countable $A \subseteq \mathrm{B}(X)$, each $f \in \overline{A}$ is a pointwise limit of a sequence of elements of A.*

Proof. We may assume, by adding the function $\mathbf{1} - f$ to all considered functions, that $f = \mathbf{1}$, the constant 1 function. For each n, let $\mathcal{U}_n = \{g^{-1}[(1 - 1/n, 1 + 1/n)] : g \in A\}$. As $\mathbf{1} \in \overline{A}$, $\mathcal{U}_n$ is a (Borel) ω-cover of X. By Theorem 2.7, there are $g_n \in A$ such that $X = \liminf_n g_n^{-1}[(1 - 1/n, 1 + 1/n)]$. Then $\mathbf{1} = \lim_n g_n$. □

This completes the proof of Theorem 2.1.

3. Baire class 1 functions (Hernańdez's problem)

The following Theorem, which strengthens Theorem 2.1 (in the realm of perfectly normal spaces), answers in the positive a question of Salvador Hernández.

A topological space X is *perfectly normal* if it is normal (any two disjoint closed sets have disjoint neighborhoods), and each open subset of X is F_σ, that is, a union of countably many closed subsets of X. For example, metric spaces are perfectly normal.

A function $f : X \to \mathbb{R}$ is of *Baire class 1* if f is the pointwise limit of a sequence of continuous real-valued functions on X. Let $\mathrm{Baire}_1(X) \subseteq \mathbb{R}^X$ denote the subspace of all Baire class 1 functions $f : X \to \mathbb{R}$.

Theorem 3.1. *Let X be a perfectly normal topological space. Assume that, for each countable $A \subseteq \mathrm{Baire}_1(X)$, each $f \in \overline{A}$ (closure in $\mathrm{Baire}_1(X)$) belongs to the closure of A under partial limits of sequences. Then for each countable $A \subseteq \mathrm{Baire}_1(X)$, each $f \in \overline{A}$ (closure in $\mathrm{Baire}_1(X)$) is a limit of a sequence of elements of A.*

Moreover, for each countable $A \subseteq \mathrm{B}(X)$, each $f \in \overline{A}$ (closure in $\mathrm{B}(X)$) is a limit of a sequence of elements of A.

Proof. As the closure in a subspace Y of $\mathbb{R}^X$ is equal to the intersection of the closure in $\mathbb{R}^X$ and Y, and $\mathrm{Baire}_1(X) \subseteq \mathrm{B}(X)$, it suffices to prove the second assertion. We follow the proof steps of Theorem 2.1, and modify them when needed.

A set $A \subseteq X$ is Δ^0_2 if both A and $X \setminus A$ are F_σ. The family $\Delta^0_2(X)$ of all Δ^0_2 subsets of X forms an algebra of sets, that is, it is closed under finite unions and complements (and therefore also under finite intersections and set differences). This fact is applied repeatedly when following the steps in the proof of Theorem 2.1.

A function $f : X \to \mathbb{R}$ is *Δ^0_2-measurable* if for each open $U \subseteq \mathbb{R}$, $f^{-1}[U]$ is Δ^0_2. For each Δ^0_2 set $U \subseteq X$, χ_U is Δ^0_2-measurable.

The following lemma is proved for the metrizable case in [11, Lemma 24.12]. The proof there uses only Urysohn's lemma, which applies for all normal spaces.

Lemma 3.2 (*Folklore*). *Let X be a normal space, and U be a Δ^0_2 subset of X. Then χ_U is of Baire class 1.*

Proof. Let $F_n \subseteq X$ be closed, and $G_n \subseteq X$ be open, such that $F_n \subseteq F_{n+1} \subseteq U \subseteq G_{n+1} \subseteq G_n$ for all n, and $U = \bigcup_n F_n = \bigcap_n G_n$. By Urysohn's lemma, there is for each n a continuous function $f_n : X \to \mathbb{R}$ such that $f_n(x) = 1$ for all $x \in F_n$ and $f_n(x) = 0$ for all $x \notin G_n$. Then $\lim_n f_n(x) = \chi_U(x)$ for all $x \in X$. □

Thus, arguing as in Step 1 of Theorem 2.1, we have that for each countable $\Delta^0_2\omega$-cover $\mathcal{U}$ of X, $X \in \mathrm{LI}(\mathcal{U})$.

The arguments of Step 2 show the following.

Proposition 3.3. *Assume that for each countable $\Delta^0_2\omega$-cover $\mathcal{U}$ of X, $X \in \mathrm{LI}(\mathcal{U})$. Then for each sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of countable Δ^0_2 covers of X, there are finite sets $\mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \ldots,$ such that for each $x \in X$, $x \in \bigcup \mathcal{F}_n$ for all but finitely many n.* □

In particular, as X is perfectly normal, X has the property in the conclusion of Proposition 3.3 for *closed* sets. We use the following strong result of Bukovský, Recław, and Repický.

Lemma 3.4 (*Bukovský–Recław–Repický [1]*)**.** *Let X be a perfectly normal space. Assume that for each sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of countable closed covers of X, there are finite sets $\mathcal{F}_1 \subseteq \mathcal{U}_1$, $\mathcal{F}_2 \subseteq \mathcal{U}_2, \ldots,$ such that for each $x \in X$, $x \in \bigcup \mathcal{F}_n$ for all but finitely many n. Then the same holds for each sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of Borel covers of X.*

The property established in Lemma 3.4 implies that every Borel subset of X is F_σ (e.g., [20]), and thus every Borel set is Δ^0_2. By the property established before Proposition 3.3, we have that, for each countable *Borel* ω-cover $\mathcal{U}$ of X, $X \in \mathrm{LI}(\mathcal{U})$.

Thus, Theorem 2.7 and Step 4 apply, and the proof is completed. □

Remark 3.5. The proof of Theorem 3.1 shows that it suffices to assume that for each countable set A of Δ^0_2-measurable real-valued functions on X, the closure of A in the space of all Δ^0_2-measurable real-valued functions on X is contained in $\mathrm{partlims}(A)$.

4. Continuous functions (Gerlits–Nagy's problem)

Thus far, we have refrained from using the notation of the field of selective properties, despite their playing important role in the proofs. However, as we are about to make a more extensive use of the theory, we give here the necessary introduction. Readers who wish to learn more on the topic and its history are referred to any of its surveys [19,12,21].

Let X be a topological space. Let $\mathrm{O}(X)$ be the family of all open covers of X. Define the following subfamilies of $\mathrm{O}(X)$: $\mathcal{U} \in \Omega(X)$ if $\mathcal{U}$ is an ω-cover of X. $\mathcal{U} \in \Gamma(X)$ if $\mathcal{U}$ is infinite, and each element of X is contained in all but finitely many members of $\mathcal{U}$.

Some of the following statements may hold for families $\mathscr{A}$ and $\mathscr{B}$ of covers of X.

$\binom{\mathscr{A}}{\mathscr{B}}$: Each element of $\mathscr{A}$ contains an element of $\mathscr{B}$.

$\mathsf{S}_1(\mathscr{A}, \mathscr{B})$: For all $\mathcal{U}_1, \mathcal{U}_2, \ldots \in \mathscr{A}$, there are $U_1 \in \mathcal{U}_1, U_2 \in \mathcal{U}_2, \ldots$ such that $\{U_n : n \in \mathbb{N}\} \in \mathscr{B}$.

$\mathsf{S}_{\mathrm{fin}}(\mathscr{A}, \mathscr{B})$: For all $\mathcal{U}_1, \mathcal{U}_2, \ldots \in \mathscr{A}$, there are finite $\mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \ldots$ such that $\bigcup_n \mathcal{F}_n \in \mathscr{B}$.

$\mathsf{U}_{\mathrm{fin}}(\mathscr{A}, \mathscr{B})$: For all $\mathcal{U}_1, \mathcal{U}_2, \ldots \in \mathscr{A}$, none containing a finite subcover, there are finite $\mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \ldots$ such that $\{\bigcup \mathcal{F}_n : n \in \mathbb{N}\} \in \mathscr{B}$.

We say, e.g., that X satisfies $\mathsf{S}_1(\mathrm{O}, \mathrm{O})$ if the statement $\mathsf{S}_1(\mathrm{O}(X), \mathrm{O}(X))$ holds. This way, $\mathsf{S}_1(\mathrm{O}, \mathrm{O})$ is a property of topological spaces, and similarly for all other statements and families of covers. Under some mild hypotheses on the considered topological spaces, each nontrivial property among these properties, where $\mathscr{A}, \mathscr{B}$ range over O, Ω, Γ, is equivalent to one in Fig. 1,

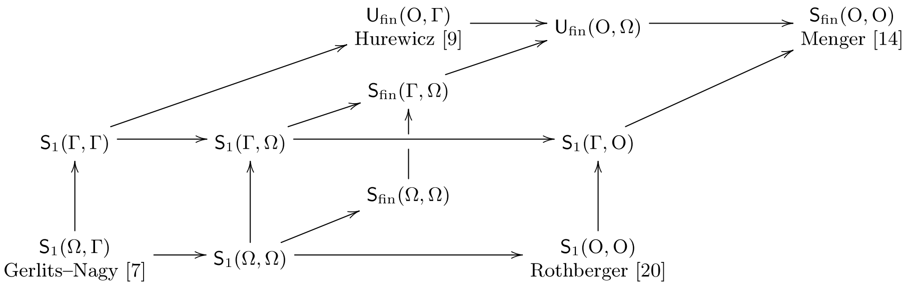

Fig. 1. The Scheepers diagram.

named after Scheepers in recognition of his seminal contribution to the field. In this diagram, an arrow denotes implication.

Other types of covers, most notably *Borel* covers, were also considered in this context. We say, for example, that X satisfies $\mathsf{S}_1(\Omega, \Omega)$ *for Borel covers* if $\mathsf{S}_1(\Omega(X), \Omega(X))$ holds, when redefining $\Omega(X)$ to consist of all countable Borel ω-covers of X.

For clarity of notation, we identify a property with the family of topological spaces (of a certain type, which should be clear from the context) satisfying it.

The property deduced in Proposition 2.6 is $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$ for Borel covers. For Borel covers, $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma) = \mathsf{S}_1(\Gamma, \Gamma)$ [20], and using this the proof of Theorem 2.7 can be slightly simplified.

Gerlits and Nagy [7] proved the following lemma for Hausdorff spaces. We will see that it holds for arbitrary topological spaces.

Lemma 4.1. $\binom{\Omega}{\Gamma} = \mathsf{S}_1(\Omega, \Gamma)$ *(for general topological spaces).*

Proof. Assume that X satisfies $\binom{\Omega}{\Gamma}$, and let $\mathcal{U}_1, \mathcal{U}_2, \ldots$ be open ω-covers of X. We may assume that for each n, $\mathcal{U}_{n+1}$ refines $\mathcal{U}_n$.

For each n, enumerate $\mathcal{U}_n = \{U^n_m : m \in \mathbb{N}\}$. Let $V_m = U^1_m$ for all m. Define

$$\mathcal{W} = \bigcup_{n\in\mathbb{N}} \{V_n \cap U^n_m : m \in \mathbb{N}\}.$$

$\mathcal{W}$ is an open ω-cover of X. Thus, there are $W_1, W_2, \ldots \in \mathcal{W}$ such that $X = \liminf_k W_k$. Fix n. As $V_n \neq X$, it is not possible that $W_k \in \{V_n \cap U^n_m : m \in \mathbb{N}\}$ for infinitely many k. Since the sets U^n_m are increasing with m, we may assume that there is at most one W_k in each set $\mathcal{W}_n = \{V_n \cap U^n_m : m \in \mathbb{N}\}$. For each n, let $r_n \geq n$ be the first such that there is some W_k in $\mathcal{W}_{r_n}$. Since the covers $\mathcal{U}_n$ get finer with n, we can pick for each n an element $U^n_{m_n} \in \mathcal{U}_n$ containing the W_k which is in $\mathcal{W}_{r_n}$. Then $X = \liminf_k W_k \subseteq \liminf_n U^n_{m_n}$, and therefore $\liminf_n U^n_{m_n} = X$. $\square$

Using Lemma 4.1, Gerlits and Nagy proved the following fundamental local-to-global correspondence result.

Theorem 4.2 (*Gerlits–Nagy [7]*)**.** *For Tychonoff spaces X, the following properties are equivalent:*

(1) *For each $A \subseteq \mathrm{C}(X)$, each $f \in \overline{A}$ is a limit of a sequence of elements of A (i.e., $\mathrm{C}(X)$ is Fréchet–Urysohn).*
(2) *X satisfies $\binom{\Omega}{\Gamma}$.*

We establish a similar result for the other major property studied in the present paper. To this end, we need the following definition and a lemma.

Definition 4.3. $\mathrm{L}(X)$ is the family of open covers of X such that $X \in \mathrm{LI}(\mathcal{U})$.

Theorem 2.7 tells that $\binom{\Omega}{\mathrm{L}} = \binom{\Omega}{\Gamma}$ for Borel covers. In particular, using that $\binom{\Omega}{\Gamma} = \mathsf{S}_1(\Omega, \Gamma)$ for Borel covers, we have that $\binom{\Omega}{\mathrm{L}} = \mathsf{S}_1(\Omega, \mathrm{L})$ for Borel covers. The last assertion also holds in the open case, but a different proof is required.

Lemma 4.4. $\binom{\Omega}{\mathrm{L}} = \mathsf{S}_1(\Omega, \mathrm{L}) = \mathsf{S}_{\mathrm{fin}}(\Omega, \mathrm{L})$.

Proof. As $\mathsf{S}_1(\Omega, \mathrm{L})$ implies $\mathsf{S}_{\mathrm{fin}}(\Omega, \mathrm{L})$, which in turn implies $\binom{\Omega}{\mathrm{L}}$, it remains to prove that $\binom{\Omega}{\mathrm{L}}$ implies $\mathsf{S}_1(\Omega, \mathrm{L})$. To this end, it suffices to prove that $\binom{\Omega}{\mathrm{L}}$ implies $\mathsf{S}_1(\Omega, \Omega)$. $\mathsf{S}_1(\Omega, \Omega)$ is equivalent to having all finite powers of X satisfy $\mathsf{S}_1(\mathrm{O}, \mathrm{O})$ [18]. Gerlits and Nagy [7] proved that $\binom{\Omega}{\mathrm{L}}$ implies $\mathsf{S}_1(\mathrm{O}, \mathrm{O})$. Thus, it remains to prove that $\binom{\Omega}{\mathrm{L}}$ is preserved by finite powers.

Assume that X satisfies $\binom{\Omega}{\mathrm{L}}$, and let $k \in \mathbb{N}$. Let $\mathcal{U}$ be an open ω-cover of X^k. Then there is an open ω-cover $\mathcal{V}$ of X such that $\mathcal{V}' = \{V^k : V \in \mathcal{V}\}$ refines $\mathcal{U}$ [10]. Then $X \in \mathrm{LI}(\mathcal{V})$. For arbitrary sets $B_1, B_2, \dots$, $\liminf_n (B_n)^k = (\liminf_n B_n)^k$. Thus, $X^k \in \{B^k : B \in \mathrm{LI}(\mathcal{V})\} = \mathrm{LI}(\mathcal{V}')$, and therefore $X^k \in \mathrm{LI}(\mathcal{U})$. $\square$

Theorem 4.5. *For Tychonoff spaces X, the following properties are equivalent:*

(1) *For each $A \subseteq \mathrm{C}(X)$, $\overline{A} \subseteq \mathrm{partlims}(A)$.*
(2) *X satisfies $\binom{\Omega}{\mathrm{L}}$ (that is, for each open ω-cover $\mathcal{U}$ of X, $X \in \mathrm{LI}(\mathcal{U})$).*

Proof. ($1 \Rightarrow 2$) For partial functions f and g, $g \circ f$ is the partial function with domain $\{x \in \mathrm{dom}(f) : f(x) \in \mathrm{dom}(g)\}$, defined as usual by $g \circ f(x) = g(f(x))$.

For a surjection $\varphi : X \to Y$ and partial functions $f_n : Y \to \mathbb{R}$, the domain of $\lim_n (f_n \circ \varphi)$ is $\varphi^{-1}[\mathrm{dom}(\lim_n f_n)]$, and $\lim_n (f_n \circ \varphi) = (\lim_n f_n) \circ \varphi$. Thus, we have the following.

Lemma 4.6. *Assume that for each $A \subseteq \mathrm{C}(X)$, each $\overline{A} \subseteq \mathrm{partlims}(A)$. Then every continuous image of X has the same property.* $\square$

A topological space is *zero-dimensional* if its clopen (simultaneously closed and open) sets form a base for its topology. An argument similar to one in [7] gives the following.

Lemma 4.7. *Let X be a Tychonoff space. Assume that for each $A \subseteq \mathrm{C}(X)$, each $f \in \overline{A}$ belongs to the closure of A under partial limits of sequences. Then X is zero-dimensional.*

Proof. It suffices to prove that $[0, 1]$ is not a continuous image of X. Indeed, for each open $U \subseteq X$ and each $a \in U$, let $\Psi : X \to [0, 1]$ be continuous, such that $\Psi(a) = 0$ and $\Psi(x) = 1$ for all $x \in X \setminus U$. Take $r \in [0, 1]$ which is not in the image of Ψ. Then $\Psi^{-1}[[0, r)]$ is a clopen neighborhood of x contained in U.

Assume that $[0, 1]$ is a continuous image of X. Let $A \subseteq \mathrm{C}([0, 1])$ be the set of all continuous $f : [0, 1] \to [0, 1]$ such that the Lebesgue measure of $f^{-1}[(1/2, 1]]$ is at most $1/2$. Then $\mathbf{1}$ is in the closure of A. Let C be the set of all *partial* $f : [0, 1] \to [0, 1]$ such that $f^{-1}[(1/2, 1]]$ is Lebesgue measurable, and its measure is at most $1/2$. C is closed under partial limits of sequences and contains A, but $\mathbf{1} \notin C$; a contradiction. $\square$

Let $\mathcal{U}$ be an open ω-cover of X. As X is zero-dimensional, $\mathcal{U}$ can be refined to a clopen ω-cover of X by replacing each $U \in \mathcal{U}$ with all finite unions of clopen subsets of U. Now, for each clopen U the function χ_U is continuous, and $\mathbf{1}$ is in the closure of $\{\chi_U : U \in \mathcal{U}\}$. By (1), $\mathbf{1}$ is in the closure of $\{\chi_U : U \in \mathcal{U}\}$ under partial limits of sequences. Continue as in the proof of Lemma 2.4.

($2 \Rightarrow 1$) In (1), by adding $\mathbf{1} - f$ to all of the involved partial functions, it suffices to consider the case $f = \mathbf{1}$. Let $A \subseteq \mathrm{C}(X)$, and assume that $\mathbf{1} \in \overline{A}$. For each n, let $\mathcal{U}_n = \{f^{-1}[(1-1/n, 1+1/n)] : f \in A\}$. $\mathcal{U}_n$ is an open ω-cover of X. By Lemma 4.4, there are $f_1, f_2, \ldots \in A$ such that $X \in \mathrm{LI}(\{f_n^{-1}[(1-1/n, 1+1/n)] : n \in \mathbb{N}\})$.

We claim that

$$\mathcal{A} = (\{f_n^{-1}[(1-1/n, 1+1/n)] : n \in \mathbb{N}\} \cup \{f^{-1}(1) : f \in \mathrm{partlims}(A)\})_{\downarrow}$$

is closed under the operator liminf. Indeed, assume that we are given a sequence of elements of $\mathcal{A}$. By thinning it out, and replacing each element by an appropriate element containing it, we may assume that this sequence is all in $\{f_n^{-1}[(1-1/n, 1+1/n)] : n \in \mathbb{N}\}$ or all in $\{f^{-1}(1) : f \in \mathrm{partlims}(A)\}$. In the first case, by thinning out further we may assume that the sequence is either constant (in which case we are done), or consists of distinct elements $f_{m_n}^{-1}[(1-1/m_n, 1+1/m_n)]$ with m_n increasing. In this case, let $f = \lim_n f_{m_n}$. For each $x \in \liminf_n f_{m_n}^{-1}[(1-1/m_n, 1+1/m_n)]$, $f(x) = \lim_n f_{m_n}(x) = 1$, and thus $\liminf_n f_{m_n}^{-1}[(1-1/m_n, 1+1/m_n)]$ is in $\mathcal{A}$. The second case is similar (and slightly easier).

Thus, $X \in \mathcal{A}$, which means that there is $f \in \mathrm{partlims}(A)$ such that $X = f^{-1}(1)$, that is, $\mathbf{1} = f \in \mathrm{partlims}(A)$. □

Clearly, $\binom{\Omega}{\Gamma}$ implies $\binom{\Omega}{\mathrm{L}}$. The original Gerlits–Nagy Problem, posed in [7], asks whether these properties are in fact equivalent (for Tychonoff X, or even for $X \subseteq \mathbb{R}$). Theorems 4.2 and 4.5 justify the reformulation given in Problem 1.5.

Originally, Gerlits and Nagy [7] studied five properties, numbered $\alpha, \beta, \gamma, \delta, \epsilon$, where each property implies the subsequent one. $\binom{\Omega}{\Gamma}$ and $\binom{\Omega}{\mathrm{L}}$ were numbered γ and δ, respectively, and are often named accordingly in the literature. Their problem was originally stated as whether property δ implies (and is therefore equivalent to) property γ.

A topological space X is said to satisfy a property P *hereditarily* if each $Y \subseteq X$ satisfies P. Pushing our methods further, we can solve the Gerlits–Nagy Problem in the affirmative for spaces X satisfying $\mathsf{S}_1(\Gamma, \Gamma)$ hereditarily. We will use the following result of Francis Jordan [9] (see also [16]), proved using a new fusion argument of his.

Lemma 4.8 (*Jordan*)**.** *Let $B = \bigcup_n B_n \subseteq X$ be an increasing union, where each B_n satisfies $\mathsf{S}_1(\Gamma, \Gamma)$. For all open sets $U^n_m \subseteq X$, $n, m \in \mathbb{N}$, with $B_n \subseteq \liminf_m U^n_m$ for each n, there are $m_1, m_2, \ldots \in \mathbb{N}$ such that $B \subseteq \liminf_n U^n_{m_n}$.*

Theorem 4.9. *For topological spaces X satisfying $\mathsf{S}_1(\Gamma, \Gamma)$ hereditarily, the following are equivalent:*

(1) *X satisfies* $\binom{\Omega}{\mathrm{L}}$.

(2) *X satisfies* $\binom{\Omega}{\Gamma}$.

Proof of (1) $\Rightarrow$ (2)**. Lemma 4.10.** *Assume that X satisfies $\mathsf{S}_1(\Gamma, \Gamma)$ hereditarily. Then X satisfies* $\binom{\mathrm{L}}{\Gamma}$.

Proof. Let $\mathcal{U}$ be an open cover of X with $X \in \mathrm{LI}(\mathcal{U})$. Define

$$\mathcal{B} = \{\liminf_n U_n : U_1, U_2, \ldots \in \mathcal{U}\}.$$

We will prove that $X \in \mathcal{B}$. To this end, it suffices to show that $\mathrm{LI}(\mathcal{B}_\downarrow) = \mathcal{B}_\downarrow$.

Let $B_1, B_2, \ldots \in \mathcal{B}_\downarrow$, and $B = \liminf_n B_n$. Replacing each B_n with $\bigcap_{m \geq n} B_m$, we may assume that $B_1 \subseteq B_2 \subseteq \ldots$, and $\bigcup_n B_n = B$. For each n, take $U_1^n, U_2^n, \ldots \in \mathcal{U}$ such that $B_n \subseteq \liminf_m U_m^n$. By the premise of the proposition, each B_n satisfies $\mathsf{S}_1(\Gamma, \Gamma)$. By Jordan's Lemma 4.8, there are $m_1, m_2, \ldots \in \mathbb{N}$ such that $B \subseteq_n \liminf_n U_{m_n}^n \in \mathcal{B}_\downarrow$, and therefore $B \in \mathcal{B}_\downarrow$. □

It remains to note that the conjunction of $\binom{\mathrm{L}}{\Gamma}$ and $\binom{\Omega}{\mathrm{L}}$ implies $\binom{\Omega}{\Gamma}$. □

Remark 4.11. For each topological space X, $\Gamma(X) \subseteq \mathrm{L}(X) \subseteq \Omega(X)$. To see the second inclusion, assume that there is a finite $F \subseteq X$ not covered by any $U \in \mathcal{U}$. Then F is not covered by any element of $\mathrm{LI}(\mathcal{U})$, and in particular, $X \notin \mathrm{LI}(\mathcal{U})$. Thus, the implication at the end of the proof of Theorem 4.9 is in fact an equivalence, that is, $\binom{\Omega}{\mathrm{L}} \cap \binom{\mathrm{L}}{\Gamma} = \binom{\Omega}{\Gamma}$.

Corollary 4.12. *For Tychonoff spaces X satisfying $\mathsf{S}_1(\Gamma, \Gamma)$, the following are equivalent:*

(1) *X satisfies $\binom{\Omega}{\mathrm{L}}$ hereditarily.*

(2) *X satisfies $\binom{\Omega}{\Gamma}$ hereditarily.*

Proof of (1) ⇒ (2)**.** By Theorem 4.9, it suffices to prove that X satisfies $\mathsf{S}_1(\Gamma, \Gamma)$ *hereditarily*.

Nowik, Scheepers and Weiss proved that $\binom{\Omega}{\mathrm{L}}$ implies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$ [15].[2] Thus, if X satisfies $\binom{\Omega}{\mathrm{L}}$ hereditarily, then X satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$ hereditarily. Fremlin and Miller [4] proved that in the latter case, X is a σ-space, that is, each Borel subset of X is F_σ. This, together with X's satisfying $\mathsf{S}_1(\Gamma, \Gamma)$, implies that X satisfies $\mathsf{S}_1(\Gamma, \Gamma)$ hereditarily [8,16]. □

Remark 4.13. The argument in the proof of Corollary 4.12 shows that, for Tychonoff *σ-spaces* X, $\binom{\Omega}{\Gamma} = \binom{\Omega}{\mathrm{L}} \cap \mathsf{S}_1(\Gamma, \Gamma)$. In this case, this joint property coincides with its hereditary version.

Assuming that the answer to the Gerlits–Nagy Problem is *negative*, the results of this section explain, to some extent, why no counter example was discovered thus far. A natural strategy would be to begin with a set $X \subseteq \mathbb{R}$ satisfying $\binom{\Omega}{\Gamma}$, and then look for a subset of X, in a way which "destroys" $\binom{\Omega}{\Gamma}$, but not too much, so that $\binom{\Omega}{\mathrm{L}}$ still holds. There are several constructions of subsets of $\mathbb{R}$ satisfying $\binom{\Omega}{\Gamma}$. The first one is due to Galvin and Miller [5]. Here, X has a countable subset Q such that $X \setminus Q$ does not satisfy $\binom{\Omega}{\Gamma}$. Unfortunately, $X \setminus Q$ does not even satisfy $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$, and in particular not $\binom{\Omega}{\mathrm{L}}$.[3] Another, substantially different, construction is due to Todorčevic [5], but this X satisfies $\binom{\Omega}{\Gamma}$ hereditarily. Finally, using a

[2] For a direct proof, see the proof of Proposition 2.6.

[3] On the other hand, we proved in [16] that any "natural" change of Galvin and Miller's construction *without* moving to a subset at the end would keep X in $\binom{\Omega}{\Gamma}$.

variation of Todorčevic's method, Miller [13] constructed $X \subseteq \mathbb{R}$ satisfying $\binom{\Omega}{\Gamma}$ for Borel covers, and a subset Y of X not satisfying $\binom{\Omega}{\Gamma}$. $\binom{\Omega}{\Gamma}$ for Borel covers, implies $\mathsf{S}_1(\Gamma, \Gamma)$ for Borel covers, which is hereditary. Thus, in this case X satisfies $\mathsf{S}_1(\Gamma, \Gamma)$ *hereditarily*, and by Theorem 4.9 no subset of X would separate $\binom{\Omega}{\mathrm{L}}$ from $\binom{\Omega}{\Gamma}$.

We conclude this section with a local reformulation of Theorem 4.9. A topological space Z has the Arhangel'skiĭ property α_2 if, for each $z \in Z$, whenever $\lim_m z^n_m = z$ for all n, there are $m_1, m_2, \ldots$ such that $\lim_n z^n_{m_n} = z$. When $Z = \mathrm{C}(X)$, we can take $z = \mathbf{1}$ in the definition. Haleš proved that, for perfectly normal spaces X, the following properties are equivalent:

(1) For each $Y \subseteq X$, $\mathrm{C}(Y)$ is an α_2 space.
(2) X satisfies $\mathsf{S}_1(\Gamma, \Gamma)$ hereditarily.

Collecting together the results of this section, we have the following.

Theorem 4.14. *Let X be a perfectly normal space, such that for each $Y \subseteq X$, $\mathrm{C}(Y)$ is an α_2 space. Then the following properties are equivalent:*

(1) *For each $A \subseteq \mathrm{C}(X)$, $\overline{A} \subseteq \mathrm{partlims}(A)$.*
(2) *For each $A \subseteq \mathrm{C}(X)$, each $f \in \overline{A}$ is a limit of a sequence of elements of A (i.e., $\mathrm{C}(X)$ is Fréchet–Urysohn).* □

5. Some results about the missing piece

The property $\binom{\mathrm{L}}{\Gamma}$ was central, implicitly or explicitly, in our proofs, for the basic reason that

$$\binom{\Omega}{\Gamma} = \binom{\Omega}{\mathrm{L}} \cap \binom{\mathrm{L}}{\Gamma}.$$

To prove that $\binom{\Omega}{\Gamma} = \binom{\Omega}{\mathrm{L}}$ (the Gerlits–Nagy Problem), it is necessary and sufficient to prove that $\binom{\Omega}{\mathrm{L}}$ implies $\binom{\mathrm{L}}{\Gamma}$. We therefore describe some fundamental properties of $\binom{\mathrm{L}}{\Gamma}$, and the ensuing open problems concerning it.

Proposition 5.1. $\binom{\mathrm{L}}{\Gamma} = \mathsf{S}_1(\mathrm{L}, \Gamma) = \mathsf{S}_{\mathrm{fin}}(\mathrm{L}, \Gamma)$. *In particular,* $\binom{\mathrm{L}}{\Gamma}$ *implies* $\mathsf{S}_1(\Gamma, \Gamma)$.

Proof. It suffices to prove the last assertion. Assume that for each n, $\mathcal{U}_n = \{U^n_m : m \in \mathbb{N}\} \in \Gamma(X)$. We may assume that the covers $\mathcal{U}_n$ get finer with n.[4]

Let $V_m = U^1_m$ for all m. Define

$$\mathcal{W} = \bigcup_{n \in \mathbb{N}} \{V_n \cap U^n_m : m \in \mathbb{N}\}.$$

Then

$$\mathrm{liminf}_n\, \mathrm{liminf}_m (V_n \cap U^n_m) = \mathrm{liminf}_n\, V_n = X,$$

and therefore $X \in \mathrm{LI}(\mathcal{W})$. By $\binom{\mathrm{L}}{\Gamma}$, there are $W_1, W_2, \ldots \in \mathcal{U}$ such that $X = \mathrm{liminf}_k\, W_k$. Fix n. As $V_n \neq X$, it is not possible that $W_k \in \{V_n \cap U^n_m : m \in \mathbb{N}\}$ for infinitely many k. Thus, by

[4] If $\{U_n : n \in \mathbb{N}\}, \{V_n : n \in \mathbb{N}\} \in \Gamma(X)$, then $\{U_n \cap V_n : n \in \mathbb{N}\} \in \Gamma(X)$ and is finer than both.

thinning out the sequence W_k if needed, we may assume that there is at most one W_k in each set $\{V_n \cap U^n_m : m \in \mathbb{N}\}$. Since the covers $\mathcal{U}_n$ get finer with n, we can pick for each n an element $U^n_{m_n} \in \mathcal{U}_n$, such that $X = \liminf_n U^n_{m_n}$. □

Proposition 5.2. *The property of satisfying* $\mathsf{S}_1(\Gamma, \Gamma)$ *hereditarily is strictly stronger than* $\binom{\mathrm{L}}{\Gamma}$.

Proof. Lemma 4.10 tells that hereditarily-$\mathsf{S}_1(\Gamma, \Gamma)$ implies $\binom{\mathrm{L}}{\Gamma}$. Assuming for example the Continuum Hypothesis, there is $X \subseteq \mathbb{R}$ and a subset Y of X such that X satisfies $\binom{\Omega}{\Gamma}$ $\left(\text{and thus also } \binom{\mathrm{L}}{\Gamma}\right)$, and Y does not even satisfy $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$, and in particular not $\mathsf{S}_1(\Gamma, \Gamma)$ [5]. Apply Proposition 5.1. □

If $\mathsf{S}_1(\Gamma, \Gamma)$ implies $\binom{\mathrm{L}}{\Gamma}$, then the word "hereditarily" can be removed from Theorem 4.9. However, we suspect that this is not the case.

Conjecture 5.3. $\binom{\mathrm{L}}{\Gamma}$ *is strictly stronger than* $\mathsf{S}_1(\Gamma, \Gamma)$.

To prove this conjecture, it suffices to construct (say using the Continuum Hypothesis) sets $X, Y \subseteq \mathbb{R}$ satisfying $\binom{\mathrm{L}}{\Gamma}$, such that $X \cup Y$ does not satisfy $\binom{\mathrm{L}}{\Gamma}$, because $\mathsf{S}_1(\Gamma, \Gamma)$ is σ-additive.

Problem 5.4. Is $\binom{\mathrm{L}}{\Gamma}$ preserved by finite unions?

If it is, then $\binom{\mathrm{L}}{\Gamma}$ is in fact σ-additive, because of the following.

Proposition 5.5. $\binom{\mathrm{L}}{\Gamma}$ *is linearly σ-additive, that is, is preserved by countable increasing unions.*

Proof. Assume that $X_1 \subseteq X_2 \subseteq \ldots$ all satisfy $\binom{\mathrm{L}}{\Gamma}$, $\bigcup_n X_n = X$, and $X \in \mathrm{LI}(\mathcal{U})$. Then for each n, $X_n \in \mathrm{LI}(\{U \cap X_n : U \in \mathcal{U}\})$, and thus there are $U^n_m \in \mathcal{U}$, $m \in \mathbb{N}$, such that $X_n \subseteq \liminf_m U^n_m$. By Jordan's Lemma 4.8, there are $m_1, m_2, \ldots \in \mathbb{N}$ such that $X = \liminf_n U^n_{m_n}$. □

The proofs of the above results are also valid in the case of Borel covers, and since $\mathsf{S}_1(\Gamma, \Gamma)$ for Borel covers is hereditary, we have the following.

Corollary 5.6. *For Borel covers,* $\binom{\mathrm{L}}{\Gamma} = \mathsf{S}_1(\Gamma, \Gamma)$. □

Thus, none of the above-mentioned problems remains open in the Borel case.

We conclude with a local characterization of $\binom{\mathrm{L}}{\Gamma}$.

Theorem 5.7. *For perfectly normal spaces X, the following are equivalent.*

(1) *For each* $A \subseteq \mathrm{C}(X)$ *each* $f \in \mathrm{C}(X) \cap \mathrm{partlims}(A)$ *is a limit of a sequence of elements of* A.

(2) *X satisfies* $\binom{\mathrm{L}}{\Gamma}$.

Proof. $(1 \Rightarrow 2)$

Lemma 5.8. *Let X be a perfectly normal space. Assume that for each* $A \subseteq \mathrm{C}(X)$, *each* $f \in \mathrm{C}(X) \cap \mathrm{partlims}(A)$ *is a limit of a sequence of elements of A. Then each element of* $\mathrm{L}(X)$ *has a clopen refinement in* $\mathrm{L}(X)$.

Proof. Indeed, this follows from a formally weaker property: Let P be the property that, for each $A \subseteq \mathrm{C}(X)$, each f in the closure of A *in* $\mathrm{C}(X)$ under limits of sequences, is a limit of a sequence of elements of A.

Fremlin [3] proved that P is equivalent to the property named wQN in [1], where it is shown that for perfectly normal spaces, wQN implies that each open set is a countable union of *clopen* sets [1, Corollary 4.6].

Now, let $\mathcal{U} \in \mathrm{L}(X)$. For each $U \in \mathcal{U}$, present U as an increasing union $U = \bigcup_n C_n(U)$ of clopen sets. Then $U = \liminf_n C_n(U)$. Let $\mathcal{V} = \{C_n(U) : U \in \mathcal{U}, n \in \mathbb{N}\}$. Then $\mathcal{V}$ is a clopen refinement of $\mathcal{U}$, and $X \in \mathrm{LI}(\mathcal{U}) \subseteq \mathrm{LI}(\mathcal{V})$, that is, $\mathcal{V} \in \mathrm{L}(X)$. □

Let $\mathcal{U} \in \mathrm{L}(X)$. By Lemma 5.8, we may assume that the elements of $\mathcal{U}$ are clopen. Let $A = \{\chi_U : U \in \mathcal{U}\}$. $A \subseteq \mathrm{C}(X)$. Let $\mathcal{V} = \{f^{-1}(1) : f \in \mathrm{partlims}(A)\}$. $\mathcal{U} \subseteq \mathcal{V}$, and $\mathcal{V}$ is closed under the operator liminf. Indeed, Let $f_1, f_2, \ldots \in \mathrm{partlims}(A)$, and $B = \liminf_n f_n^{-1}(1)$. As $f = \lim_n f_n \in \mathrm{partlims}(A)$, $B = f^{-1}(1) \in \mathcal{V}$.

Thus, $X \in \mathcal{V}$, and therefore $\mathbf{1} \in \mathrm{partlims}(A)$. By (1), there are $U_n \in \mathcal{U}$ such that $\lim_n \chi_{U_n} = \mathbf{1}$, that is, $\liminf_n U_n = X$.

($2 \Rightarrow 1$) Assume that $\mathbf{1} \in \mathrm{partlims}(A)$. For each n, let $\mathcal{U}_n = \{f^{-1}[(1 - 1/n, 1 + 1/n)] : f \in A\}$. $\mathcal{U}_n \in \mathrm{L}(X)$. Indeed, let C be the family of all partial $f : X \to \mathbb{R}$, such that $f^{-1}[(1 - 1/n, 1+1/n)] \in \mathrm{LI}(\mathcal{U}_n)$. Then $A \subseteq C$, and C is closed under partial limits of sequences. Thus, $\mathbf{1} \in C$, that is, $X = \mathbf{1}^{-1}[(1 - 1/n, 1 + 1/n)] \in \mathrm{LI}(\mathcal{U}_n)$.

By Proposition 5.1, there are $f_1, f_2, \ldots \in A$ such that $\liminf_n f_n^{-1}[(1 - 1/n, 1 + 1/n)] = X$. In particular, $\lim_n f_n = \mathbf{1}$. □

The notation used below is available, e.g., in the survey [21].

Proposition 5.9. *The minimal cardinality of a set $X \subseteq \mathbb{R}$ such that X does not satisfy $\binom{\mathrm{L}}{\Gamma}$ is $\mathfrak{b}$ (the minimal cardinality of a subset of $\mathbb{N}^{\mathbb{N}}$ which is not bounded, with respect to eventual dominance).*

Proof. If $|X| < \mathfrak{b}$, then X satisfies $\mathsf{S}_1(\Gamma, \Gamma)$ [10]. Thus, X satisfies $\mathsf{S}_1(\Gamma, \Gamma)$ hereditarily, and by Lemma 4.10, X satisfies $\binom{\mathrm{L}}{\Gamma}$. On the other hand, there is $X \subseteq \mathbb{R}$ with $|X| = \mathfrak{b}$, such that X does not satisfy $\mathsf{S}_1(\Gamma, \Gamma)$ [10]. By Proposition 5.1, this X does not satisfy $\binom{\mathrm{L}}{\Gamma}$. □

The proof of the main theorem in [16], with trivial modifications, gives the first item of the following theorem. The other items are easy consequences.

Theorem 5.10. (1) *For each unbounded tower T of cardinality $\mathfrak{b}$ in $[\mathbb{N}]^\infty$, $T \cup [\mathbb{N}]^{<\infty}$ satisfies $\binom{\mathrm{L}}{\Gamma}$.*

(2) *If $\mathfrak{t} = \mathfrak{b}$, then there are subsets of $\mathbb{R}$ of cardinality $\mathfrak{b}$, satisfying $\binom{\mathrm{L}}{\Gamma}$.*

(3) *There are subsets of $\mathbb{R}$ of cardinality $\mathfrak{t}$, satisfying $\binom{\mathrm{L}}{\Gamma}$.* □

The assumption $\mathfrak{t} = \mathfrak{b}$ is known to be strictly weaker than the Continuum Hypothesis or even Martin's Axiom, but it is open whether it is weaker than $\mathfrak{p} = \mathfrak{b}$, which implies that the sets mentioned in Theorem 5.10 actually have the stronger property $\binom{\Omega}{\Gamma}$ [16].

Acknowledgments

We thank Salvador Hernańdez for inviting the second author to give a lecture on the topic in the conference *Functional Analysis in Valencia 2010*, dedicated to the 80'th birthday of Manuel Valdivia. We also thank Lyubomyr Zdomskyy for reading the paper and making useful comments.

This work is an extension of a part of the first author's M.Sc. thesis at the Weizmann Institute of Science, supervised by Gady Kozma and the second author. We thank Gady Kozma for useful discussions, and the Weizmann Institute of Science for the stimulating atmosphere.

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