TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 368, Number 4, April 2016, Pages 2865–2889
http://dx.doi.org/10.1090/tran/6581
Article electronically published on October 2, 2015

SELECTIVE COVERING PROPERTIES OF PRODUCT SPACES, II: γ SPACES

ARNOLD W. MILLER, BOAZ TSABAN, AND LYUBOMYR ZDOMSKYY

Abstract. We study productive properties of γ spaces and their relation to other, classic and modern, selective covering properties. Among other things, we prove the following results:

(1) Solving a problem of F. Jordan, we show that for every unbounded tower set $X \subseteq \mathbb{R}$ of cardinality $\aleph_1$, the space $\mathrm{C_p}(X)$ is productively Fréchet–Urysohn. In particular, the set X is productively γ.
(2) Solving problems of Scheepers and Weiss and proving a conjecture of Babinkostova–Scheepers, we prove that, assuming the Continuum Hypothesis, there are γ spaces whose product is not even Menger.
(3) Solving a problem of Scheepers–Tall, we show that the properties γ and Gerlits–Nagy (*) are preserved by Cohen forcing. Moreover, every Hurewicz space that remains Hurewicz in a Cohen extension must be Rothberger (and thus (*)).

We apply our results to solve a large number of additional problems and use Arhangel'skiĭ duality to obtain results concerning local properties of function spaces and countable topological groups.

1. Introduction

For a Tychonoff space X, let $\mathrm{C_p}(X)$ be the space of continuous real-valued functions on X, endowed with the topology of pointwise convergence, that is, the topology inherited from the Tychonoff product $\mathbb{R}^X$. In their seminal paper [14], Gerlits and Nagy characterized the property that the space $\mathrm{C_p}(X)$ is *Fréchet–Urysohn*—that every point in the closure of a set is the limit of a sequence of elements from that set—in terms of a covering property of the domain space X. We study the behavior of this covering property under taking products with spaces possessing related covering properties.

By *space* we mean an infinite topological space. Whenever the space $\mathrm{C_p}(X)$ is considered, we tacitly restrict our scope to Tychonoff spaces. The concrete examples constructed in this paper are all subsets of the real line.

The covering property introduced by Gerlits and Nagy is best viewed in terms of its relation to other, selective covering properties. The framework of *selection principles* was introduced by Scheepers in [29] to study, in a uniform manner, a variety of properties introduced in several mathematical contexts since the early 1920's. Detailed introductions are available in [19, 28, 34, 43]. We provide here a brief one, adapted from [26].

Received by the editors November 13, 2013 and, in revised form, September 2, 2014.
2010 *Mathematics Subject Classification.* Primary 26A03; Secondary 03E75, 03E17.
Key words and phrases. Gerlits–Nagy property γ, Gerlits–Nagy property (*), Menger property, Hurewicz property, Rothberger property, Sierpiński set, selection principles, Cohen forcing, special sets of real numbers, $\mathrm{C_p}$ theory, selectively separable, countable fan tightness.

Let X be a space. We say that $\mathcal{U}$ is a *cover* of X if $X = \bigcup \mathcal{U}$, but X is not covered by any single member of $\mathcal{U}$. Let $\mathrm{O}(X)$ be the family of all open covers of X. When X is considered as a subspace of a larger space Y, the family $\mathrm{O}(X)$ consists of the covers of X by open subsets of Y. Define the following subfamilies of $\mathrm{O}(X)$: $\mathcal{U} \in \Omega(X)$ if each finite subset of X is contained in some member of $\mathcal{U}$, $\mathcal{U} \in \Gamma(X)$ if $\mathcal{U}$ is infinite, and each element of X is contained in all but finitely many members of $\mathcal{U}$.

Some of the following statements may hold for families $\mathscr{A}$ and $\mathscr{B}$ of covers of X.

- $\binom{\mathscr{A}}{\mathscr{B}}$**:** Each member of $\mathscr{A}$ contains a member of $\mathscr{B}$.
- $\mathsf{S}_1(\mathscr{A}, \mathscr{B})$**:** For each sequence $\langle \mathcal{U}_n \in \mathscr{A} : n \in \mathbb{N} \rangle$, there is a selection $\langle U_n \in \mathcal{U}_n : n \in \mathbb{N} \rangle$ such that $\{U_n : n \in \mathbb{N}\} \in \mathscr{B}$.
- $\mathsf{S}_{\mathrm{fin}}(\mathscr{A}, \mathscr{B})$**:** For each sequence $\langle \mathcal{U}_n \in \mathscr{A} : n \in \mathbb{N} \rangle$, there is a selection of finite sets $\langle \mathcal{F}_n \subseteq \mathcal{U}_n : n \in \mathbb{N} \rangle$ such that $\bigcup_n \mathcal{F}_n \in \mathscr{B}$.
- $\mathsf{U}_{\mathrm{fin}}(\mathscr{A}, \mathscr{B})$**:** For each sequence $\langle \mathcal{U}_n \in \mathscr{A} : n \in \mathbb{N} \rangle$, where no $\mathcal{U}_n$ contains a finite subcover, there is a selection of finite sets $\langle \mathcal{F}_n \subseteq \mathcal{U}_n : n \in \mathbb{N} \rangle$ such that $\{\bigcup \mathcal{F}_n : n \in \mathbb{N}\} \in \mathscr{B}$.

We say, e.g., that X satisfies $\mathsf{S}_1(\mathrm{O}, \mathrm{O})$ if the statement $\mathsf{S}_1(\mathrm{O}(X), \mathrm{O}(X))$ holds. This way, the notation $\mathsf{S}_1(\mathrm{O}, \mathrm{O})$ stands for a property (or a class) of spaces. An analogous convention is followed for all other selection principles and families of covers. Each nontrivial property among these properties, where $\mathscr{A}$ and $\mathscr{B}$ range over O, Ω and Γ, is equivalent to one in Figure 1 [18, 29]. Some of the equivalences request that the space be Lindelöf. All spaces constructed in this paper to satisfy properties in the Scheepers Diagram are Lindelöf. Moreover, they are all subspaces of $\mathbb{R}$.

In the Scheepers Diagram, an arrow denotes implication. We indicate below each class P its *critical cardinality* $\mathrm{non}(P)$, the minimal cardinality of a space not in the class. These cardinals are all combinatorial cardinal characteristics of the continuum, details about which are available in [8]. Following the convention in the field of selection principles, influenced by the monograph [5], we deviate from the notation in [8] by denoting the family of meager (Baire first category) sets in $\mathbb{R}$ by $\mathcal{M}$.

$$\begin{tikzcd}
 & & \begin{matrix}\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)\\ \mathfrak{b}\end{matrix} \arrow[r] & \begin{matrix}\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Omega)\\ \mathfrak{d}\end{matrix} \arrow[r] & \begin{matrix}\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})\\ \mathfrak{d}\end{matrix} \\
 & & \begin{matrix}\mathsf{S}_{\mathrm{fin}}(\Gamma,\Omega)\\ \mathfrak{d}\end{matrix} \arrow[ur] & & \\
\begin{matrix}\mathsf{S}_1(\Gamma,\Gamma)\\ \mathfrak{b}\end{matrix} \arrow[r] \arrow[uurr] & \begin{matrix}\mathsf{S}_1(\Gamma,\Omega)\\ \mathfrak{d}\end{matrix} \arrow[rr] \arrow[ur] & & \begin{matrix}\mathsf{S}_1(\Gamma,\mathrm{O})\\ \mathfrak{d}\end{matrix} \arrow[uur] & \\
 & & \begin{matrix}\mathsf{S}_{\mathrm{fin}}(\Omega,\Omega)\\ \mathfrak{d}\end{matrix} \arrow[uu] & & \\
\begin{matrix}\binom{\Omega}{\Gamma}\\ \mathfrak{p}\end{matrix} \arrow[r] \arrow[uu] & \begin{matrix}\mathsf{S}_1(\Omega,\Omega)\\ \mathrm{cov}(\mathcal{M})\end{matrix} \arrow[rr] \arrow[uu] \arrow[ur] & & \begin{matrix}\mathsf{S}_1(\mathrm{O},\mathrm{O})\\ \mathrm{cov}(\mathcal{M})\end{matrix} \arrow[uu] &
\end{tikzcd}$$

FIGURE 1. The Scheepers Diagram

The properties $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$, $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$ and $\mathsf{S}_1(\mathrm{O},\mathrm{O})$ were first studied by Hurewicz, Menger and Rothberger, respectively. γ *spaces* were introduced by Gerlits and Nagy [14] as the spaces satisfying $\binom{\Omega}{\Gamma}$. Gerlits and Nagy proved that, for a space X, the space $\mathrm{C_p}(X)$ is Fréchet–Urysohn if and only if X is a γ space.

We also consider the classes of covers B, B_Ω and B_Γ, defined as O, Ω and Γ were defined, replacing *open cover* by *countable Borel cover*. The properties thus obtained have a rich history of their own [38], and the Borel variants of the studied properties are strictly stronger than the open ones [38]. Many additional—classic and new—properties are studied in relation to the Scheepers Diagram.

Definition 1.1. Let P be a property (or class) of spaces. A space X is *productively* P if $X \times Y$ has the property P for each space Y satisfying P.

In Section 2 we construct productively γ spaces in $\mathbb{R}$ from a weak hypothesis. In Section 3 we construct, using the Continuum Hypothesis, two γ spaces in $\mathbb{R}$ whose product is not Menger. In Section 4 we use our results to solve a large number of problems from the literature and from the folklore of selection principles. In Section 5 we determine the effect of Cohen forcing on γ spaces, Hurewicz spaces, and Gerlits–Nagy (*) spaces. In Section 6 we use our results together with $\mathrm{C_p}$ theory to obtain new results concerning local and density properties of function spaces. In the last section, we prove that every product of an unbounded tower set and a Sierpiński set satisfies $\mathsf{S}_1(\Gamma,\Gamma)$.

2. Productively γ spaces in $\mathbb{R}$

Recall the Gerlits–Nagy Theorem that a space X is a γ space if and only if the space $\mathrm{C_p}(X)$ is Fréchet–Urysohn. In his papers [16, 17], F. Jordan studied the property that $\mathrm{C_p}(X)$ is *productively* Fréchet–Urysohn. (In this case, it is said in [16, 17] that the space X is a *productive γ-space*. Since this terminology is admitted in [17] to be confusing, we avoid it here.) We begin with a short proof of a result of Jordan. In the proof, and later on, we use the following observations.

Lemma 2.1. *Let P be a class of spaces that is hereditary for closed subsets and is preserved by finite powers. Then for all spaces X and Y such that the disjoint union space $X \sqcup Y$ satisfies P, the product space $X \times Y$ satisfies P, too. In particular, if P is preserved by finite unions, then it is preserved by finite products.*

Proof. We prove the first assertion. As P is preserved by finite powers, the space $(X \sqcup Y)^2$ satisfies P. As $X \times Y$ is a closed subset of $(X \sqcup Y)^2$, it satisfies P, too. $\square$

If the disjoint union space $X \sqcup Y$ is a γ space, then so is the product space $X \times Y$ [23, Proposition 2.3].

Corollary 2.2. *Let X and Y be spaces. The disjoint union space $X \sqcup Y$ is a γ space if and only if the product space $X \times Y$ is.* $\square$

The following observation is made in [16, Corollary 24].

Proposition 2.3 (Jordan)**.** *Let X be a space. If the space $\mathrm{C_p}(X)$ is productively Fréchet–Urysohn, then the space X is productively γ.*

Proof. Let Y be a γ space. To prove that $X \times Y$ is a γ space, we may assume that the spaces X and Y are disjoint. By the Gerlits–Nagy Theorem, the space $\mathrm{C_p}(Y)$

is Fréchet–Urysohn. Thus, the space $\mathrm{C_p}(X \sqcup Y) = \mathrm{C_p}(X) \times \mathrm{C_p}(Y)$ is Fréchet–Urysohn. Applying the Gerlits–Nagy Theorem again, we have that $X \sqcup Y$ is a γ space. Apply Corollary 2.2. $\square$

Some of the major results concerning the property that $\mathrm{C_p}(X)$ is productively Fréchet–Urysohn are collected in the following theorem.

Theorem 2.4 (Jordan)**.**

(1) *Assuming the Continuum Hypothesis, there is an uncountable set $X \subseteq \mathbb{R}$ such that $\mathrm{C_p}(X)$ is productively Fréchet–Urysohn* [16, Theorem 33].
(2) *There is no uncountable set $X \subseteq \mathbb{R}$, of cardinality smaller than $\mathfrak{b}$, such that $\mathrm{C_p}(X)$ is productively Fréchet–Urysohn* [16, Theorem 34].
(3) *The minimal cardinality of a set $X \subseteq \mathbb{R}$ such that $\mathrm{C_p}(X)$ is not productively Fréchet–Urysohn is $\aleph_1$* [16, Corollary 35].
(4) *Every uncountable set $X \subseteq \mathbb{R}$ has a co-countable subset Y such that $\mathrm{C_p}(Y)$ is not productively Fréchet–Urysohn* [17, Theorem 1].
(5) *If $\mathrm{C_p}(X)$ is productively Fréchet–Urysohn, then so is $\mathrm{C_p}(A)$ for every F_σ subset A of X* [17, Proof of Theorem 1].

Items (4) and (5) of Jordan's Theorem 2.4 solved Problems 1 and 4 of Jordan's earlier paper [16]. The following problem—Problem 3 of [16]—remains open.

Problem 2.5 (Jordan)**.** Is the existence of uncountable set $X \subseteq \mathbb{R}$ with $\mathrm{C_p}(X)$ productively Fréchet–Urysohn compatible with Martin's Axiom and the negation of the Continuum Hypothesis?

Problem 2 of Jordan [16] asks whether the Continuum Hypothesis is necessary in item (1). We solve this problem. To this end, we use the following characterization of Jordan [16, Corollary 23]. For families of sets A and B, let

$$\mathrm{A} \wedge \mathrm{B} = \{\, \mathcal{B} \cap \mathcal{A} : \mathcal{B} \in \mathrm{B}, \mathcal{A} \in \mathrm{A} \,\}.$$

A family of sets is *centered* if every intersection of finitely many elements from this family is infinite. A *pseudointersection* of a family $\mathcal{F}$ of sets is an infinite set A such that $A \subseteq^* B$ for each element $B \in \mathcal{F}$.

Theorem 2.6 (Jordan)**.** *Let X be a space and $\mathcal{O}$ be the family of all open subsets of X. The following two assertions are equivalent:*

(1) *The space $\mathrm{C_p}(X)$ is productively Fréchet–Urysohn.*
(2) *For each family $\mathrm{A} \subseteq \Omega(X)$ that is closed under finite intersections, the first property below implies the second:*
    (P1) *For every countable family $\mathrm{B} \subseteq P(\mathcal{O})$ with $\mathrm{B} \wedge \mathrm{A}$ centered, the family $\mathrm{B} \wedge \mathrm{A}$ has a pseudointersection.*
    (P2) *The family* A *has a pseudointersection $\mathcal{U}$ such that $\mathcal{U} \in \Gamma(X)$.*

Lemma 2.7. *Let X be a space and $\mathrm{A} \subseteq \Omega(X)$ be closed under finite intersections and such that* (P1) *holds. Then:*

(1) *For each countable set $C \subseteq X$ such that C is not contained in any element of any member of* A*, the family* A *has a pseudointersection $\mathcal{U}$ such that $\mathcal{U} \in \Gamma(C)$.*
(2) *For every sequence $\langle \mathcal{U}_n \in P(\mathcal{O}) : n \in \mathbb{N} \rangle$ with $\{\mathcal{U}_n\} \wedge \mathrm{A}$ centered for each n, there is a selection of finite sets $\langle \mathcal{F}_n \subseteq \mathcal{U}_n : n \in \mathbb{N} \rangle$ such that the family $\bigcup_n \mathcal{F}_n$ is a pseudointersection of* A.

Proof. (1) For each finite $F \subseteq C$, we have that

$$[F] := \{ U \subseteq X : U \text{ is open and } F \subseteq U \} \in \Omega(X).$$

Let

$$\mathrm{B} = \{ [F] : F \in [C]^{<\infty} \}.$$

As $\mathrm{B} \wedge \mathrm{A}$ is centered, it has a pseudointersection $\mathcal{U}$. In particular, the family $\mathcal{U}$ is a pseudointersection of B, and thus $\mathcal{U} \in \Gamma(C)$.

(2) For each n, let $\mathcal{V}_n = \bigcup_{m\geq n} \mathcal{U}_m$. Let $\mathrm{B} = \{ \mathcal{V}_n : n \in \mathbb{N} \}$. By (P1), the set $\mathrm{B} \wedge \mathrm{A}$ has a pseudointersection $\mathcal{U}$. Represent $\mathcal{U} = \bigcup_n \mathcal{F}_n$ such that $\mathcal{F}_n$ is a finite subset of $\mathcal{U}_n$ for all n. □

The following theorem is the main theorem of this section. Identify $P(\mathbb{N})$ with the Cantor space $\{0,1\}^{\mathbb{N}}$ via characteristic functions. The space $P(\mathbb{N})$ is homeomorphic to the Cantor set and can be viewed as a subset of $\mathbb{R}$. Naturally, the space $P(\mathbb{N})$ is the union of $[\mathbb{N}]^{\infty}$ and $[\mathbb{N}]^{<\infty}$, the family of infinite subsets of $\mathbb{N}$ and the family of finite subsets of $\mathbb{N}$, respectively. We identify elements $x \in [\mathbb{N}]^{\infty}$ with increasing elements of $\mathbb{N}^{\mathbb{N}}$ by letting $x(n)$ be the nth element in the increasing enumeration of x. A subset of $[\mathbb{N}]^{\infty}$ is *unbounded* if it is unbounded (with respect to $\leq^*$) when viewed as a subset of $\mathbb{N}^{\mathbb{N}}$. An enumerated set $T = \{ x_\alpha : \alpha < \kappa \}$ is a *tower* if the sequence $\langle x_\alpha : \alpha < \kappa \rangle$ is decreasing with respect to $\subseteq^*$. Unbounded towers of cardinality $\aleph_1$ exist if and only if $\mathfrak{b} = \aleph_1$ (cf. [26, Lemma 3.3]).

Theorem 2.8. *For each unbounded tower $T = \{ x_\alpha : \alpha < \aleph_1 \}$, the space $\mathrm{C_p}(T \cup [\mathbb{N}]^{<\infty})$ is productively Fréchet–Urysohn. In particular, the space $T \cup [\mathbb{N}]^{<\infty}$ is productively γ.*

Proof. Let $X = T \cup [\mathbb{N}]^{<\infty}$. For each $\alpha < \aleph_1$, let $X_\alpha = \{ x_\beta : \beta < \alpha \} \cup [\mathbb{N}]^{<\infty}$. We may assume that there is $\alpha_0 < \aleph_1$ such that X_{α_0} is not contained in any member of any of the considered covers. Indeed, let $\{ V_n : n \in \mathbb{N} \}$ be the set of all finite unions of basic open sets. We may restrict our attention to open covers contained in $\{ V_n : n \in \mathbb{N} \}$. For each n, using that X is not a subset of V_n, let $\beta_n < \aleph_1$ be such that $X_{\beta_n} \not\subseteq V_n$. Take $\alpha_0 = \sup_n \beta_n$. Let $\mathrm{A} \subseteq \Omega(X)$ be closed under finite intersections and such that (P1) holds.

By (P1) and Lemma 2.7(1), there is a pseudointersection $\mathcal{U}$ of A such that $\mathcal{U} \in \Gamma(X_{\alpha_0})$. By [12, Lemma 1.2], there are $m^0_0 < m^0_1 < \dots$ and distinct elements $U^0_0, U^0_1, \dots \in \mathcal{U}$ (so that $\{ U^0_n : n \in \mathbb{N} \} \in \Gamma(X_{\alpha_0})$) such that, for each $x \in P(\mathbb{N})$ and each n with $x \cap (m^0_n, m^0_{n+1}) = \emptyset$, we have that $x \in U^0_n$. Note that $\{ U^0_n : n \in \mathbb{N} \}$ is a pseudointersection of A. Let $I_0 = \mathbb{N}$.

As $\alpha_0 < \aleph_1$, the set $\{ x_\alpha : \alpha_0 < \alpha < \aleph_1 \}$ is unbounded. Thus (e.g., [26, Lemma 3.1]), there is $\alpha_1 > \alpha_0$ such that the set $I_1 := \{ n : x_{\alpha_1} \cap (m^0_n, m^0_{n+1}) = \emptyset \}$ is infinite.

By (P1) and Lemma 2.7(1), there is a pseudointersection $\mathcal{U}$ of A such that $\mathcal{U} \in \Gamma(X_{\alpha_1})$. By [12, Lemma 1.2], there are $1 < m^1_0 < m^1_1 < \dots$ and distinct elements $U^1_0, U^1_1, \dots \in \mathcal{U}$ (so that $\{ U^1_n : n \in \mathbb{N} \} \in \Gamma(X_{\alpha_1})$) such that, for each $x \in P(\mathbb{N})$ and each n with $x \cap (m^1_n, m^1_{n+1}) = \emptyset$, we have that $x \in U^1_n$. Here too, the set $\{ U^1_n : n \in \mathbb{N} \}$ is a pseudointersection of A.

Continue in the same manner to define, for each $k > 0$, elements with the following properties:

(1) $\alpha_k > \alpha_{k-1}$;
(2) $I_k := \{ n : x_{\alpha_k} \cap (m_n^{k-1}, m_{n+1}^{k-1}) = \emptyset \}$ is infinite;
(3) $k < m_0^k < m_1^k < \dots$;
(4) $\{ U_n^k : n \in \mathbb{N} \} \in \Gamma(X_{\alpha_k})$ and is a bijectively enumerated pseudointersection of A;
(5) for each $x \in P(\mathbb{N})$ and each n with $x \cap (m_n^k, m_{n+1}^k) = \emptyset$, we have that $x \in U_n^k$.

Let $\alpha = \sup_k \alpha_k$. Then $\alpha < \aleph_1$, the set X_α is countable, and $X_{\alpha_k} \subseteq X_{\alpha_{k+1}}$ for all k. Thus, there are for each k a finite set $F_k \subseteq X_{\alpha_k}$ such that $F_k \subseteq F_{k+1}$ for all k and $X_\alpha = \bigcup_k F_k$. For each k, by removing finitely many elements from the set I_k, we may assume that $F_k \subseteq U_n^k$ for all $n \in I_k$.

Fix $k \in \mathbb{N}$. By removing finitely many more elements from each set I_{k+1}, we may assume that $x_\alpha \setminus [0, m_n^k) \subseteq x_{\alpha_{k+1}}$ for all $n \in I_{k+1}$. As $x_{\alpha_{k+1}} \cap (m_n^k, m_{n+1}^k)$ is empty for $n \in I_{k+1}$, we have that

$$x_\alpha \cap (m_n^k, m_{n+1}^k) = \emptyset$$

for all $n \in I_{k+1}$.

For each k, let $\mathcal{U}_k = \{ U_n^k : n \in I_{k+1} \}$. By thinning out the sets I_k, we may assume that the families $\mathcal{U}_k$ are pairwise disjoint. By Lemma 2.7(2), there are finite sets $\mathcal{F}_k \subseteq \mathcal{U}_k$ for $k \in \mathbb{N}$ such that $\mathcal{U} := \bigcup_k \mathcal{F}_k$ is a pseudointersection of A. It remains to show that $\mathcal{U} \in \Gamma(X)$. Let $x \in X_\alpha$. Let N be such that $x \in F_N$. Then, for each $k \geq N$ and each $U_n^k \in \mathcal{F}_k$, we have that

$$x \in F_N \subseteq F_k \subseteq U_n^k.$$

This shows that $\mathcal{U} \in \Gamma(X_\alpha)$.

It remains to consider the elements x_β for $\beta \geq \alpha$. Let $\beta \geq \alpha$. Then $x_\beta \subseteq^* x_\alpha$. Let k be such that $x_\beta \setminus [0, k) \subseteq x_\alpha$. For each element $U_n^k \in \mathcal{F}_k$, we have that $n \in I_{k+1}$ and $m_n^k > k$. Thus,

$$x_\beta \cap (m_n^k, m_{n+1}^k) \subseteq x_\alpha \cap (m_n^k, m_{n+1}^k) = \emptyset,$$

and therefore $x_\beta \in U_n^k$. □

Our proof method cannot produce sets of cardinality greater than $\aleph_1$, since the countability of the initial sets X_α (for $\alpha < \aleph_1$) is used in an essential manner.

Corollary 2.9. *The following assertions are equivalent:*

(1) $\mathfrak{b} = \aleph_1$.
(2) *There is a set $X \subseteq \mathbb{R}$, of cardinality $\aleph_1$, such that $\mathrm{C_p}(X)$ is productively Fréchet–Urysohn.*

Proof. If $\mathfrak{b} = \aleph_1$, then there is an unbounded tower of cardinality $\aleph_1$, and Theorem 2.8 applies. The remaining implication follows from Jordan's Theorem 2.4(2). □

The partial orders $\leq^*$ and $\subseteq^*$, and their inverses, all have the property mentioned in the following result, which rules out the possibility of our method producing examples of cardinality greater than $\aleph_1$. This is in contrast to [26, Theorem 3.6], which implies that γ spaces $X \subseteq \mathbb{R}$ of cardinality $\mathfrak{p}$ exist whenever $\mathfrak{p} = \mathfrak{b}$.

Proposition 2.10. *Assume that $\mathfrak{b} > \aleph_1$. Let $\preceq$ be a partial order on $[\mathbb{N}]^\infty$ such that, for each $a \in [\mathbb{N}]^\infty$, the set $\{\, b \in [\mathbb{N}]^\infty : b \preceq a \,\}$ is F_σ in $[\mathbb{N}]^\infty$. Let $T = \{\, x_\alpha : \alpha < \kappa \,\}$ be strictly $\preceq$-increasing with α. Then the space $\mathrm{C_p}(T \cup [\mathbb{N}]^{<\infty})$ is* not *productively Fréchet–Urysohn.*

Proof. Assume that $\mathrm{C_p}(T \cup [\mathbb{N}]^{<\infty})$ is productively Fréchet–Urysohn.

If $\kappa = \aleph_1$, then $\kappa < \mathfrak{b}$ and Jordan's Theorem 2.4(2) applies. Thus, assume that $\kappa > \aleph_1$. Let $A = \{\, x_\alpha : \alpha \leq \aleph_1 \,\}$. As

$$A = T \cap \{\, x \in [\mathbb{N}]^\infty : x \preceq x_{\aleph_1} \,\},$$

the set A is F_σ in T. As $|A \cup [\mathbb{N}]^{<\infty}| = \aleph_1 < \mathfrak{b}$, the set $A \cup [\mathbb{N}]^{<\infty}$ is a σ-set; that is, all subsets of this set are relatively F_σ. In particular, the set A is F_σ in $A \cup [\mathbb{N}]^{<\infty}$. Let F_1 and F_2 be F_σ subsets of $P(\mathbb{N})$ such that $F_1 \cap T = A$ and $F_2 \cap (A \cup Q) = A$. Then

$$F_1 \cap F_2 \cap (T \cup Q) = (F_1 \cap F_2 \cap T) \cup (F_1 \cap F_2 \cap Q) = A \cup \emptyset = A.$$

It follows that A is F_σ in $T \cup Q$. By Jordan's Theorem 2.4(5), the space $\mathrm{C_p}(A)$ is productively Fréchet–Urysohn and has cardinality $\aleph_1$, in contradiction to Jordan's Theorem 2.4(2). $\square$

Problem 2.11. Is the assumption $\mathfrak{b} = \aleph_1$ necessary for the existence of uncountable sets $X \subseteq \mathbb{R}$ such that $\mathrm{C_p}(X)$ is productively Fréchet–Urysohn?

By Jordan's Theorem 2.4(2), if the answer to Problem 2.11 is "No", then the answer to the following problem is "Yes".

Problem 2.12. Are there, consistently, sets $X \subseteq \mathbb{R}$ of cardinality greater than $\aleph_1$ such that $\mathrm{C_p}(X)$ is productively Fréchet–Urysohn?

Problem 2.13. Are there, consistently, sets $X \subseteq \mathbb{R}$ such that X is productively γ but $\mathrm{C_p}(X)$ is not productively Fréchet–Urysohn?

3. A product of γ spaces need not have Menger's property

Rothberger's property $\mathsf{S}_1(\mathrm{O}, \mathrm{O})$ implies Borel's closely related property of *strong measure zero*. Weiss [48] and, independently, Scheepers [32] proved that every metric space satisfying $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$ and $\mathsf{S}_1(\mathrm{O}, \mathrm{O})$ is productively strong measure zero.

Problem 3.1 (Scheepers [32])**.** Assume that $X \subseteq \mathbb{R}$ satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$ and $\mathsf{S}_1(\mathrm{O}, \mathrm{O})$. Must X be productively $\mathsf{S}_1(\mathrm{O}, \mathrm{O})$?

In [2], Babinkostova and Scheepers conjecture that a very strong negative answer to the Scheepers Problem holds, namely, that assuming the Continuum Hypothesis, there are γ spaces $X, Y \subseteq \mathbb{R}$ such that the product space $X \times Y$ does not satisfy $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$. By Theorem 2.8, the unbounded tower method from [12, 26, 40] cannot be used to establish this conjecture. Here, we use the Aronszajn tree method of Todorčević [9, 12, 22, 41] to prove the Babinkostova–Scheepers Conjecture.

Theorem 3.2 (CH)**.** *There are sets $X, Y \subseteq \mathbb{R}$ satisfying $\binom{\mathrm{B}_\Omega}{\mathrm{B}_\Gamma}$ such that the product space $X \times Y$ does not satisfy $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$.*

In the proof of Theorem 3.2, we work in $\{0, 1\}^\mathbb{N}$ instead of $\mathbb{R}$. We construct an Aronszajn tree of perfect sets determined by Silver forcing [15].

Definition 3.3. The partially ordered set $\mathbb{P}$ is the set of conditions p such that there is a co-infinite set $D \subseteq \mathbb{N}$ with $p\colon D \to \{0,1\}$. For $p \in \mathbb{P}$,

$$[p] := \left\{\, x \in \{0,1\}^{\mathbb{N}} : p \subseteq x \,\right\}.$$

A condition $p \in \mathbb{P}$ is *stronger* than a condition $q \in \mathbb{P}$, denoted $p \leq q$, if $p \supseteq q$ or, equivalently, if $[p] \subseteq [q]$. For $n \in \mathbb{N}$, the relation $p \leq_n q$ holds if $p \leq q$ and the first n elements of ${D_p}^{\mathsf{c}}$ are the same as the first n elements of ${D_q}^{\mathsf{c}}$.

The following important lemma is folklore.

Lemma 3.4 (Fusion Lemma)**.** *Let $\langle\, p_n : n \in \mathbb{N} \,\rangle$ be a sequence in $\mathbb{P}$ such that $p_{n+1} \leq_n p_n$ for all n. Then the fusion $q = \bigcup_n p_n$ is in $\mathbb{P}$, and $q \leq_n p_n$ for all n.*

Proof of Theorem 3.2. Define the following countable dense subsets of $[p]$:

$$\begin{aligned} Q^0(p) &= \{\, x \in [p] : \forall^\infty n \in {D_p}^{\mathsf{c}},\ x(n) = 0 \,\}; \\ Q^1(p) &= \{\, x \in [p] : \forall^\infty n \in {D_p}^{\mathsf{c}},\ x(n) = 1 \,\}. \end{aligned}$$

Define $q \leq_n^* p$ if and only if $q \leq_n p$ and q is identically zero on $D_q \setminus D_p$.

Lemma 3.5. *Let $\mathcal{U} \in \Omega(Q^0(p))$. For each n, there are $U \in \mathcal{U}$ and $q \leq_n^* p$ such that $[q] \subseteq U$.*

Proof. Let F be the set consisting of the first n elements of ${D_p}^{\mathsf{c}}$. For each $s \in \{0,1\}^F$, let $x_s \in Q^0(p)$ be such that $x_s \restriction F = s$ and $x_s(k) = 0$ for every $k \in {D_p}^{\mathsf{c}} \setminus F$. Take $U \in \mathcal{U}$ with $\{\, x_s : s \in \{0,1\}^F \,\} \subseteq U$. Since U is open there is $N \in \mathbb{N}$ with $[x_s \restriction N] \subseteq U$ for all $s \in \{0,1\}^F$. Define $q \leq_n^* p$ by

$$q = p \cup \{\, \langle k, 0\rangle : k < N \text{ and } k \in ({D_p}^{\mathsf{c}} \setminus F) \,\}. \qquad \square$$

Lemma 3.6. *Let $p_n \in \mathbb{P}$, $k_n \in \mathbb{N}$ for $n < N$, and $\mathcal{U} \in \Omega(\bigcup_{n<N} Q^0(p_n))$. Then there are $U \in \mathcal{U}$ and $\langle\, q_n \leq_{k_n}^* p_n : n < N \,\rangle$ such that*

$$\bigcup_{n<N} [q_n] \subseteq U.$$

Proof. Let F_n be the set consisting of the first k_n elements of ${D_{p_n}}^{\mathsf{c}}$. For $s \in \{0,1\}^{F_n}$, define $x_s^n \in Q^0(p_n)$ as in the proof of Lemma 3.5. Let $H \subseteq \bigcup_{n<N} Q^0(p_n)$ be a finite set containing all such x_s^n. Choose $U \in \mathcal{U}$ with $H \subseteq U$ and determine the q_n for $n < N$ as in Lemma 3.5. $\square$

Remark 3.7. If $q \leq_k^* p$, then $Q^0(q) \subseteq Q^0(p)$ and hence any $\Omega(Q^0(p)) \subseteq \Omega(Q^0(q))$. In these two lemmata, the q we obtain are also equal mod finite to the p, which also implies this.

Lemma 3.8. *Let $\langle\, (p_n, k_n) : n \in \mathbb{N} \,\rangle$ be a sequence in $\mathbb{P} \times \mathbb{N}$ and $\langle\, \mathcal{U}_n : n \in \mathbb{N} \,\rangle$ be a sequence in $\Omega(Q)$, where $Q = \bigcup_{n\in\mathbb{N}} Q^0(p_n)$. Then there are sequences $\langle U_m \in \mathcal{U}_m : m \in \mathbb{N}\rangle$ and $\langle\, q_n \leq_{k_n} p_n : n \in \mathbb{N} \,\rangle$ such that*

$$(\forall n, \forall m \geq n) \quad [q_n] \subseteq U_m.$$

Proof. Construct $\langle\, q_n^m : n, m \in \mathbb{N} \,\rangle$ and $\langle\, U_m \in \mathcal{U}_m : m \in \mathbb{N} \,\rangle$ by induction on m. Set $q_n^1 = p_n$ for all n. Given $\langle\, q_n^m : n \in \mathbb{N} \,\rangle$ and $\langle\, U_n : n < m \,\rangle$, construct q_n^{m+1} and

$U_m \in \mathcal{U}_m$ so that

(1) $q_n^{m+1} = p_n$ for $n \geq m+1$,
(2) $q_n^{m+1} \leq^*_{k_n+m} q_n^m$ for $n \leq m$, and
(3) $[q_n^{m+1}] \subseteq U_m$ for $n \leq m$.

Let $q_n = \bigcup_{m>n} q_n^m$ be the fusion. We have that $q_n \leq_{k_n} q_n^n = p_n$ and $[q_n] \subseteq U_m$ whenever $m \geq n$. ☐

Remark 3.9.

(1) The analogue of this lemma for Q^1 is also true.
(2) The proof of the lemma above only uses the fact that $[p_n] \cap U$ is open in $[p_n]$ for all n and U appearing in some $\mathcal{U}_m$.

Lemma 3.10. *Let $p \in \mathbb{P}$, $n \in \mathbb{N}$, and $B \subseteq \{0,1\}^{\mathbb{N}}$ be a Borel set. Then there exists $q \leq_n p$ such that $[q] \cap B$ is clopen in $[q]$.*

Proof. Let F be the set consisting of the first n elements of ${D_p}^{\mathsf{c}}$ and let $\phi\colon \mathbb{N} \to ({D_p}^{\mathsf{c}} \setminus F)$ be a bijection. For $I \subseteq \mathbb{N}$ let $\psi_I\colon ({D_p}^{\mathsf{c}} \setminus F) \to \{0,1\}$ be the restriction of the characteristic function of $\phi(I)$. For each $s \in \{0,1\}^F$ define

$$C_s = \{\, I \in [\mathbb{N}]^\infty : (p \cup s \cup \psi_I) \in B \,\}.$$

Since these are Borel sets, by the Galvin–Prikry Theorem [13] there exists $H \in [\mathbb{N}]^\infty$ such that for each $s \in \{0,1\}^F$ either $[H]^\infty \subseteq C_s$ or $[H]^\infty \cap C_s = \emptyset$. Let $H_1 \subseteq H$ be infinite such that $H \setminus H_1$ is also infinite. Let

$$q = p \cup (\phi(H^{\mathsf{c}}) \times \{0\}) \cup (\phi(H_1) \times \{1\}).$$

Note that ${D_q}^{\mathsf{c}} = F \cup \phi(H \setminus H_1)$. We claim that given any $x, y \in [q]$, if $x \restriction F = y \restriction F = s$, then $x \in B$ if and only if $y \in B$. Letting $H_x = \phi^{-1}(x^{-1}(1))$, we have that $H_1 \subseteq H_x \subseteq H$ and so H_x is an infinite subset of H. Similarly for H_y. By the choice of H we have that $H_x \in C_s$ if and only if $H_y \in C_s$, and the claim follows. ☐

Lemma 3.11. *Let $\langle (p_n, k_n) : n \in \mathbb{N} \rangle$ be a sequence in $\mathbb{P} \times \mathbb{N}$. Then there is a sequence $\langle q_n \leq_{k_n} p_n : n \in \mathbb{N} \rangle$ such that for $n \neq m$, q_n and q_m are strongly disjoint; i.e., there are infinitely many $k \in (D_{q_n} \cap D_{q_m})$ with $q_n(k) \neq q_m(k)$.*

Proof. Given p_1, p_2 and n it is easy to find $q_1 \leq_n p_1$ and $q_2 \leq_n p_2$ which are strongly disjoint. A fusion argument produces a sequence $\langle q_n : n \in \mathbb{N} \rangle$ where all pairs have been considered and made strongly disjoint. ☐

We construct an Aronszajn tree of Silver conditions. Let B^β for $\beta < \aleph_1$ list all Borel sets. Let $\mathcal{B}_\alpha = \langle \mathcal{B}_\alpha^n : n \in \mathbb{N} \rangle$ for $\alpha < \aleph_1$ be all countable sequences of countable families of Borel sets. We may assume that each element of $\bigcup_n \mathcal{B}_\alpha^n$ is equal to B^β for some $\beta < \alpha$. We may also assume that each such sequence occurs as an element $\mathcal{B}_\alpha$ for both α even and α odd.

We construct a tree $T \subseteq \mathbb{N}^{<\aleph_1}$ and $\langle p_s \in \mathbb{P} : s \in T \rangle$ with the following properties:

(1) $T \subseteq \mathbb{N}^{<\aleph_1}$ is a subtree; i.e., $s \subseteq t \in T$ implies $s \in T$.
(2) $T_\alpha = T \cap \mathbb{N}^\alpha$ is countable for each $\alpha < \aleph_1$.
(3) $s \subseteq t \in T$ implies $p_t \leq p_s$.
(4) If $s, t \in T$ are incomparable, then p_s and p_t are strongly disjoint (as in Lemma 3.11).
(5) For any $\alpha < \beta < \aleph_1$ and any $s \in T_\alpha$ and $n \in \mathbb{N}$ there is $t \in T_\beta$ with $p_t \leq_n p_s$.

(6) For any $\beta < \alpha$ and $s \in T_\alpha$, $[p_s] \cap B^\beta$ is clopen in $[p_s]$.

(7) Define

$$\begin{aligned} Q^0_\alpha &= \bigcup \{\, Q^0(p_t) : t \in T_{\leq \alpha} \,\}; \\ Q^1_\alpha &= \bigcup \{\, Q^1(p_t) : t \in T_{\leq \alpha} \,\}. \end{aligned}$$

(a) For an even ordinal α, if $\mathcal{B}_\alpha = \langle\, \mathcal{B}^\alpha_n : n \in \mathbb{N} \,\rangle$ is a sequence in $\Omega(Q^0_\alpha)$, then there is a family

$$\langle\, U_n \in \mathcal{U}^\alpha_n : n \in \mathbb{N} \,\rangle \in \Gamma(Q^0_\alpha \cup \bigcup \{\, [p_s] : s \in T_{\alpha+1} \,\}).$$

(b) For α odd, the analogous statement is true with Q^1_α in place of Q^0_α.

(8) Let $D = \{\, D_{p_s}{}^{\mathsf{c}} : s \in T \,\} \subseteq [\mathbb{N}]^\infty$. Then D is dominating.

To construct T_λ and p_s for $s \in T_\lambda$ where λ is a countable limit ordinal, proceed as follows. For any $s \in T_{<\lambda}$ and $N \in \mathbb{N}$ choose a strictly increasing sequence $\langle\, \lambda_n : n \in \mathbb{N} \,\rangle$ co-final in λ with $s \in T_{\lambda_1}$. Let $t_1 = t_1^{s,N}$ be equal to s. By the inductive hypothesis we can find $t_n = t_n^{s,N} \in T_{\lambda_n}$ with $p_{t_{n+1}} \leq_{N+n} p_{t_n}$ for all n. Set $t^{s,N} = \bigcup_n t_n^{s,N}$ and $T_\lambda = \{\, t^{s,N} : s \in T_{<\lambda}, N \in \mathbb{N} \,\}$. For every $t = t^{s,N} \in T_\lambda$, let p_t be the fusion of the sequence $\langle\, p_{t_n^{s,N}} : n \in \mathbb{N} \,\rangle$, i.e., $p_t = \bigcup_n p_{t_n^{s,N}}$.

At successor stages for α even, check to see if $\mathcal{B}_\alpha$ is a sequence in $\Omega(Q^0_\alpha)$. If it is not, we need never worry about it since the set we are building will contain Q^0_α. If it is, let $\{\, x_n : n \in \mathbb{N} \,\} = Q^0_\alpha$ and let

$$\mathcal{B}_n = \{\, B \in \mathcal{B}^\alpha_n : \{\, x_i : i < n \,\} \subseteq B \,\}.$$

Let $\langle\, p_n, k_n : n \in \mathbb{N} \,\rangle$ list all elements of

$$\{\, p_s : s \in T_\alpha \,\} \times \mathbb{N}$$

with infinite repetitions. Combining the fact that only B^β's for $\beta < \alpha$ may occur in some $\mathcal{B}^\alpha_n$, Lemma 3.8 (see also Remark 3.9), and Lemma 3.11, we can find sequences $\langle\, q_n \leq_{k_n} p_n : n \in \mathbb{N} \,\rangle$ and $\langle\, B_m \in \mathcal{B}_m : m \in \mathbb{N} \,\rangle$ such that $[q_n] \subseteq B_m$ for all $n < m$ and q_{n_1}, q_{n_2} are strongly disjoint for all distinct $n_1, n_2 \in \mathbb{N}$. As a result, for every $s \in T_\alpha$ and $k \in \mathbb{N}$ there is some $q_{s,k} \leq_k s$ such that $[q_{s,k}] \subseteq B_m$ for all but finitely many m. By Lemma 3.10, for such s and k there is $p \leq_k q_{s,k}$ such that $[p] \cap B^\alpha$ is clopen in $[p]$. We denote this p by $p_{s\frown\langle k\rangle}$.

This concludes our inductive construction, which ensures conditions (1)–(7). Obtaining condition (8) is easy to satisfy. Set

$$\begin{aligned} X &= \bigcup_{s \in T} Q^0(p_s); \\ Y &= \bigcup_{s \in T} Q^1(p_s). \end{aligned}$$

By condition (7), the sets X and Y satisfy $\binom{\mathrm{B}_\Omega}{\mathrm{B}_\Gamma}$. For all $x \in X$ and $y \in Y$, there are infinitely many n with $x(n) \neq y(n)$. Indeed, if $x \in Q^0(p_s)$ and $y \in Q^1(p_t)$, and s and t are incomparable, then p_s and p_t are strongly disjoint. On the other hand, if s and t are comparable, for example, if $s \subseteq t$, then since $p_t \leq p_s$, we have that $D_{p_t}{}^{\mathsf{c}} \subseteq D_{p_s}{}^{\mathsf{c}}$. Thus, for all but finitely many $n \in D_{p_t}{}^{\mathsf{c}}$, we have that $y(n) = 1$ and $x(n) = 0$.

Condition (8) provides a continuous map from $X \times Y$ onto a dominating set $D \subseteq \mathbb{N}^\mathbb{N}$. Namely, if $x_0 \in Q^0(p_s)$ is identically zero on $D_{p_s}{}^{\mathsf{c}}$ and $x_1 \in Q^1(p_s)$ is identically one on ${D_{p_s}}^{\mathsf{c}}$, then ${D_{p_s}}^{\mathsf{c}} = \{\, n : x_0(n) \neq x_1(n) \,\}$. Thus, the continuous map $\Phi\colon X \times Y \to \mathbb{N}^{\mathbb{N}}$ defined by $\Phi(x,y) = \{\, n : x(n) \neq y(n) \,\}$ is as required. □

4. Applications

The conjunction of Hurewicz's property $\mathsf{U}_{\text{fin}}(\mathrm{O},\Gamma)$ and Rothberger's property $\mathsf{S}_1(\mathrm{O},\mathrm{O})$, shown in [25, Theorems 14 and 19] to be equivalent to the Gerlits–Nagy property (*), is of growing importance in the area of selection principles [35]. In an unpublished manuscript [49], Weiss proposed a plan to prove that the Gerlits–Nagy property (*) is preserved by finite products. By Lemma 2.1, this problem is equivalent to the following one.

Problem 4.1 (Weiss)**.** Is the conjunction of $\mathsf{U}_{\text{fin}}(\mathrm{O},\Gamma)$ and $\mathsf{S}_1(\mathrm{O},\mathrm{O})$ preserved by finite powers?

A negative solution of Weiss's Problem was proposed in [36] and later withdrawn [37]. A set $S \subseteq \mathbb{R}$ is *Sierpiński* if the set S is uncountable and its intersection with every Lebesgue measure zero set is countable. The solution proposed in [36] was based on the assumption that if $S \subseteq \mathbb{R}$ is a Sierpiński set, then S continues to satisfy the Hurewicz property $\mathsf{U}_{\text{fin}}(\mathrm{O},\Gamma)$ in extensions of the universe by Cohen forcing [36, Theorem 40]. It turns out that this assumption is not provable (Theorem 5.2 below).[1]

Theorem 4.2 provides an alternative solution to Weiss's Problem, also in the negative. In particular, the answer to Problem 6.6 in [46] is "No". It was, thus far, open whether the Gerlits–Nagy property (*) implies $\mathsf{S}_1(\Omega,\Omega)$. Theorem 4.2 solves this problem in the negative. It also shows that the answer to Problem 4.1(j) in [46], concerning the realization of a certain setting in the Borel version of the Scheepers Diagram, is "Yes". This theorem solves 8 out of the 55 problems that remained open in Mildenberger–Shelah–Tsaban [21], concerning potential implications between covering properties (details are provided below). It also solves, in the negative, all 5 problems in [46, Problem 7.6(2)], concerning the preservation of certain covering properties under finite powers.

An element $\mathcal{U} \in \mathrm{O}(X)$ is in $\mathrm{T}(X)$ if every member of X is a member of infinitely many elements of $\mathcal{U}$ and, for all $x, y \in X$, either $x \in U$ implies $y \in U$ for all but finitely many $U \in \mathcal{U}$, or $y \in U$ implies $x \in U$ for all but finitely many $U \in \mathcal{U}$. Figure 2 contains all new properties introduced by the inclusion of T into the framework, together with their critical cardinalities [20, 21, 39, 42] and a serial number to be used below.

---

[1] The gap in the proof of Theorem 40 in [36] may be the following one. It seems that, in item 6) on page 30, the definition of $\dot{V}^n_j$ should be $\dot{V}^n_{j-1} \cap \big(\bigcap_{i\leq \ell^n_j} \dot{V}^n_{m^n_i+\cdots+m^n_{j-1}+1,x^{n,j}_i}\big)$, not $\dot{V}^n_{j-1} \cap \big(\bigcap_{i\leq \ell^n_j} \dot{V}^n_{j,x^{n,j}_i}\big)$. Given that, the claim "By 3), 5), 6) and 8) above, the set F_k is disjoint from $\bigcup_{n\geq k} C_n$" at the end of page 30 is unclear. Indeed, to make it true, one should have in $V[G]$ that $\overline{V^n_t} \supseteq C_n$. By the definition of V^n_j, this would require that, in $V[G]$, $C_n \subseteq \overline{V^n_{m^n_i+\cdots+m^n_{j-1}+1,x^{n,j}_i}}$ for all $i < \ell^n_j$. For each individual $i < \ell^n_j$, every element p of $F_{m^n_i+\cdots+m^n_{j-1}+1}(x^{n,j}_i, \dot{C}_n)$ indeed forces that $\dot{C}_n \subseteq \dot{V}_{p,x^{n,j}_i}(\dot{C}_n)$. However, the elements of $F_{m^n_i+\cdots+m^n_{j-1}+1}(x^{n,j}_i, \dot{C}_n)$ may be incompatible. As, in $V[G]$, we have that

$$V^n_{m^n_i+\cdots+m^n_{j-1}+1,x^{n,j}_i} = \bigcap \{\, V_{p,x^{n,j}_i}(\dot{C}_n) : p \in F_{m^n_i+\cdots+m^n_{j-1}+1}(x^{n,j}_i, \dot{C}_n) \,\},$$

it is unclear why $C_n \subseteq V^n_{m^n_i+\cdots+m^n_{j-1}+1,x^{n,j}_i}$ there.

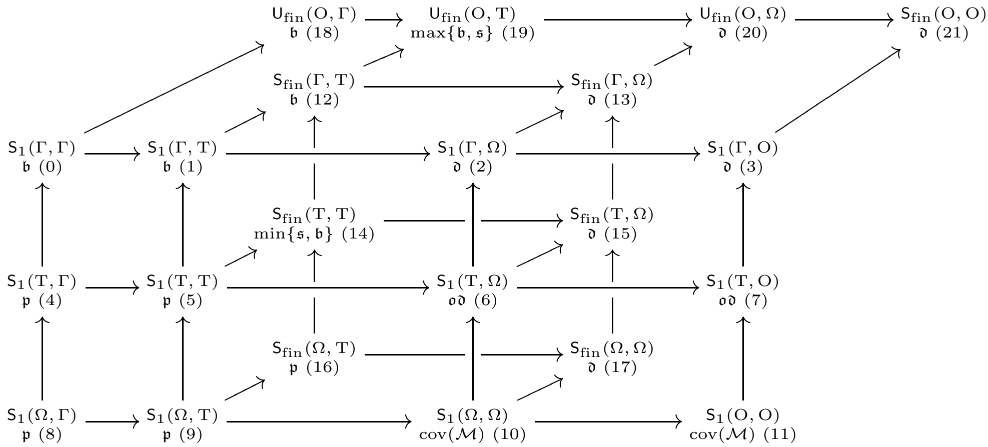


FIGURE 2. The Extended Scheepers Diagram

Theorem 4.2 (CH). *There are sets* $X_0, X_1 \subseteq \mathbb{R}$ *satisfying* $\binom{\mathrm{B}_\Omega}{\mathrm{B}_\Gamma}$ *such that the set* $X = X_0 \cup X_1$ *has the following properties:*

(1) X *satisfies* $\mathsf{S}_1(\mathrm{B}_\mathrm{T}, \mathrm{B}_\Gamma)$ *and* $\mathsf{S}_1(\mathrm{B}, \mathrm{B})$ *(and, in particular, the Gerlits–Nagy property* (*)*);*
(2) X *does not satisfy* $\mathsf{S}_{\mathrm{fin}}(\Omega, \Omega)$*;*
(3) *the square space* X^2 *does not satisfy* $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$.

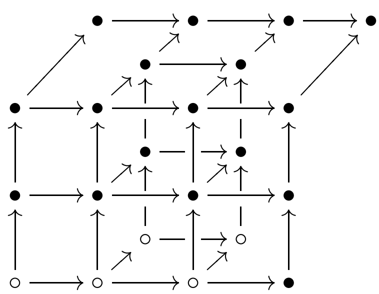
Proof. Let $X_0, X_1 \subseteq \mathbb{R}$ be as in Theorem 3.2, i.e., both satisfying $\binom{\mathrm{B}_\Omega}{\mathrm{B}_\Gamma}$, and such that the product space $X_0 \times X_1$ does not satisfy $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$. We may assume, by taking a homeomorphic image, that $X_0 \subseteq (0, 1)$ and $X_1 \subseteq (2, 3)$. Let $X = X_0 \cup X_1$.

(1) As both properties $\mathsf{S}_1(\mathrm{B}_\mathrm{T}, \mathrm{B}_\Gamma)$ and $\mathsf{S}_1(\mathrm{B}, \mathrm{B})$ are preserved by finite unions (e.g., [44]), X satisfies $\mathsf{S}_1(\mathrm{B}_\mathrm{T}, \mathrm{B}_\Gamma)$ and $\mathsf{S}_1(\mathrm{B}, \mathrm{B})$.

(2) This follows from (3), since $\mathsf{S}_{\mathrm{fin}}(\Omega, \Omega)$ is equivalent to being $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$ in all finite powers [18, Theorem 3.9].

(3) The product space $X_0 \times X_1$ is closed in X^2. Since Menger's property $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$ is hereditary for closed subsets, the space X^2 does not satisfy $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$. $\square$

The set in Theorem 4.2 realizes the following setting in the Extended Scheepers Diagram:



Consider the serial numbers in the Extended Scheepers Diagram. Table 1 describes all known implications and nonimplications among the properties, so that entry (i, j) indicates whether property (i) implies property (j). The framed entries remained open in [21]. Their solution follows from Theorem 4.2. This gives a complete understanding of which properties in the Extended Scheepers Diagram imply $\mathsf{S}_{\mathrm{fin}}(\Omega, \Omega)$ and which properties are implied by $\mathsf{S}_{\mathrm{fin}}(\mathrm{T}, \Omega)$.

TABLE 1. Known implications and nonimplications

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**
0	✓	✓	✓	✓	×	×	×	×	×	×	×	×	✓	✓	×	?	×	×	✓	✓	✓	✓
1	?	✓	✓	✓	×	×	×	×	×	×	×	×	✓	✓	×	?	×	×	?	✓	✓	✓
2	×	×	✓	✓	×	×	×	×	×	×	×	×	×	✓	×	?	×	×	×	×	✓	✓
3	×	×	×	✓	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	✓
4	✓	✓	✓	✓	✓	✓	✓	✓	×	×	$\boxed{\times}$	?	✓	✓	✓	✓	×	$\boxed{\times}$	✓	✓	✓	✓
5	?	✓	✓	✓	?	✓	✓	✓	×	×	$\boxed{\times}$	?	✓	✓	✓	✓	×	$\boxed{\times}$	?	✓	✓	✓
6	×	×	✓	✓	×	×	✓	✓	×	×	$\boxed{\times}$	?	×	✓	×	✓	×	$\boxed{\times}$	×	×	✓	✓
7	×	×	×	✓	×	×	×	✓	×	×	×	?	×	×	×	×	×	×	×	×	×	✓
8	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
9	?	✓	✓	✓	?	✓	✓	✓	?	✓	✓	✓	✓	✓	✓	✓	✓	✓	?	✓	✓	✓
10	×	×	✓	✓	×	×	✓	✓	×	×	✓	✓	×	✓	×	✓	×	✓	×	×	✓	✓
11	×	×	×	✓	×	×	×	✓	×	×	×	✓	×	×	×	×	×	×	×	×	×	✓
12	?	?	?	?	×	×	×	×	×	×	×	×	✓	✓	×	?	×	×	?	✓	✓	✓
13	×	×	×	×	×	×	×	×	×	×	×	×	×	✓	×	?	×	×	×	×	✓	✓
14	?	?	?	?	×	×	×	×	×	×	×	×	✓	✓	✓	✓	×	$\boxed{\times}$	?	✓	✓	✓
15	×	×	×	×	×	×	×	×	×	×	×	×	×	✓	×	✓	×	$\boxed{\times}$	×	×	✓	✓
16	?	?	?	?	?	?	?	?	?	?	?	?	✓	✓	✓	✓	✓	✓	?	✓	✓	✓
17	×	×	×	×	×	×	×	×	×	×	×	×	×	✓	×	✓	×	✓	×	×	✓	✓
18	×	×	×	×	×	×	×	×	×	×	×	×	×	?	×	?	×	×	✓	✓	✓	✓
19	×	×	×	×	×	×	×	×	×	×	×	×	×	?	×	?	×	×	×	✓	✓	✓
20	×	×	×	×	×	×	×	×	×	×	×	×	×	?	×	?	×	×	×	×	✓	✓
21	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	✓

5. PRESERVATION UNDER FORCING EXTENSIONS

Scheepers proved in [35] that random real forcing preserves being a γ space. We will show that this is also the case for Cohen's forcing. We say that a property is *preserved by Cohen forcing* if, whenever a space X has this property in the ground model, it will have this property in any extension by Cohen forcing, adding any number of Cohen reals.

Theorem 5.1. *The property γ is preserved by Cohen forcing.*

Proof. Let M be the ground model and X be a γ space in M. Let G be $\mathbb{P}$-generic over M and $\kappa > 0$ be an arbitrary, possibly finite, cardinal. Let $\mathbb{P}$ be the poset adding κ Cohen reals. In $M[G]$, let $\mathcal{U} \in \Omega(X)$ be a cover consisting of open sets in M.

According to Lemma 3.3 of [11], the Lindelöf property is preserved by adding uncountably many Cohen reals. The proof of that lemma also shows that the Lindelöf property is preserved by adding countably many Cohen reals. Thus, in $M[G]$, all finite powers of X are Lindelöf, and therefore $\mathcal{U}$ contains a countable member of $\Omega(X)$. Thus, we may assume that $\mathcal{U}$ is countable and hence is determined in an extension by countably many Cohen reals. As the poset for adding countably many Cohen reals is countable, it is isomorphic to $\{0,1\}^{<\aleph_0}$. Thus, we may assume that $\mathbb{P} = \{0,1\}^{<\aleph_0}$. Let $p_0 \in \mathbb{P}$ be a condition forcing the above-mentioned properties of $\mathcal{U}$. To simplify our notation, assume that p_0 is the trivial condition or replace $\mathbb{P}$ by the conditions stronger than p_0. Work in M.

Fix $p \in \mathbb{P}$. Let

$$\mathcal{U}_p = \left\{ U : \exists q \leq p,\ q \Vdash U \in \dot{\mathcal{U}} \right\}.$$

Then $\mathcal{U}_p \in \Omega(X)$. As X is a γ space, we may, by thinning out $\mathcal{U}_p$, assume that $\mathcal{U}_p \in \Gamma(X)$. Thus, by further thinning out, we may assume that the sets $\mathcal{U}_p$, for $p \in \mathbb{P}$, are pairwise disjoint. As X satisfies $\mathsf{S}_1(\Omega,\Gamma)$ (the property $\mathsf{S}_1(\Gamma,\Gamma)$ suffices here), there are elements $U_p \in \mathcal{U}_p$ for $p \in \mathbb{P}$ such that $\{\, U_p : p \in \mathbb{P} \,\} \in \Gamma(X)$. As the families $\mathcal{U}_p$ are pairwise disjoint, the sets U_p are distinct for distinct conditions $p \in \mathbb{P}$. For each $p \in \mathbb{P}$, pick a condition $q_p \leq p$ forcing that $U_p \in \dot{\mathcal{U}}$.

As the set $\{\, q_p : p \in \mathbb{P} \,\}$ is dense in $\mathbb{P}$, its intersection with G is infinite. Thus, the family $\{\, U_p : q_p \in G \,\}$, which is a subset of $\mathcal{U}$, is infinite. As $\{\, U_p : p \in \mathbb{P} \,\} \in \Gamma(X)$, we have that $\{\, U_p : q_p \in G \,\} \in \Gamma(X)$. $\square$

In [36, Theorem 37], Scheepers and Tall show that the negation of Hurewicz's property $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$ is preserved by Cohen forcing. In [36, page 26], it is shown that adding a Cohen real destroys the property that the ground model's Cantor set satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$. Problem 6 in [36] asks whether $\mathsf{U}_{\mathrm{fin}}(\mathrm{B},\mathrm{B}_\Gamma)$, the Hurewicz property for countable Borel covers, is preserved by Cohen forcing. The following theorem shows, in particular, that the answer is "No". It is well known that Sierpiński sets, which have positive outer measure, satisfy $\mathsf{U}_{\mathrm{fin}}(\mathrm{B},\mathrm{B}_\Gamma)$. (A simple proof is given, e.g., in [47].) As Rothberger's property $\mathsf{S}_1(\mathrm{O},\mathrm{O})$ implies Lebesgue measure zero, Sierpiński sets cannot satisfy $\mathsf{S}_1(\mathrm{O},\mathrm{O})$.

In the proof of our theorem, we use a technical lemma whose proof applies to the Rothberger game $\mathsf{G}_1(\mathrm{O},\mathrm{O})$. This is a game for two players, ONE and TWO, with an inning per each natural number n. In the nth inning, ONE picks a cover $\mathcal{U}_n \in \mathrm{O}(X)$, and TWO responds by picking an element $U_n \in \mathcal{U}_n$. ONE wins if $\{\, U_n : n \in \mathbb{N} \,\}$ is not a cover of X. Otherwise, TWO wins. Pawlikowski proved in [27, Theorem 1] that, for spaces X with points G_δ, the space X satisfies $\mathsf{S}_1(\mathrm{O},\mathrm{O})$ if and only if ONE does not have a winning strategy in the game $\mathsf{G}_1(\mathrm{O},\mathrm{O})$.

Theorem 5.2. *For $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$ spaces X with points G_δ, the following assertions are equivalent:*

(1) *X remains $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$ in every forcing extension by adding Cohen reals.*
(2) *X remains $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$ in every forcing extension by adding one Cohen real.*
(3) *X satisfies $\mathsf{S}_1(\mathrm{O},\mathrm{O})$.*

Proof. The implication (1) $\Rightarrow$ (2) is trivial. The implication (2) $\Rightarrow$ (1) is proved as in the proof of Theorem 5.1; namely, a counter-example to $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$ in the extension is determined in an extension by a single Cohen real, and the negation of $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$ is preserved by Cohen forcing [36, Theorem 37].

(2) $\Rightarrow$ (3): Let M be the ground model. The property $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$ implies, in particular, that the space X is Lindelöf in M. Let $\mathbb{P} = \mathbb{N}^{<\aleph_0}$, the poset adding one Cohen real $g \in \mathbb{N}^\mathbb{N}$.

Let $\langle\, \mathcal{U}_n : n \in \mathbb{N} \,\rangle \in M$ be a sequence of open covers of X. Since X is Lindelöf, we may assume that, for each n, there is an enumeration $\mathcal{U}_n = \{\, U^n_m : m \in \mathbb{N} \,\}$. Let G be $\mathbb{P}$-generic over M, and let $g = \bigcup G \in \mathbb{N}^\mathbb{N}$ be the corresponding Cohen real. By genericity, the family $\{\, U^n_{g(n)} : n \geq k \,\}$ is a cover of X for each k. If the family $\{\, U^n_{g(n)} : n \in \mathbb{N} \,\}$ has a finite subcover $\{\, U^n_{g(n)} : n < k \,\}$, then (since the restriction of g to $\{0, \ldots, k-1\}$ is in M) this finite subcover is in M, and we are done. Thus, assume that this is not the case.

By (2), there is a function $f \in \mathbb{N}^{\mathbb{N}} \cap M[G]$ such that

$$\left\{ \bigcup_{k \leq n < f(k)} U^n_{g(n)} : k \in \mathbb{N} \right\} \in \Gamma(X).$$

Work in the ground model. For $p \in \mathbb{P}$ and $K \in \mathbb{N}$, let

$$X(p, K) = \left\{ x \in X : p \Vdash \forall k \geq K,\ x \in \bigcup_{k \leq n < \dot{f}(k)} U^n_{\dot{g}(n)} \right\}.$$

Then $X = \bigcup_{(p,K) \in \mathbb{P} \times \mathbb{N}} X(p, K)$, a countable union. We may assume that, for each k, $\mathcal{U}_{k+1}$ is a refinement of $\mathcal{U}_k$.

Claim 5.3. In M, for each pair $(p, K) \in \mathbb{P} \times \mathbb{N}$ and each $K_0 \in \mathbb{N}$, there are $K_1 \in \mathbb{N}$ and a sequence $\langle\, m_n : K_0 \leq n < K_1 \,\rangle$ such that $X(p, K) \subseteq \bigcup_{K_0 \leq n < K_1} U^n_{m_n}$.

Proof. If $X(p, K) \subseteq \bigcup_{K_0' \leq n < K_1} U^n_{m_n}$ for some $K_0' \geq K_0$, then

$$X(p, K) \subseteq \bigcup_{K_0 \leq n < K_1} U^n_{m_n}.$$

Thus, we may assume that $K_0 \geq K$. Take $q \leq p$ and K_1 such that $q \Vdash \dot{f}(K_0) = K_1$. Extend q so that K_1 is in the domain of q. Then

$$\begin{aligned} X(p, K) \subseteq &\ \left\{ x \in X : p \Vdash x \in \bigcup_{K_0 \leq n < \dot{f}(K_0)} U^n_{\dot{g}(n)} \right\} \\ \subseteq &\ \left\{ x \in X : q \Vdash x \in \bigcup_{K_0 \leq n < \dot{f}(K_0)} U^n_{\dot{g}(n)} \right\} \\ = &\ \bigcup_{K_0 \leq n < K_1} U^n_{q(n)}. \end{aligned}$$ □

Enumerate $\mathbb{P} \times \mathbb{N} = \langle\, (p_i, N_i) : i \in \mathbb{N} \,\rangle$. Using the claim, pick numbers K_1 and m_n for $n < K_1$ such that $X(p_0, N_0) \subseteq \bigcup_{n < K_1} U^n_{m_n}$. Pick numbers K_2 and m_n for $K_1 \leq n < K_2$ such that $X(p_1, N_1) \subseteq \bigcup_{K_1 \leq n < K_2} U^n_{m_n}$. Pick numbers K_3 and m_n for $K_2 \leq n < K_3$ such that $X(p_2, N_2) \subseteq \bigcup_{K_2 \leq n < K_3} U^n_{m_n}$. Continuing in this manner, we obtain a sequence $\langle\, m_n : n \in \mathbb{N} \,\rangle \in M$ in $\mathbb{N}$ such that

$$X = \bigcup_{i \in \mathbb{N}} X(p_i, K_i) \subseteq \bigcup_{n \in \mathbb{N}} U^n_{m_n}.$$

(3) $\Rightarrow$ (2): We will use the following lemma.

Lemma 5.4. *Assume that, in the ground model, a space X with points G_δ satisfies $\mathsf{S}_1(\mathrm{O}, \mathrm{O})$. Assume that $\mathbb{P}$ is a poset and $\dot{\mathcal{U}}$ is a $\mathbb{P}$-name for an open cover of X, consisting of open sets from the ground model. For each $p \in \mathbb{P}$, there are a decreasing sequence $\langle\, q_m : m \in \mathbb{N} \,\rangle$ in $\mathbb{P}$ and a sequence $\langle\, U_m : m \in \mathbb{N} \,\rangle$ of sets open in the ground model such that:*

1. *$q_0 = p$;*
2. *$q_{m+1} \Vdash U_m \in \dot{\mathcal{U}}$ for all m; and*
3. *$\{\, U_m : m \in \mathbb{N} \,\}$ is a cover of X.*

Proof. For each condition $q \in \mathbb{P}$, let

$$\mathcal{U}_q = \left\{ U : \exists r \leq q,\ r \Vdash U \in \dot{\mathcal{U}} \right\}.$$

Then $\mathcal{U}_q \in M$ and is a cover of X.

Define a strategy for ONE in the Rothberger game $\mathsf{G}_1(\mathrm{O}, \mathrm{O})$ on X, as follows. Let $q_0 = p$. ONE's first move is the cover $\mathcal{U}_{q_0}$. Suppose that TWO responds with an element $U_0 \in \mathcal{U}_{q_0}$. Then ONE picks, using a fixed choice function on the nonempty subsets of $\mathbb{P}$, a condition $q_1 \leq q_0$, forcing that $U_0 \in \dot{\mathcal{U}}$, and plays $\mathcal{U}_{q_1}$. If TWO responds with an element $U_1 \in \mathcal{U}_{q_1}$, then ONE picks $q_2 \leq q_1$, forcing that $U_1 \in \dot{\mathcal{U}}$, and plays $\mathcal{U}_{q_2}$, and so on.

By Pawlikowski's Theorem [27, Theorem 1], since X satisfies $\mathsf{S}_1(\mathrm{O}, \mathrm{O})$, the strategy thus defined is not a winning strategy. Let $\langle\, q_m : m \in \mathbb{N} \,\rangle$ and $\langle\, U_m : m \in \mathbb{N} \,\rangle$ be the sequences occurring during a play lost by ONE. Then (1)–(3) hold. □

Let $\mathbb{P} = \{0,1\}^{<\aleph_0}$. Let $\langle \dot{\mathcal{U}}_n : n \in \mathbb{N} \rangle$ be a sequence of $\mathbb{P}$-names for open covers of X consisting of ground model open sets.

Fix n and a condition $p \in \mathbb{P}$. By Lemma 5.4, there are a decreasing sequence $\langle\, q^{n,p}_m \in \mathbb{P} : m \in \mathbb{N} \,\rangle$ and a sequence $\langle\, U^{n,p}_m : m \in \mathbb{N} \,\rangle \in M$ of open subsets of X such that

(1) $q^{n,p}_0 = p$;
(2) $q^{n,p}_{m+1} \Vdash U^{n,p}_m \in \dot{\mathcal{U}}_n$ for all m; and
(3) $\{\, U^{n,p}_m : m \in \mathbb{N} \,\}$ is a cover of X.

As X satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$, there are for each pair $(n,p) \in \mathbb{N} \times \mathbb{P}$ a number $k(n,p)$ such that

$$\left\{ \bigcup_{m<k(n,p)} U^{n,p}_m : (n,p) \in \mathbb{N} \times \mathbb{P} \right\} \in \Gamma(X).$$

By enlarging the numbers $k(n,p)$, we may assume that the displayed enumeration is bijective.

Let G be $\mathbb{P}$-generic over M. Fix n. The set $\{\, q^{n,p}_{k(n,p)} : p \in \mathbb{P} \,\}$ is dense in $\mathbb{P}$. Let p_n be a condition such that $q^{n,p_n}_{k(n,p_n)} \in G$. Then, in $M[G]$, we have that

$$\{\, U^{n,p_n}_m : m < k(n,p_n) \,\} \subseteq \mathcal{U}_n.$$

As our enumeration is bijective, we have that

$$\left\{ \bigcup_{m<k(n,p_n)} U^{n,p_n}_m : n \in \mathbb{N} \right\} \in \Gamma(X).$$

This completes the proof. □

Remark 5.5. In Theorem 5.2, the only implication that uses the premise that the points of the space are G_δ is "(3) ⇒ (2)". Since this hypothesis is very mild, we have not tried to eliminate it.

Theorem 5.2 has the following corollary.

Corollary 5.6. *For spaces with points G_δ, the Gerlits–Nagy property* (*) *(equivalently, the conjunction of* $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$ *and* $\mathsf{S}_1(\mathrm{O}, \mathrm{O})$*) is preserved by Cohen forcing.* □

6. $\mathrm{C_p}$ THEORY AND MORE APPLICATIONS

For a space X, let $\mathrm{D}(X)$ be the family of all dense subsets of X. Spaces satisfying $\mathsf{S}_{\mathrm{fin}}(\mathrm{D},\mathrm{D})$ are also called *selectively separable* or *M-separable*, and spaces satisfying $\mathsf{S}_1(\mathrm{D},\mathrm{D})$ are also called *R-separable*; see [10] for a summary and references.[2] For a space X and a point $x\in X$, let $\Omega_x(X)$ be the family of all sets $A\subseteq X$ with $x\in\overline{A}\setminus A$. A space X has *countable fan tightness* if $\mathsf{S}_{\mathrm{fin}}(\Omega_x,\Omega_x)$ holds at all points $x\in X$. It has *strong countable fan tightness* if $\mathsf{S}_1(\Omega_x,\Omega_x)$ holds at all points $x\in X$. When the space X is a topological group, it suffices to consider $\mathsf{S}_{\mathrm{fin}}(\Omega_x,\Omega_x)$ and $\mathsf{S}_1(\Omega_x,\Omega_x)$ at the neutral element of that group.

Generalizing results of Scheepers [33, Theorems 13 and 35], Bella, Bonanzinga, Matveev and Tkachuk prove in [7, Corollary 2.10] that the following assertions are equivalent for every space X and each $\mathsf{S}\in\{\mathsf{S}_1,\mathsf{S}_{\mathrm{fin}}\}$:

(1) $\mathrm{C_p}(X)$ satisfies $\mathsf{S}(\mathrm{D},\mathrm{D})$;
(2) $\mathrm{C_p}(X)$ is separable and satisfies $\mathsf{S}(\Omega_0,\Omega_0)$;
(3) X has a coarser, second countable topology and satisfies $\mathsf{S}(\Omega,\Omega)$.

Corollary 6.1 (CH)**.** *There are sets $X,Y\subseteq\mathbb{R}$ such that the spaces $\mathrm{C_p}(X)$ and $\mathrm{C_p}(Y)$ are Fréchet–Urysohn, and their product $\mathrm{C_p}(X)\times\mathrm{C_p}(Y)$ does not satisfy $\mathsf{S}_{\mathrm{fin}}(\mathrm{D},\mathrm{D})$ (or, equivalently, $\mathsf{S}_{\mathrm{fin}}(\Omega_0,\Omega_0)$).*

Proof. By Theorem 4.2, there are γ spaces $X,Y\subseteq\mathbb{R}$ (so that $\mathrm{C_p}(X)$ and $\mathrm{C_p}(Y)$ are Fréchet–Urysohn) such that $X\sqcup Y$ does not satisfy $\mathsf{S}_{\mathrm{fin}}(\Omega,\Omega)$, and hence $\mathrm{C_p}(X)\times\mathrm{C_p}(Y)=\mathrm{C_p}(X\sqcup Y)$ does not satisfy $\mathsf{S}_{\mathrm{fin}}(\mathrm{D},\mathrm{D})$ [33, Theorem 35]. $\square$

Corollary 6.1 strengthens Babinkostova's Corollary 2.5 in [1], where the spaces $\mathrm{C_p}(X)$ and $\mathrm{C_p}(Y)$ satisfy the weaker property $\mathsf{S}_1(\mathrm{D},\mathrm{D})$. In fact, Babinkostova's spaces are provably not Fréchet–Urysohn. When this extra feature is taken into account, the results become incomparable. Corollary 6.1 can be used to reproduce a result of Barman and Dow [3, Theorem 2.24]. The Barman–Dow Theorem is identical to Corollary 6.2 below, except that their countable spaces are not topological groups.

Corollary 6.2 (CH)**.** *There are countable abelian Fréchet–Urysohn topological groups A and B such that the product group $A\times B$ does not satisfy $\mathsf{S}_{\mathrm{fin}}(\mathrm{D},\mathrm{D})$ or $\mathsf{S}_{\mathrm{fin}}(\Omega_0,\Omega_0)$.*

Proof. It suffices to consider $\mathsf{S}_{\mathrm{fin}}(\mathrm{D},\mathrm{D})$. Indeed, according to [7, Proposition 2.3(2)], every separable space with countable fan tightness satisfies $\mathsf{S}_{\mathrm{fin}}(\mathrm{D},\mathrm{D})$.

Let X and Y be as in Corollary 6.1. Let $\langle\, D_n : n\in\mathbb{N}\,\rangle$ be a sequence of countable dense subsets of $\mathrm{C_p}(X)\times\mathrm{C_p}(Y)$ witnessing the failure of $\mathsf{S}_{\mathrm{fin}}(\mathrm{D},\mathrm{D})$ for $\mathrm{C_p}(X)\times\mathrm{C_p}(Y)$. Let A and B be the groups generated by the projections of $\bigcup_n D_n$ on the first and second coordinates, respectively. As being Fréchet–Urysohn is hereditary, the countable groups A and B are Fréchet–Urysohn. As $A\times B$ contains D_0, it is dense in $\mathrm{C_p}(X)\times\mathrm{C_p}(Y)$. The sets D_n are contained in $A\times B$ and are dense (in particular) there. Assume that there are finite sets $F_n\subseteq D_n$ for $n\in\mathbb{N}$ such that $\bigcup_n F_n$ is dense in $A\times B$. Then $\bigcup_n F_n$ is dense in $\mathrm{C_p}(X)\times\mathrm{C_p}(Y)$, a contradiction. $\square$

[2] In the paper [10], the family D is defined differently in order to study additional properties in a uniform manner. The change in the definition of D does not change the property $\mathsf{S}_{\mathrm{fin}}(\mathrm{D},\mathrm{D})$.

To what extent is the Continuum Hypothesis necessary for Theorem 3.2? Typically, in the field of selection principles, Martin's Axiom suffices to establish consequences of the Continuum Hypothesis. Surprisingly, this is not the case here. The following theorem is an immediate consequence of a result of Barman and Dow [4, Theorem 3.3]. PFA stands for the Proper Forcing Axiom, an axiom that is strictly stronger than Martin's Axiom.

Theorem 6.3 (PFA). *All finite products of separable metric γ spaces satisfy* $\mathsf{S}_{\mathrm{fin}}(\Omega, \Omega)$.

Proof. According to a result of Barman and Dow [4, Theorem 3.3], PFA implies that all finite products of countable Fréchet–Urysohn spaces satisfy $\mathsf{S}_{\mathrm{fin}}(\mathrm{D}, \mathrm{D})$. We consider products of two sets. The generalization to arbitrary finite products is straightforward.

Assume that X and Y are separable metric γ spaces and $X \times Y$ does not satisfy $\mathsf{S}_{\mathrm{fin}}(\Omega, \Omega)$. As the property $\mathsf{S}_{\mathrm{fin}}(\Omega, \Omega)$ is preserved by finite powers [18, Theorem 2.5], Lemma 2.1 implies that $X \sqcup Y$ does not satisfy $\mathsf{S}_{\mathrm{fin}}(\Omega, \Omega)$. Thus, by Scheepers's Theorem, the space $\mathrm{C_p}(X) \times \mathrm{C_p}(Y) = \mathrm{C_p}(X \sqcup Y)$ does not satisfy $\mathsf{S}_{\mathrm{fin}}(\mathrm{D}, \mathrm{D})$. Continuing as in the proof of Corollary 6.2, we obtain two countable Fréchet–Urysohn spaces whose product is not $\mathsf{S}_{\mathrm{fin}}(\mathrm{D}, \mathrm{D})$, a contradiction. □

By the above-mentioned theorem of Bella, Bonanzinga, Matveev and Tkachuk [7, Corollary 2.10], it suffices to assume in Theorem 6.3 that the γ spaces have a coarser, second countable topology.

In the Cohen model, a result stronger than Theorem 6.3 follows from another result of Barman and Dow [4].

Theorem 6.4. *In the Cohen model, obtained by adding at least $\aleph_2$ Cohen reals to a model of the Continuum Hypothesis, all Tychonoff γ spaces X have cardinality at most $\aleph_1$.*

Proof. Let X be a Tychonoff γ space. Then $\mathrm{C_p}(X)$ is Fréchet–Urysohn. Fix a countable dense subset D of $\mathrm{C_p}(X)$. Then D is Fréchet–Urysohn. According to [4, Theorem 3.1], in the Cohen model, all countable Fréchet–Urysohn spaces having π-weight at most $\aleph_1$. It follows that the π-weight of D is at most $\aleph_1$. By the density of D, the π-weight of $\mathrm{C_p}(X)$ is at most $\aleph_1$. In a topological group, if $\mathcal{U}$ is a pseudo-base, then the set $\{\, U^{-1} \cdot U : U \in \mathcal{U} \,\}$ is a local base at the neutral element. Thus, the cardinality of X, which is equal to the character of $\mathrm{C_p}(X)$, is at most $\aleph_1$. □

As $\aleph_1 < \mathfrak{d}$ in the Cohen model, the consequence that products of γ spaces in $\mathbb{R}$ satisfy $\mathsf{S}_{\mathrm{fin}}(\Omega, \Omega)$ there is trivial, i.e., follows from sheer cardinality considerations.

The following theorem solves, in the negative, Problem 3.1 (and thus also Problems 3.2 and 3.3) of Samet–Tsaban [45, §3]. This problem asks whether every set $X \subseteq \mathbb{R}$ with the Hurewicz property, and with Menger's property in all finite powers, necessarily has the Hurewicz property in all finite powers. Theorem 6.5 also provides a consistently positive answer to Problem 3.4 there, since adding $\aleph_1$ Cohen reals to a model of the Continuum Hypothesis preserves the Continuum Hypothesis. A proposed solution of these problems in [36] is withdrawn in [37], for the reasons in the discussion following Problem 4.1.

Theorem 6.5. *In any model obtained by adding uncountably many Cohen reals to a model of the Continuum Hypothesis, there is a set $X \subseteq \mathbb{R}$ such that X satisfies $\mathsf{S}_1(\mathrm{T},\Gamma)$ and $\mathsf{S}_1(\Omega,\Omega)$, but its square X^2 does not satisfy $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$.*

Proof. In the ground model, using the Continuum Hypothesis, let X be the set in the proof of Theorem 4.2. Move to the generic extension. By Theorem 5.1, the set X remains the union of two γ spaces. Thus, X satisfies $\mathsf{S}_1(\mathrm{T},\Gamma)$. All finite powers of ground model sets, including X, satisfy $\mathsf{S}_1(\mathrm{O},\mathrm{O})$ in the extension [36, Theorem 11]. Equivalently, X satisfies $\mathsf{S}_1(\Omega,\Omega)$. By Theorem 4.2, in the ground model, the square X^2 does not satisfy $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$, and thus does not satisfy $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$. It follows that, in the extension, the square X^2 does not satisfy $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$ [36, Theorem 37]. $\square$

Similarly, we have the following.

Theorem 6.6. *In any model obtained by adding uncountably many Cohen reals to a model of the Continuum Hypothesis, there are γ spaces $X, Y \subseteq \mathbb{R}$ such that $X \times Y$ satisfies $\mathsf{S}_1(\Omega,\Omega)$ but not $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$.* $\square$

For a space X, let $\mathcal{D} \in \mathrm{D}_\Gamma(X)$ if $\mathcal{D}$ is infinite, and for each open set U in X, U intersects all but finitely many members of $\mathcal{D}$. Spaces satisfying $\mathsf{S}_{\mathrm{fin}}(\mathrm{D},\mathrm{D}_\Gamma)$ are also called *H-separable* (e.g., [6]). Also, for $x \in X$, let Γ_x be the family of all countable sets converging to x. Spaces satisfying $\mathsf{S}_1(\Gamma_x,\Gamma_x)$ are also called α_2 *spaces.*

Corollary 6.7. *In any model obtained by adding uncountably many Cohen reals to a model of the Continuum Hypothesis, there is a set $X \subseteq \mathbb{R}$ such that the space $\mathrm{C_p}(X)$ satisfies $\mathsf{S}_1(\mathrm{D},\mathrm{D})$ and $\mathsf{S}_1(\Gamma_0,\Gamma_0)$, but not $\mathsf{S}_{\mathrm{fin}}(\mathrm{D},\mathrm{D}_\Gamma)$.*

Proof. Let X be the set from Theorem 6.5. As X satisfies $\mathsf{S}_1(\Omega,\Omega)$, the space $\mathrm{C_p}(X)$ satisfies $\mathsf{S}_1(\mathrm{D},\mathrm{D})$ [33, Theorem 13]. As X satisfies $\mathsf{S}_1(\Gamma,\Gamma)$, the space $\mathrm{C_p}(X)$ satisfies $\mathsf{S}_1(\Gamma_0,\Gamma_0)$ [30, Theorem 4]. As X^2 does not satisfy $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$, the space $\mathrm{C_p}(X)$ does not satisfy $\mathsf{S}_{\mathrm{fin}}(\mathrm{D},\mathrm{D}_\Gamma)$ [6, Theorem 40]. $\square$

By the usual method used in the earlier proofs, Corollary 6.7 has the following consequence.

Corollary 6.8. *In any model obtained by adding uncountably many Cohen reals to a model of the Continuum Hypothesis, there is a countable abelian topological group A satisfying $\mathsf{S}_1(\mathrm{D},\mathrm{D})$ and $\mathsf{S}_1(\Gamma_0,\Gamma_0)$, but not $\mathsf{S}_{\mathrm{fin}}(\mathrm{D},\mathrm{D}_\Gamma)$.* $\square$

7. The product of an unbounded tower set and a Sierpiński set

We conclude this paper with a proof that, for each unbounded tower $T = \{ x_\alpha : \alpha < \mathfrak{b} \} \subseteq [\mathbb{N}]^\infty$ and each Sierpiński set S, the product space $(T \cup [\mathbb{N}]^{<\infty}) \times S$ satisfies $\mathsf{S}_1(\Gamma,\Gamma)$. In fact, we prove a more general result.

For each unbounded tower $T = \{ x_\alpha : \alpha < \mathfrak{b} \} \subseteq [\mathbb{N}]^\infty$, the set $T \cup [\mathbb{N}]^{<\infty}$ satisfies $\mathsf{S}_1(\Gamma,\Gamma)$ (implicitly in [31, Theorem 6] and explicitly in [38, Proposition 2.5]). The existence of unbounded towers of cardinality $\mathfrak{b}$ follows from the existence of unbounded towers of any cardinality [24, Proposition 2.4]. Examples of hypotheses implying the existence of unbounded towers are $\mathfrak{t} = \mathfrak{b}$ or $\mathfrak{b} < \mathfrak{d}$ [24, Lemma 2.2].

The property $\mathsf{S}_1(\mathrm{B}_\Gamma,\mathrm{B}_\Gamma)$ is equivalent to the Hurewicz property for countable Borel covers and also to the property that all Borel images in the Baire space $\mathbb{N}^\mathbb{N}$ are bounded [38, Theorem 1].

Theorem 7.1. *Let $T = \{ x_\alpha : \alpha < \mathfrak{b} \} \subseteq [\mathbb{N}]^\infty$ be an unbounded tower. For every space Y satisfying $\mathsf{S}_1(\mathrm{B}_\Gamma, \mathrm{B}_\Gamma)$, the product space $(T \cup [\mathbb{N}]^{<\infty}) \times Y$ satisfies $\mathsf{S}_1(\Gamma, \Gamma)$.*

Proof. Let $\mathcal{U} = \{ U_n : n \in \mathbb{N} \} \in \Gamma((T \cup [\mathbb{N}]^{<\infty}) \times Y)$. For a finite set $s \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, let

$$[s, n] = \{ x \subseteq \mathbb{N} : x \cap \{0, \dots, n-1\} = s \} \cap (T \cup [\mathbb{N}]^{<\infty}).$$

By shrinking the elements of $\mathcal{U}$, we may assume that $U_n \cap (\{n\} \times Y) = \emptyset$ for all n. Consider the functions $f, g\colon Y \to \mathbb{N}^\mathbb{N}$, defined by

$$\begin{aligned} f(y)(n) &= \max \{ k : P(\{0, \dots, k-1\}) \times \{y\} \subseteq U_n \}, \\ g(y)(n) &= \min \{ l \geq n : \forall s \in P(\{0, \dots, f(y)-1\}),\ [s, l] \times \{y\} \subseteq U_n \}. \end{aligned}$$

By our assumption on $\mathcal{U}$, we have that $f(y)(n) \leq n$. As $\mathcal{U} \in \Gamma((T \cup [\mathbb{N}]^{<\infty}) \times Y)$, the sequence $\langle f(y)(n) : n \in \mathbb{N} \rangle$ converges to infinity for each $y \in Y$.

Claim 7.2. The function f is Borel, and there is a Borel function $h\colon Y \to \mathbb{N}^\mathbb{N}$ such that $g(y)(n) \leq h(y)(n)$ for all $y \in Y$ and all n.

Proof. The function f is Borel, since the preimages under f of the standard basic open subsets of $\mathbb{N}^\mathbb{N}$ are finite intersections of subsets of Y which are either closed or open.

Represent each open set U_n as an increasing union $\bigcup_k U_{n,k}$ of clopen sets. Let $\overline{\mathbb{N}}$ be the set $\mathbb{N} \cup \{\infty\}$, with the discrete topology. Define a function $\Phi\colon Y \to \left(\overline{\mathbb{N}}^\mathbb{N}\right)^\mathbb{N}$ as follows: $\Phi(y)(n)(k) = \infty$ if $P(\{0, \dots, f(y)(n)-1\}) \times \{y\} \not\subseteq U_{n,k}$, and if not, then $\Phi(y)(n)(k)$ is the minimal l such that $[s, l] \times \{y\} \subseteq U_{n,k}$ for all $s \subseteq \{0, \dots, f(y)(n)-1\}$. Since $P(\{0, \dots, f(y)(n)-1\}) \times \{y\} \subseteq U_n$, by the definition of f, there is k such that $P(\{0, \dots, f(y)(n)-1\}) \times \{y\} \subseteq U_{n,k}$. Thus, the set $\{ k : \Phi(y)(n)(k) = \infty \}$ is finite. Moreover, the sequence $\langle \Phi(y)(n)(k) : k \in \mathbb{N} \rangle$ is nonincreasing (we assume that $i < \infty$ for all i), and $\Phi(y)(n)(k) \geq g(y)(n)$ for all k. Set $h(y)(n) = \min \{ \Phi(y)(n)(k) : k \in \mathbb{N} \}$. It follows that $h(y)(n) \geq g(y)(n)$ for all n. Thus, it suffices to prove that $h\colon Y \to \mathbb{N}^\mathbb{N}$ is Borel, which follows as soon as we prove that $\Phi\colon Y \to \left(\overline{\mathbb{N}}^\mathbb{N}\right)^\mathbb{N}$ is Borel.

Fix $n, k \in \mathbb{N}$ and $m \in \overline{\mathbb{N}}$. We need to show that the set $A = \{y \in Y : \Phi(y)(n)(k) = m\}$ is Borel. Consider the two possible cases.

Case 1: $m = \infty$. In this case,

$$\begin{aligned} A &= \{ y : P(\{0, \dots, f(y)(n)-1\}) \times \{y\} \not\subseteq U_{n,k} \} \\ &= \bigcup_{l<n} \big(\{ y \in Y : f(y)(n) = l \} \cap \{ y \in Y : P(\{0, \dots, l-1\}) \times \{y\} \not\subseteq U_{n,k} \} \big) \\ &= \bigcup_{l<n} \big(\{ y \in Y : f(y)(n) = l \} \cap \bigcup_{s \subseteq \{0, \dots, l-1\}} \{ y \in Y : (s, y) \notin U_{n,k} \} \big). \end{aligned}$$

As the function f is Borel, the set $\{ y \in Y : f(y)(n) = l \}$ is Borel. The set $\{y \in Y : (s, y) \notin U_{n,k}\}$ is a clopen subset of Y for all $s \subseteq l$. Thus, A is Borel.

Case 2: $m \in \mathbb{N}$. In this case,

$$\begin{aligned} A &= \{ y \in Y : \forall s \subseteq \{0, \dots, f(y)(n)-1\},\ ([s,m] \times \{y\} \subseteq U_{n,k}) \} \\ &\quad \cap \{ y \in Y : \exists s \subseteq \{0, \dots, f(y)(n)-1\},\ ([s,m-1] \times \{y\} \not\subseteq U_{n,k}) \} \\ &= \bigcup_{l<n} \big(\{ y \in Y : \forall s \subseteq \{0, \dots, l-1\},\ ([s,m] \times \{y\} \subseteq U_{n,k}) \} \\ &\quad \cap \{ y \in Y : f(y)(n) = l \} \big) \\ &\quad \cap \bigcup_{l<n} \big(\{ y \in Y : \exists s \subseteq \{0, \dots, l-1\},\ ([s,m-1] \times \{y\} \not\subseteq U_{n,k}) \} \\ &\quad \cap \{ y \in Y : f(y)(n) = l \} \big). \end{aligned}$$

As the function f is Borel, the latter set is Borel. Indeed, for each $V \subseteq T \cup [\mathbb{N}]^{<\infty}$, the set $\{ y \in Y : V \times \{y\} \subseteq U \}$ is closed whenever $U \subseteq (T \cup [\mathbb{N}]^{<\infty}) \times Y$ is closed. □

Claim 7.3. There is an increasing function $c \in \mathbb{N}^{\mathbb{N}}$ such that, for each $y \in Y$,

$$c(n) \leq f(y)(c(n+1)) \leq h(y)(c(n+1)) < c(n+2)$$

for all but finitely many n.

Proof. Consider the map $f' \colon Y \to \mathbb{N}^{\mathbb{N}}$, defined by $f'(y)(n) = \min \{ f(y)(l) : l \geq n \}$. Then the set $f'(Y) \subseteq \mathbb{N}^{\mathbb{N}}$ consists of nondecreasing unbounded sequences. Set

$$f''(y)(k) = \min \{ n : f'(y)(n) \geq k \}.$$

Then $f'' \colon Y \to \mathbb{N}^{\mathbb{N}}$ is a Borel map, and hence $f''(Y)$ is bounded by some increasing function $a' \in \mathbb{N}^{\mathbb{N}}$. Let $a(n) = \min \{ k : a'(k) \geq n \}$. Then $a \leq^* f'(y) \leq^* f(y)$ for all $y \in Y$.

Since Y satisfies $\mathsf{S}_1(\mathrm{B}_\Gamma, \mathrm{B}_\Gamma)$ and h is Borel, there is an increasing $b \in \mathbb{N}^{\mathbb{N}}$ such that $h(y) \leq^* b$ for all $y \in Y$. Let $c(0) = 1$, and

$$c(n+1) = \max\{\min \{ l : a(l) \geq c(n) \}, b(c(n))\} + 1.$$

We claim that c is as required. Indeed, fix $y \in Y$ and find n such that $a(m) \leq h(y)(m) \leq g(y)(m) \leq b(m)$ for all $m \geq n$. For $m \geq n$, as $c(m+1) \geq \min \{ l : a(l) \geq c(m) \}$ and a is nondecreasing, we have that $f(y)(c(m+1)) \geq a(c(m+1)) \geq c(m)$, and the inequality $h(y)(c(m+1)) \leq b(c(m+1)) < c(m+2)$ follows. □

Let $\langle \mathcal{U}_k : k \in \mathbb{N} \rangle$ be a sequence in $\Gamma((T \cup [\mathbb{N}]^{<\infty}) \times Y)$, where $\mathcal{U}_k = \langle U^k_n : n \in \mathbb{N} \rangle$ for all k.

Claim 7.4. Suppose that for every sequence $\langle \mathcal{V}_k : k \in \mathbb{N} \rangle$ in $\Gamma((T \cup [\mathbb{N}]^{<\infty}) \times Y)$, where $\mathcal{V}_k = \langle V^k_n : n \in \mathbb{N} \rangle$ for all k, there exists a sequence $\langle n_k : k \in \mathbb{N} \rangle$ in $\mathbb{N}$ such that $\langle V^k_{n_k} : k \in \mathbb{N} \rangle \in \Gamma(A \times Y)$ for some A containing $[\mathbb{N}]^{<\infty}$ with $|T \setminus A| < \mathfrak{b}$. Then $(T \cup [\mathbb{N}]^{<\infty}) \times Y$ is $S_1(\Gamma, \Gamma)$.

Proof. First let us note that the following statement may be obtained simply by splitting each $\mathcal{V}_n$ into countably many disjoint infinite pieces and applying the assumption to the sequence of pieces: for every sequence $\langle \mathcal{V}_k : k \in \mathbb{N} \rangle$ in $\Gamma((T \cup [\mathbb{N}]^{<\infty}) \times Y)$ there exists a sequence $\langle \mathcal{V}'_k : k \in \mathbb{N} \rangle$ such that $\mathcal{V}'_k$ is an infinite subset of $\mathcal{V}_k$ and $\bigcup_k \mathcal{V}'_k \in \Gamma(A \times Y)$ for some A containing $[\mathbb{N}]^{<\infty}$ with $|T \setminus A| < \mathfrak{b}$.

Fix $\alpha_0 < \mathfrak{b}$ and a sequence $\langle \mathcal{V}_k : k \in \mathbb{N} \rangle$ in $\Gamma((T \cup [\mathbb{N}]^{<\infty}) \times Y)$. Since the set $\{ x_\xi : \xi < \alpha_0 \} \times Y$ is $S_1(\mathrm{B}_\Gamma, \mathrm{B}_\Gamma)$, there exists a sequence $\langle \mathcal{W}^0_k : k \in \mathbb{N} \rangle$ such that $\mathcal{W}_k^0$ is an infinite subset of $\mathcal{V}_k$ and $\bigcup\{\,\mathcal{W}_k^0 : k \in \mathbb{N}\,\} \in \Gamma(\{\,x_\xi : \xi < \alpha_0\,\} \times Y)$. Applying (the reformulation of) our assumption to the sequence $\langle\,\mathcal{W}_k^0 : k \in \mathbb{N}\,\rangle$ in $\Gamma\big((T \cup [\mathbb{N}]^{<\infty}) \times Y\big)$, we can find a sequence $\langle\,\mathcal{V}_k^0 : k \in \mathbb{N}\,\rangle$ such that $\mathcal{V}_k^0$ is an infinite subset of $\mathcal{W}_k^0$ and $\bigcup_k \mathcal{V}_k^0 \in \Gamma(A \times Y)$ for some A containing $[\mathbb{N}]^{<\infty}$ with $|T \setminus A| < \mathfrak{b}$. It follows that $\bigcup_k \mathcal{V}_k^0 \in \Gamma((\{\,x_\xi : \xi < \alpha_0\,\} \cup \{\,x_\xi : \xi > \alpha_1\,\} \cup [\mathbb{N}]^{<\infty}) \times Y)$ for some $\alpha_1 > \alpha_0$.

Applying the same argument infinitely many times we can get an increasing sequence $\langle\,\alpha_n : n \in \mathbb{N}\,\rangle$ of ordinals below $\mathfrak{b}$, and for every n a sequence $\langle\,\mathcal{V}_k^n : k \in \mathbb{N}\,\rangle$ such that $\mathcal{V}_k^n$ is an infinite subset of $\mathcal{V}_k^{n-1}$ and

$$\bigcup_k \mathcal{V}_k^n \in \Gamma((\{\,x_\xi : \xi < \alpha_n\,\} \cup \{\,x_\xi : \xi > \alpha_{n+1}\,\} \cup [\mathbb{N}]^{<\infty}) \times Y).$$

Let us select $V_k \in \mathcal{V}_k^k \setminus \{V_0, \dots, V_{k-1}\}$ for all k. Then $V_k \in \mathcal{V}_k$ and $\{\,V_k : k \in \mathbb{N}\,\}$ is easily seen to be in $\Gamma((T \cup [\mathbb{N}]^{<\infty}) \times Y)$. ☐

By Lemma 7.4, it suffices to find a sequence $\langle\,n_k : k \in \mathbb{N}\,\rangle$ in $\mathbb{N}$ such that $\langle\,U_{n_k}^k : k \in \mathbb{N}\,\rangle \in \Gamma(A \times Y)$ for some A containing $[\mathbb{N}]^{<\infty}$ with $|T \setminus A| < \mathfrak{b}$.

For each $k \in \mathbb{N}$, let $c_k \in \mathbb{N}^\mathbb{N}$ be such as in Claim 7.3, where $\mathcal{U}$ is replaced with $\mathcal{U}_k$, and let f_k and h_k be the associated functions. Consider the function $d \colon Y \to \mathbb{N}^\mathbb{N}$ defined by

$$\begin{aligned} d(y)(k) = \min\{n : \forall m \geq n,\ \big(c_k(m) \leq f_k(y)(c_k(m+1))& \\ < h_k(y)(c_k(m+1)) < c_k(m+2)\big)\}.& \end{aligned}$$

Since the functions f_k and h_k are Borel, so is the function d, and hence there is an increasing $x \in \mathbb{N}^\mathbb{N}$ such that $d(y) \leq^* x$ for all $y \in Y$. We may assume that $c_{k+1}(x(k+1)) > c_k(x(k)+2)$ for all k. Let $\alpha < \mathfrak{b}$ be such that the set $I = \{\,k : x_\alpha \cap [c_k(x(k)), c_k(x(k)+2)) = \emptyset\,\}$ is infinite. Fix $\beta \geq \alpha$ and $y \in Y$, and find k_0 such that $x_\beta \setminus x_\alpha \subseteq k_0$ and $d(y)(k) \leq x(k)$ for all $k \geq k_0$. Then, for all $k \geq k_0$ in I, we have that $x_\beta \cap [c_k(x(k)), c_k(x(k)+2)) \subseteq x_\alpha \cap [c_k(x(k)), c_k(x(k)+2)) = \emptyset$. Consequently, $x_\beta \cap [f_k(y)(c_k(x(k)+1)), h_k(y)(c_k(x(k)+1))) = \emptyset$, and hence $x_\beta \cap [f_k(y)(c_k(x(k)+1)), g_k(y)(c_k(x(k)+1))) = \emptyset$. Thus,

$$(x_\beta, y) \in [x_\beta \cap f_k(y)(c_k(x(k)+1)), g_k(y)(c_k(x(k)+1))].$$

By the definitions of f_k and g_k, the latter open set is a subset of $U^k_{c_k(x(k)+1)}$. Therefore, for every $\beta \geq \alpha$ and $y \in Y$, we have that $(x_\beta, y) \in U^k_{c_k(x(k)+1)}$ for all but finitely many $k \in I$. As the covers $\mathcal{U}_k$ get finer with k, this completes our proof. ☐

As the unbounded set T in Theorem 7.1 is a Borel subset of the space $T \cup [\mathbb{N}]^{<\infty}$, the latter space does not satisfy $\mathsf{S}_1(\mathrm{B}_\Gamma, \mathrm{B}_\Gamma)$. In particular, it is not productively $\mathsf{S}_1(\mathrm{B}_\Gamma, \mathrm{B}_\Gamma)$.

Problem 7.5. Let $T = \{\,x_\alpha : \alpha < \mathfrak{b}\,\}$ be an unbounded tower. Is the space $T \cup [\mathbb{N}]^{<\infty}$, provably, productively $\mathsf{S}_1(\Gamma, \Gamma)$? Is this the case assuming the Continuum Hypothesis?

Acknowledgments

The research of the third-named author was supported by FWF Grant I 1209-N25 and the program APART of the Austrian Academy of Sciences. Parts of the work reported here were carried out during three visits of the second-named author to the Kurt Gödel Research Center for Mathematical Logic. These visits were partially supported by the above-mentioned grants and by the Research Networking Programme *New Frontiers of Infinity* (INFTY), funded by the European Science Foundation. The second-named author thanks the third-named author for his kind hospitality and the Kurt Gödel Research Center director, researchers and staff for the excellent academic and friendly atmosphere. The authors thank Francis Jordan for useful discussions and the referee for reviewing this paper.

References

[1] Liljana Babinkostova, *On some questions about selective separability*, MLQ Math. Log. Q. **55** (2009), no. 5, 539–541, DOI 10.1002/malq.200810010. MR2568764 (2010i:54026)

[2] Liljana Babinkostova and Marion Scheepers, *Products and selection principles*, Topology Proc. **31** (2007), no. 2, 431–443. MR2476622 (2010b:54015)

[3] Doyel Barman and Alan Dow, *Selective separability and* SS^+, Topology Proc. **37** (2011), 181–204. MR2678950 (2011i:54024)

[4] Doyel Barman and Alan Dow, *Proper forcing axiom and selective separability*, Topology Appl. **159** (2012), no. 3, 806–813, DOI 10.1016/j.topol.2011.11.048. MR2868880

[5] Tomek Bartoszyński and Haim Judah, *Set theory*, On the structure of the real line, A K Peters, Ltd., Wellesley, MA, 1995. MR1350295 (96k:03002)

[6] Angelo Bella, Maddalena Bonanzinga, and Mikhail Matveev, *Variations of selective separability*, Topology Appl. **156** (2009), no. 7, 1241–1252, DOI 10.1016/j.topol.2008.12.029. MR2502000 (2010d:54038)

[7] Angelo Bella, Maddalena Bonanzinga, Mikhail V. Matveev, and Vladimir V. Tkachuk, *Selective separability: general facts and behavior in countable spaces*, Spring Topology and Dynamics Conference, Topology Proc. **32** (2008), Spring, 15–30. MR1500068 (2010a:54007)

[8] Andreas Blass, *Combinatorial cardinal characteristics of the continuum*, Handbook of set theory. Vols. 1, 2, 3, Springer, Dordrecht, 2010, pp. 395–489, DOI 10.1007/978-1-4020-5764-9_7. MR2768685

[9] Jörg Brendle, *Generic constructions of small sets of reals*, Topology Appl. **71** (1996), no. 2, 125–147, DOI 10.1016/0166-8641(95)00071-2. MR1399552 (97i:03030)

[10] Maddalena Bonanzinga, Filippo Cammaroto, Bruno Antonio Pansera, and Boaz Tsaban, *Diagonalizations of dense families*, Topology Appl. **165** (2014), 12–25, DOI 10.1016/j.topol.2014.01.001. MR3174063

[11] A. Dow, *Two applications of reflection and forcing to topology*, General topology and its relations to modern analysis and algebra, VI (Prague, 1986), Res. Exp. Math., vol. 16, Heldermann, Berlin, 1988, pp. 155–172. MR952602 (89k:54007)

[12] Fred Galvin and Arnold W. Miller, *γ-sets and other singular sets of real numbers*, Topology Appl. **17** (1984), no. 2, 145–155, DOI 10.1016/0166-8641(84)90038-5. MR738943 (85f:54011)

[13] Fred Galvin and Karel Prikry, *Borel sets and Ramsey's theorem*, J. Symbolic Logic **38** (1973), 193–198. MR0337630 (49 #2399)

[14] J. Gerlits and Zs. Nagy, *Some properties of $C(X)$. I*, Topology Appl. **14** (1982), no. 2, 151–161, DOI 10.1016/0166-8641(82)90065-7. MR667661 (84f:54021)

[15] Serge Grigorieff, *Combinatorics on ideals and forcing*, Ann. Math. Logic **3** (1971), no. 4, 363–394. MR0297560 (45 #6614)

[16] Francis Jordan, *Productive local properties of function spaces*, Topology Appl. **154** (2007), no. 4, 870–883, DOI 10.1016/j.topol.2006.09.012. MR2294635 (2008b:54032)

[17] Francis Jordan, *There are no hereditary productive γ-spaces*, Topology Appl. **155** (2008), no. 16, 1786–1791, DOI 10.1016/j.topol.2008.05.016. MR2445301 (2009f:54029)

[18] Winfried Just, Arnold W. Miller, Marion Scheepers, and Paul J. Szeptycki, *The combinatorics of open covers. II*, Topology Appl. **73** (1996), no. 3, 241–266, DOI 10.1016/S0166-8641(96)00075-2. MR1419798 (98g:03115a)

[19] Ljubiša D. R. Kočinac, *Selected results on selection principles*, Proceedings of the 3rd Seminar on Geometry & Topology, Azarb. Univ. Tarbiat Moallem, Tabriz, 2004, pp. 71–104. MR2090207 (2005g:54038)

[20] M. Malliaris and S. Shelah, *Cofinality spectrum theorems in model theory, set theory and general topology*, arXiv eprint 1208.5424, 2012.

[21] Heike Mildenberger, Saharon Shelah, and Boaz Tsaban, *The combinatorics of τ-covers*, Topology Appl. **154** (2007), no. 1, 263–276, DOI 10.1016/j.topol.2006.04.011. MR2271787 (2007i:03059)

[22] Arnold W. Miller, *A nonhereditary Borel-cover γ-set*, Real Anal. Exchange **29** (2003/04), no. 2, 601–606. MR2083799 (2005g:03079)

[23] Arnold W. Miller, *A hodgepodge of sets of reals*, Note Mat. **27** (2007), suppl. 1, 25–39. MR2406153 (2009e:03087)

[24] Arnold W. Miller and Boaz Tsaban, *Point-cofinite covers in the Laver model*, Proc. Amer. Math. Soc. **138** (2010), no. 9, 3313–3321, DOI 10.1090/S0002-9939-10-10407-9. MR2653961 (2011j:03119)

[25] Andrej Nowik, Marion Scheepers, and Tomasz Weiss, *The algebraic sum of sets of real numbers with strong measure zero sets*, J. Symbolic Logic **63** (1998), no. 1, 301–324, DOI 10.2307/2586602. MR1610427 (99c:54049)

[26] Tal Orenshtein and Boaz Tsaban, *Linear σ-additivity and some applications*, Trans. Amer. Math. Soc. **363** (2011), no. 7, 3621–3637, DOI 10.1090/S0002-9947-2011-05228-1. MR2775821 (2012e:54028)

[27] Janusz Pawlikowski, *Undetermined sets of point-open games*, Fund. Math. **144** (1994), no. 3, 279–285. MR1279482 (95i:54043)

[28] Masami Sakai and Marion Scheepers, *The combinatorics of open covers*, Recent progress in general topology. III, Atlantis Press, Paris, 2014, pp. 751–799, DOI 10.2991/978-94-6239-024-9_18. MR3205498

[29] Marion Scheepers, *Combinatorics of open covers. I. Ramsey theory*, Topology Appl. **69** (1996), no. 1, 31–62, DOI 10.1016/0166-8641(95)00067-4. MR1378387 (97h:90123)

[30] Marion Scheepers, *A sequential property of $C_p(X)$ and a covering property of Hurewicz*, Proc. Amer. Math. Soc. **125** (1997), no. 9, 2789–2795, DOI 10.1090/S0002-9939-97-03897-5. MR1396994 (97j:54018)

[31] Marion Scheepers, *$C_p(X)$ and Arhangel′skiĭ's α_i-spaces*, Topology Appl. **89** (1998), no. 3, 265–275, DOI 10.1016/S0166-8641(97)00218-6. MR1645184 (99g:54018)

[32] Marion Scheepers, *Finite powers of strong measure zero sets*, J. Symbolic Logic **64** (1999), no. 3, 1295–1306, DOI 10.2307/2586631. MR1779763 (2001m:03096)

[33] Marion Scheepers, *Combinatorics of open covers. VI. Selectors for sequences of dense sets*, Quaest. Math. **22** (1999), no. 1, 109–130, DOI 10.1080/16073606.1999.9632063. MR1711901 (2001j:91031)

[34] Marion Scheepers, *Selection principles and covering properties in topology*, Note Mat. **22** (2003/04), no. 2, 3–41. MR2112729 (2005h:54024)

[35] Marion Scheepers, *Gerlits and function spaces*, Studia Sci. Math. Hungar. **47** (2010), no. 4, 529–557, DOI 10.1556/SScMath.47.2010.4.1161. MR2883130 (2012i:54001)

[36] Marion Scheepers and Franklin D. Tall, *Lindelöf indestructibility, topological games and selection principles*, Fund. Math. **210** (2010), no. 1, 1–46, DOI 10.4064/fm210-1-1. MR2720214 (2012b:54006)

[37] Marion Scheepers and Franklin D. Tall, *Errata to "Lindelöf indestructibility, topological games and selection principles" (Fund. Math. 210 (2010), 1–46) [MR 2720214]*, Fund. Math. **224** (2014), no. 2, 203–204, DOI 10.4064/fm224-2-5. MR3180588

[38] Marion Scheepers and Boaz Tsaban, *The combinatorics of Borel covers*, Topology Appl. **121** (2002), no. 3, 357–382, DOI 10.1016/S0166-8641(01)00078-5. MR1908999 (2003e:03091)

[39] S. Shelah and B. Tsaban, *Critical cardinalities and additivity properties of combinatorial notions of smallness*, J. Appl. Anal. **9** (2003), no. 2, 149–162, DOI 10.1515/JAA.2003.149. MR2021285 (2004j:03058)

[40] Franklin D. Tall and Boaz Tsaban, *On productively Lindelöf spaces*, Topology Appl. **158** (2011), no. 11, 1239–1248, DOI 10.1016/j.topol.2011.04.003. MR2806359 (2012h:54059)

[41] S. Todorčević, *Aronszajn orderings*, Đuro Kurepa memorial volume, Publ. Inst. Math. (Beograd) (N.S.) **57(71)** (1995), 29–46. MR1387352 (98g:03116)

[42] Boaz Tsaban, *Selection principles and the minimal tower problem*, Note Mat. **22** (2003/04), no. 2, 53–81. MR2112731 (2006b:03054)

[43] Boaz Tsaban, *Some new directions in infinite-combinatorial topology*, Set theory, Trends Math., Birkhäuser, Basel, 2006, pp. 225–255, DOI 10.1007/3-7643-7692-9_7. MR2267150 (2007f:03064)

[44] Boaz Tsaban, *Additivity numbers of covering properties*, Selection principles and covering properties in topology, Quad. Mat., vol. 18, Dept. Math., Seconda Univ. Napoli, Caserta, 2006, pp. 245–282. MR2395757 (2009e:03088)

[45] N. Samet and B. Tsaban, *Problem of the issue*, SPM Bulletin **18** (2006), 7.

[46] B. Tsaban, *Selection principles and special sets of reals*, in Open Problems in Topology II (E. Pearl, ed.), Elsevier B.V. 2007, pp. 91–108.

[47] Boaz Tsaban, *Menger's and Hurewicz's problems: solutions from "the book" and refinements*, Set theory and its applications, Contemp. Math., vol. 533, Amer. Math. Soc., Providence, RI, 2011, pp. 211–226, DOI 10.1090/conm/533/10509. MR2777750 (2012f:03096)

[48] T. Weiss, *Martin's axiom for σ-centered partial orders and some properties of special subsets of the set of the real numbers*, doctoral dissertation, Pennsylvania State University, 1989.

[49] T. Weiss, *On finite products of special sets of real numbers*, unpublished manuscript, 2003.

Department of Mathematics, University of Wisconsin-Madison, Van Vleck Hall, 480 Lincoln Drive, Madison, Wisconsin 53706-1388

E-mail address: miller@math.wisc.edu

URL: http://www.math.wisc.edu/~miller

Department of Mathematics, Bar-Ilan University, Ramat Gan 5290002, Israel – and – Faculty of Mathematics and Computer Science, Weizmann Institute of Science, Rehovot 7610001, Israel

E-mail address: tsaban@math.biu.ac.il

URL: http://math.biu.ac.il/~tsaban

Kurt Gödel Research Center for Mathematical Logic, University of Vienna, Währinger Strasse 25, 1090 Vienna, Austria

E-mail address: lzdomsky@logic.univie.ac.at

URL: http://www.logic.univie.ac.at/~lzdomsky