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Topology and its Applications 121 (2002) 357–382

TOPOLOGY AND ITS APPLICATIONS

www.elsevier.com/locate/topol

The combinatorics of Borel covers

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Received 28 October 1999; received in revised form 12 December 2000

Abstract

In this paper we extend previous studies of selection principles for families of open covers of sets of real numbers to also include families of countable Borel covers. The main results of the paper could be summarized as follows:

(1) Some of the classes which were different for open covers are equal for Borel covers—Section 1.
(2) Some Borel classes coincide with classes that have been studied under a different guise by other authors—Section 4.

AMS classification: 03E05; 54D20; 54D80

Keywords: Rothberger property C''; Gerlits–Nagy; Property γ-sets; γ-cover; ω-cover; Sierpiński set; Lusin set; Selection priniciples; Borel covers

1. Introduction

Let X be a topological space. Let $\mathcal{O}$ denote the collection of all countable open covers of X. According to [5] an open cover $\mathcal{U}$ of X is said to be an ω-cover if X is not a member of $\mathcal{U}$, but for each finite subset F of X there is a $U \in \mathcal{U}$ such that $F \subseteq U$. It is shown in [5] that every ω-cover of X has a countable subset which is an ω-cover of X if, and only if, all finite powers of X have the Lindelöf property. All finite powers of sets of real numbers have the Lindelöf property. The symbol Ω denotes the collection of all *countable* ω-covers of X. Acording to [8,18] an open cover of X is said to be a γ-cover if it is infinite and each element of X is a member of all but finitely many members of the cover. Since each

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0166-8641/02/$ – see front matter
PII: S0166-8641(01)00078-5

infinite subset of a γ-cover is a γ-cover, each γ-cover has a countable subset which is a γ-cover. The symbol Γ denotes the collection of all *countable* γ-covers of X.

Let $\mathcal{A}$ and $\mathcal{B}$ be collections of subsets of X. The following two selection hypotheses have a long history for the case when $\mathcal{A}$ and $\mathcal{B}$ are collections of topologically significant subsets of a space. Early instances of these can be found in [6,16]; many papers since then have studied these selection hypotheses in one form or another.

$\mathsf{S}_1(\mathcal{A},\mathcal{B})$: For each sequence $(A_n\colon n\in\mathbb{N})$ of members of $\mathcal{A}$, there is a sequence $(b_n\colon n\in\mathbb{N})$ such that for each n $b_n\in A_n$, and $\{b_n\colon n\in\mathbb{N}\}\in\mathcal{B}$.

$\mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$: For each sequence $(A_n\colon n\in\mathbb{N})$ of members of $\mathcal{A}$, there is a sequence $(B_n\colon n\in\mathbb{N})$ such that each B_n is a finite subset of $\mathcal{A}_n$, and $\bigcup_{n\in\mathbb{N}}B_n\in\mathcal{B}$.

These selection hypotheses are monotonic in the second variable and antimonotonic in the first. Moreover, each has a naturally associated game:

In the game $\mathsf{G}_1(\mathcal{A},\mathcal{B})$ ONE chooses in the nth inning an element O_n of $\mathcal{A}$ and then TWO responds by choosing $T_n\in O_n$. They play an inning per natural number. A play $(O_1,T_1,\ldots,O_n,T_n,\ldots)$ is won by TWO if $\{T_n\colon n\in\mathbb{N}\}$ is a member of $\mathcal{B}$, otherwise, ONE wins. If ONE does not have a winning strategy in $\mathsf{G}_1(\mathcal{A},\mathcal{B})$, then $\mathsf{S}_1(\mathcal{A},\mathcal{B})$ holds. The converse is not always true; when it is true, the game is a powerful tool for studying the combinatorial properties of $\mathcal{A}$ and $\mathcal{B}$.

The game $\mathsf{G}_{fin}(\mathcal{A},\mathcal{B})$ is played similarly. In the nth inning ONE chooses an element O_n of $\mathcal{A}$ and TWO responds with a finite set $T_n\subseteq O_n$. A play $(O_1,T_1,\ldots,O_n,T_n,\ldots)$ is won by TWO if $\bigcup_{n\in\mathbb{N}}T_n$ is in $\mathcal{B}$, otherwise, ONE wins. As above: If ONE has no winning strategy in $\mathsf{G}_{fin}(\mathcal{A},\mathcal{B})$, then $\mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$ holds; when the converse is also true the game is a powerful tool for studying $\mathcal{A}$ and $\mathcal{B}$.

A third selection hypothesis, introduced by Hurewicz in [6], is as follows:

$\mathsf{U}_{fin}(\mathcal{A},\mathcal{B})$: For each sequence $(A_n\colon n\in\mathbb{N})$ of members of $\mathcal{A}$, there is a sequence $(B_n\colon n\in\mathbb{N})$ such that for each n B_n is a finite subset of A_n, and either $\bigcup B_n=X$ for all but finitely many n, or else $\{\bigcup B_n\colon n\in\mathbb{N}\}\setminus\{X\}\in\mathcal{B}$.

The three classes of open covers above are related: $\Gamma\subseteq\Omega\subseteq\mathcal{O}$. This and the properties of the selection hypotheses lead to a complicated diagram depicting how the classes defined this way interrelate. However, only a few of these classes are really distinct, as was shown in [8,18]. Fig. 1 (borrowed from [8]) contains the distinct ones among these classes (it is not known if the class $\mathsf{S}_{fin}(\Gamma,\Omega)$ is $\mathsf{U}_{fin}(\Gamma,\Omega)$, or if it contains $\mathsf{U}_{fin}(\Gamma,\Gamma)$). In this diagram, as in the ones to follow, an arrow denotes implication.

Now we consider the following covers of X. The symbol $\mathcal{B}$ denotes the family of all *countable* covers of X by *Borel sets*; call elements of $\mathcal{B}$ countable Borel covers of X. A countable Borel cover of X is said to be a *Borel ω-cover* of X if X is not a member of it but for each finite subset of X there is a member of the cover which contains the finite set. The symbol $\mathcal{B}_\Omega$ denotes the collection of Borel ω-covers of X. A countable Borel cover of X is said to be a *Borel γ-cover* of X if it is infinite and each element of X belongs to

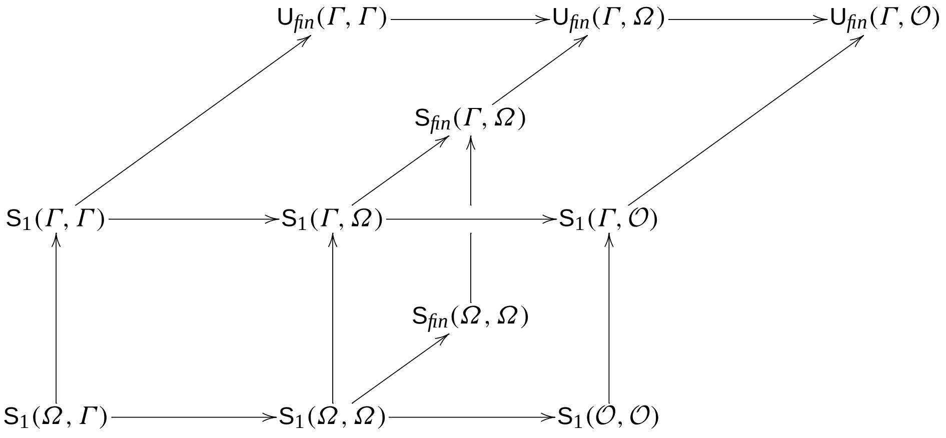

Fig. 1. The open covers diagram.

all but finitely many members of the cover. The symbol $\mathcal{B}_\Gamma$ denotes the collection of Borel γ-covers of X. It is evident that the following inclusions hold:

$$\mathcal{B}_\Gamma \subseteq \mathcal{B}_\Omega \subseteq \mathcal{B}; \qquad \Gamma \subseteq \mathcal{B}_\Gamma; \qquad \Omega \subseteq \mathcal{B}_\Omega \quad \text{and} \quad \mathcal{O} \subseteq \mathcal{B}.$$

On account of these inclusions and monotonicity properties of the selection principles we have: $\mathsf{S}_1(\mathcal{B}, \mathcal{B}) \subseteq \mathsf{S}_1(\mathcal{O}, \mathcal{O})$; $\mathsf{S}_{fin}(\mathcal{B}, \mathcal{B}) \subseteq \mathsf{S}_{fin}(\mathcal{O}, \mathcal{O})$; $\mathsf{U}_{fin}(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma) \subseteq \mathsf{U}_{fin}(\Gamma, \Gamma)$; $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma) \subseteq \mathsf{S}_1(\Omega, \Gamma)$; and so on.

The methods of [8,18] can be used to show that a diagram obtained from Fig. 1 by substituting all the open classes by their corresponding Borel versions summarizes all the interrelationships among these.

But there are big differences about what is provable in these two situations. For example, it has been shown in [8,20] that there always is an uncountable set of real numbers in the class $\mathsf{S}_1(\Gamma, \Gamma)$ and thus in $\mathsf{U}_{fin}(\Gamma, \Gamma)$. According to a result of [9] it is consistent that no uncountable set of real numbers has property $\mathsf{U}_{fin}(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$. Thus it is consistent that some of the classes which provably do not coincide in the open covers diagram, do coincide in the Borel covers diagram.

It must be checked which, if any, of the classes in the Borel covers diagram are provably equal; this is our first task.

2. Characterizations and equivalence of properties

In this section we give a number of characterizations for some of the Borel classes above. In particular, we get that some of the new properties are equivalent, even though their "open" versions are not provably equivalent.

The classes $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$, $\mathsf{S}_{fin}(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$, *and* $\mathsf{U}_{fin}(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$

Theorem 1. *For a set X of real numbers, the following are equivalent*:

(1) *X has property $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$.*
(2) *X has property $\mathsf{S}_{fin}(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$.*
(3) *X has property $\mathsf{U}_{fin}(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$.*
(4) *Every Borel image of X in ${}^{\mathbb{N}}\mathbb{N}$ is bounded.*

Proof. We must show that (3) ⇒ (4) and (4) ⇒ (1).

(3) ⇒ (4): This is a theorem of [2]. In short, note that the collections $\mathcal{U}_n = \{U^n_m\colon m \in \mathbb{N}\}$, where $U^n_m = \{f \in {}^{\mathbb{N}}\mathbb{N}\colon f(n) < m\}$, are open γ-covers of ${}^{\mathbb{N}}\mathbb{N}$. Assume that Ψ is a Borel function from X to ${}^{\mathbb{N}}\mathbb{N}$. Then the collections $\mathcal{B}_n = \{\Psi^{-1}[U^n_m]\colon m \in \mathbb{N}\}$ are in $\mathcal{B}_\Gamma$ for X. For all n, the sequence U^n_m is monotonically increasing with respect to m. Thus, we may use (1) instead of (3) to get a sequence $\Psi^{-1}[U^n_{m_n}] \in \mathcal{B}_n$ which is in $\mathcal{B}_\Gamma$ for X. Then the sequence m_n bounds $\Psi[X]$.

(4) ⇒ (1): Assume that $\mathcal{B}_n = \{B^n_k\colon k \in \mathbb{N}\}$, are in $\mathcal{B}_\Gamma$ for X. Define a function Ψ from X to ${}^{\mathbb{N}}\mathbb{N}$ so that for each x and n:

$$\Psi(x)(n) = \min\bigl\{k\colon (\forall m \geqslant k)\ x \in B^n_m\bigr\}.$$

Then Ψ is a Borel map, and so $\Psi[X]$ is bounded, say by the sequence m_n. Then the sequence $(B^n_{m_n}\colon n \in \mathbb{N})$ is in $\mathcal{B}_\Gamma$ for X. □

Corollary 2. *For a set X of real numbers, the following are equivalent*:

(1) *X has property $\mathsf{U}_{fin}(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$.*
(2) *Every Borel image of X has property $\mathsf{U}_{fin}(\Gamma, \Gamma)$.*

Proof. An old theorem of Hurewicz [7] asserts that X has property $\mathsf{U}_{fin}(\Gamma, \Gamma)$ if, and only if, every continuous image of X in ${}^{\mathbb{N}}\mathbb{N}$ is bounded. □

Theorem 3. *For a set X of real numbers the following are equivalent*:

(1) *X has property $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$.*
(2) *Each subset of X has property $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$.*
(3) *For each measure zero set N of real numbers, $X \cap N$ has property $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$.*

Proof. (1) ⇒ (2): This follows immediately from Theorem 1 and the fact that for sets of real numbers a function on a subspace which is Borel on the subspace, extends to one which is Borel on the whole space.

(3) ⇒ (1): Let X be as in (3), and let Ψ be a Borel function from X to ${}^{\mathbb{N}}\mathbb{N}$. We may assume that X is a subset of $[0, 1]$, the unit interval (as was shown in [20], the property $\mathsf{S}_1(\Gamma, \Gamma)$ is preserved by countable unions). Let Φ be a Borel function from $[0, 1]$ to ${}^{\mathbb{N}}\mathbb{N}$ whose restriction to X is Ψ.

By Lusin's Theorem choose for each n a closed subset C_n of the unit interval such that $\mu(C_n) \geqslant 1 - (\frac{1}{2})^n$, and such that Φ is continuous on C_n. Since C_n is compact, the image of Φ on C_n is bounded in ${}^{\mathbb{N}}\mathbb{N}$, say by h_n. The set $N = [0, 1] \setminus \bigcup_{n\in\mathbb{N}} C_n$ has measure zero, and so $X \cap N$ has property $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$. It follows that the image under Ψ of $X \cap N$ is bounded, say by h. Now let f be a function which eventually dominates each h_n, and h. Then f eventually dominates each member of $\Psi[X]$.

Since Ψ was an arbitrary Borel function from X to ${}^{\mathbb{N}}\mathbb{N}$, Theorem 1 implies that X has property $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$. □

Proposition 4. *If a set X of real numbers has the $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$ property, then it is a σ-set.*

Proof. We show that each G_δ-subset of X is an F_σ-subset. Thus, let A be a G_δ-subset of X, say $A = \bigcap_{n\in\mathbb{N}} U_n$ where for all n $U_n \supseteq U_{n+1}$ are open subsets of X. Since X is metrizable, each U_n is an F_σ-set. Write, for each n,

$$U_n = \bigcup_{k\in\mathbb{N}} C_k^n,$$

where for all m, $C_m^n \subseteq C_{m+1}^n$ are closed sets. Then for each n $\mathcal{B}_n := (C_m^n\colon m \in \mathbb{N})$ is in $\mathcal{B}_\Gamma$ for A. Since $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$ is hereditary, A has this property and we find for each n an m_n such that $(C_{m_n}^n\colon n \in \mathbb{N})$ is a γ-cover of A. For each k define

$$F_k := \bigcap_{n\geqslant k} C_{m_n}^n.$$

Then each F_k is closed and $A = \bigcup_{k\in\mathbb{N}} F_k$. □

According to Besicovitch [3] a set X of real numbers is *concentrated on* a set Q if for every open set U containing Q, the set $X \setminus U$ is countable.

Corollary 5. *If an uncountable set of real numbers is concentrated on a countable subset of itself, then it does not have property $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$.*

The classes $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B})$, $\mathsf{S}_{fin}(\mathcal{B}_\Gamma, \mathcal{B})$, and $\mathsf{U}_{fin}(\mathcal{B}_\Gamma, \mathcal{B})$

Theorem 6. *The following are equivalent*:

(1) *X has property $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B})$.*
(2) *X has property $\mathsf{S}_{fin}(\mathcal{B}_\Gamma, \mathcal{B})$.*
(3) *X has property $\mathsf{U}_{fin}(\mathcal{B}_\Gamma, \mathcal{B})$.*
(4) *No Borel image of X in ${}^{\mathbb{N}}\mathbb{N}$ is dominating.*

Proof. The proof is similar to that of Theorem 1.

(3) ⇒ (4): Given a Borel function Ψ from X to ${}^{\mathbb{N}}\mathbb{N}$, define $\mathcal{B}_n$ as in the proof of Theorem 1. Let A_k, $k \in \mathbb{N}$, be a partition of $\mathbb{N}$ into infinitely many infinite sets. From each sequence of covers $\mathcal{B}_n$, $n \in A_k$, we can extract by (1) a cover $B_{m_n}^n$ $(n \in A_k)$. Taken together, $B_{m_n}^n$ $(n \in \mathbb{N})$ form a *large* cover of X. Recalling that $B_{m_n}^n = \Psi^{-1}[U_{m_n}^n]$, we get that the sequence m_n witnesses that $\Psi[X]$ is not dominating.

(4) ⇒ (1): With notation as in the proof of Theorem 1, we get that if m_n witnesses that $\Psi[X]$ is not dominating, then $(B_{m_n}^n\colon n \in \mathbb{N})$ is a (large) cover of X. □

Corollary 7. *For a set X of real numbers, the following are equivalent*:

(1) *X has property $\mathsf{U}_{fin}(\mathcal{B}_\Gamma, \mathcal{B})$.*

(2) *Every Borel image of X in ${}^{\mathbb{N}}\mathbb{N}$ has property $\mathsf{U}_{fin}(\Gamma, \mathcal{O})$.*

Proof. A Theorem of Hurewicz [7] asserts that a set X is $\mathsf{U}_{fin}(\Gamma, \mathcal{O})$ if, and only if, every continuous image of X in ${}^{\mathbb{N}}\mathbb{N}$ is not dominating. □

The classes $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Omega)$, $\mathsf{S}_{fin}(\mathcal{B}_\Gamma, \mathcal{B}_\Omega)$, and $\mathsf{U}_{fin}(\mathcal{B}_\Gamma, \mathcal{B}_\Omega)$

The characterization of these classes is best stated in the language of filters. Let $\mathcal{F}$ be a filter over $\mathbb{N}$. An equivalence relation $\sim_\mathcal{F}$ is defined on ${}^{\mathbb{N}}\mathbb{N}$ by

$$f \sim_\mathcal{F} \iff \{n\colon f(n) = g(n)\} \in \mathcal{F}.$$

The equivalence class of f is denoted $[f]_\mathcal{F}$, and the set of these equivalence classes is denoted ${}^{\mathbb{N}}\mathbb{N}/\mathcal{F}$. Using this terminology, $[f]_\mathcal{F} < [g]_\mathcal{F}$ means

$$\{n\colon f(n) < g(n)\} \in \mathcal{F}.$$

The following combinatorial notion and the accompanying Lemma 8 will be to used get a technical version of the filter-based characterization.

For a family $Y \subset {}^{\mathbb{N}}\mathbb{N}$, define maxfin$(Y)$ to be the set of elements f in ${}^{\mathbb{N}}\mathbb{N}$ for which there is a finite set $F \subset Y$ such that

$$f(n) = \max\{h(n)\colon h \in F\}$$

for all n.

Lemma 8. *Let $Y \subset {}^{\mathbb{N}}\mathbb{N}$ be such that for each n the set $\{h(n)\colon h \in Y\}$ is infinite. Then the following are equivalent*:

(1) maxfin(Y) *is not a dominating family.*

(2) *There is a non-principal filter $\mathcal{F}$ on $\mathbb{N}$ such that the subset $\{[f]_\mathcal{F}\colon f \in Y\}$ of the reduced product ${}^{\mathbb{N}}\mathbb{N}/\mathcal{F}$ is bounded.*

Proof. (1) ⇒ (2): Choose an $h \in {}^{\mathbb{N}}\mathbb{N}$ which is strictly increasing, and which is not eventually dominated by any element of maxfin(Y). For any finite subset F of Y, put $f_F(n) = \max\{g(n)\colon g \in F\}$ for each n, and then define the set

$$A_F = \{n \in \mathbb{N}\colon f_F(n) \leqslant h(n)\}.$$

Observe that for finite subsets F and G of Y, if $F \subset G$, then $A_G \subseteq A_F$. Thus, the family $\{A_F\colon F \subset Y \text{ finite}\}$ is a basis for a filter $\mathcal{F}$ on $\mathbb{N}$. By the hypothesis on Y this filter is non-principal. It is evident that $[h]/\mathcal{F}$ is an upper bound for $Y/\mathcal{F}$.

(2) ⇒ (1): Let $\mathcal{F}$ be a non-principal filter on $\mathbb{N}$ such that $Y/\mathcal{F}$ is bounded, and choose a function h in ${}^{\mathbb{N}}\mathbb{N}$ such that for each $f \in Y$ we have $[f]_\mathcal{F} < [h]_\mathcal{F}$. Then for each $f \in Y$ the set $\{n\colon f(n) \leqslant h(n)\}$ is in $\mathcal{F}$ and is infinite (since $\mathcal{F}$ is non-principal). Since $\mathcal{F}$ has the finite intersection property it follows that for each finite subset F of Y the set

$S_F = \{n\colon (\forall f \in F)(f(n) \leqslant h(n))\}$ is in $\mathcal{F}$. But then h is not eventually dominated by any element of $\mathrm{maxfin}(Y)$. $\square$

Theorem 9. *For a set X of real numbers, the following are equivalent*:

(1) *X has property $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Omega)$.*
(2) *X has property $\mathsf{S}_{fin}(\mathcal{B}_\Gamma, \mathcal{B}_\Omega)$.*
(3) *X has property $\mathsf{U}_{fin}(\mathcal{B}_\Gamma, \mathcal{B}_\Omega)$.*
(4) *For each Borel function Ψ from X to ${}^{\mathbb{N}}\mathbb{N}$, $\mathrm{maxfin}(\Psi[X])$ is not a dominating family.*
(5) *For each Borel function Ψ from X to ${}^{\mathbb{N}}\mathbb{N}$, either there is a principal filter $\mathcal{G}$ for which $\Psi[X]/\mathcal{G}$ is finite, or else there is a non-principal filter $\mathcal{F}$ on $\mathbb{N}$ such that the subset $\Psi[X]/\mathcal{F}$ of the reduced product ${}^{\mathbb{N}}\mathbb{N}/\mathcal{F}$ is bounded.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are immediate. We will first show that (3) $\Rightarrow$ (4) $\Rightarrow$ (1), and then use Lemma 8 to establish the equivalence of (4) and (5). As in the previous proof, for any finite subset F of Y, put $f_F(n) = \max\{g(n)\colon g \in F\}$ for each n.

(3) $\Rightarrow$ (4): Let $Y = \Psi[X]$. By the upcoming Theorem 48, Y has property $\mathsf{U}_{fin}(\mathcal{B}_\Gamma, \mathcal{B}_\Omega)$. For each n and each k, define $U^n_k := \{f\colon f(n) < k\}$; then set $\mathcal{U}_n := \{U^n_k\colon k \in \mathbb{N}\}$. Each $\mathcal{U}_n$ is a γ-cover of ${}^{\mathbb{N}}\mathbb{N}$ since for each n and for $k < j$ we have $U^n_k \subset U^n_j$. Let A_k, $k \in \mathbb{N}$, be a partition of $\mathbb{N}$ into infinitely many infinite sets. From each sequence of γ-covers $\mathcal{U}_n$, $n \in A_k$, we can use the $\mathsf{U}_{fin}(\mathcal{B}_\Gamma, \mathcal{B}_\Omega)$ property of Y to extract an ω-cover $(U^n_{m_n}\colon n \in A_k)$. Then for each finite $F \subseteq X$, we have for each $k \in \mathbb{N}$ an $n \in A_k$ such that $\Psi[F] \subseteq U^n_{m_n}$, i.e., $f_{\Psi[F]}(n) \leqslant m_n$. Thus, the sequence m_n witnesses that $\mathrm{maxfin}(\Psi[X])$ is not a dominating family.

(4) $\Rightarrow$ (1): Assume that $\mathcal{B}_n = \{B^n_m\colon m \in \mathbb{N}\}$ are in $\mathcal{B}_\Gamma$ for X. Define a Borel function Ψ from X to ${}^{\mathbb{N}}\mathbb{N}$ so that for each x and n:

$$\Psi(x)(n) = \min\bigl\{k\colon (\forall m \geqslant k)\ x \in B^n_m\bigr\}.$$

Note that if $F \subseteq X$ is finite, then for all $m \geqslant f_{\Psi[F]}(n)$, $F \subseteq B^n_m$. Let the sequence m_n witness that $\mathrm{maxfin}(\Psi[X])$ is not dominating. Then for all finite $F \subseteq X$, $F \subseteq B^n_{m_n}$ infinitely many times. That is, $(B^n_{m_n}\colon n \in \mathbb{N})$ is in $\mathcal{B}_\Omega$ for X.

(4) $\Rightarrow$ (5): There are two cases to consider:

Case 1. There is an n such that $\{\Psi(x)(n)\colon x \in X\}$ is finite. Then the principal filter generated by $\{n\}$ does the job.

Case 2. For each n the set $\{\Psi(x)(n)\colon x \in X\}$ is infinite. Apply Lemma 8.

(5) $\Rightarrow$ (4): Again consider two cases, and apply Lemma 8. $\square$

Remark 10. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (5) in Theorem 9 can be proved for the open version of these properties in a similar manner. The implication (3) $\Rightarrow$ (2) in the open case is counter-exampled by the Cantor set [8]. We do not know whether the open version of (4) $\Rightarrow$ (3) is true.

This gives the following characterization of $\mathfrak{d}$:

Corollary 11. *For an infinite cardinal number κ the following are equivalent*:
(1) $\kappa < \mathfrak{d}$.
(2) *For each subset X of ${}^{\mathbb{N}}\mathbb{N}$ of cardinality at most κ, there is a non-principal filter $\mathcal{F}$ on $\mathbb{N}$ such that in the reduced product ${}^{\mathbb{N}}\mathbb{N}/\mathcal{F}$ the set $X/\mathcal{F}$ is bounded.*

Proof. By Theorem 9, (2) implies (1). To see that (1) implies (2), consider an infinite $\kappa < \mathfrak{d}$ and a subset X of ${}^{\mathbb{N}}\mathbb{N}$ which is of cardinality κ. We may assume that $y \in X$ whenever there is an $x \in X$ such that y differs from x in only finitely many points. Then $\text{maxfin}(X)$ also has cardinality κ. By Lemma 8 there exists non-principal filter $\mathcal{F}$ on $\mathbb{N}$ such that $X/\mathcal{F}$ is bounded in ${}^{\mathbb{N}}\mathbb{N}/\mathcal{F}$. □

Theorem 12. *For a set X of real numbers, the following are equivalent*:
(1) *X has property $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Omega)$.*
(2) *For each Borel mapping Ψ of X into ${}^{\mathbb{N}}\mathbb{Z}$ there is a non-principal filter $\mathcal{F}$ such that the subring generated by $\Psi[X]/\mathcal{F}$ in the reduced power ${}^{\mathbb{N}}\mathbb{Z}/\mathcal{F}$ is bounded below and above.*

Proof. That (2) implies (1) is proved as before. Regarding (1) implies (2): It is evident that if we confine attention to the ring ${}^{\mathbb{N}}\mathbb{Z}$ with pointwise operations, then a subset Y of it would have property $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Omega)$ if, and only if, there is a non-principal filter $\mathcal{F}$ such that $Y/\mathcal{F}$ is bounded from below and from above in ${}^{\mathbb{N}}\mathbb{Z}$. Let g be an element of ${}^{\mathbb{N}}\mathbb{N}$ such that $\Psi[X]/\mathcal{F}$ is bounded by $[g]$. Since the set $\{n \cdot g\colon n \in \mathbb{Z}\} \cup \{g^n\colon n \in \mathbb{N}\}$ is countable, we find a single h such that for all n h eventually dominates each of $n \cdot g$ and g^n. But then in the reduced power ${}^{\mathbb{N}}\mathbb{Z}/\mathcal{F}$ the element $[-h]$ is a lower bound and the element $[h]$ is an upper bound for the ring generated by $\Psi[X]/\mathcal{F}$. □

The class $\mathsf{S}_1(\mathcal{B}, \mathcal{B})$

The classes $\mathsf{S}_1(\mathcal{B}, \mathcal{B})$ and $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$ appear to be each other's "duals".

Theorem 13. *For a set X of real numbers, the following are equivalent*:
(1) *X has property $\mathsf{S}_1(\mathcal{B}, \mathcal{B})$.*
(2) *Every subset of X has property $\mathsf{S}_1(\mathcal{B}, \mathcal{B})$.*
(3) *For each meager set $M \subset \mathbb{R}$, $X \cap M$ has property $\mathsf{S}_1(\mathcal{B}, \mathcal{B})$.*

Proof. We must show that (1) implies (2), and that (3) implies (1).

(1) ⇒ (2): This is immediate from the equivalence of $\mathsf{S}_1(\mathcal{B}, \mathcal{B})$ with another notion (see Section 5). However, we give a direct proof.

Let M be a subset of X, and assume that X has property $\mathsf{S}_1(\mathcal{B}, \mathcal{B})$. For each n let $\mathcal{U}_n$ be a countable cover of M by Borel subsets of M. For each $U \in \mathcal{U}_n$ let B_U be a Borel subset of X such that $U = M \cap B_U$. Then $X_n := \bigcup\{B_U\colon U \in \mathcal{U}_n\}$ is a Borel subset of X since $\mathcal{U}_n$ is countable. In turn, $\widetilde{X} := \bigcap_{n\in\mathbb{N}} X_n$ is a Borel subset of X.

For each n let $\widetilde{\mathcal{U}}_n$ be $\{B_U\colon U \in \mathcal{U}_n\} \cup \{X\setminus\widetilde{X}\}$. Then $(\mathcal{U}_n\colon n \in \mathbb{N})$ is a sequence of countable Borel covers of X. For each n choose a $V_n \in \widetilde{\mathcal{U}}_n$ such that $\{V_n\colon n \in \mathbb{N}\}$ is a cover of X. For each n for which $V_n \neq \widetilde{X}$, choose $U_n \in \mathcal{U}_n$ such that $V_n = B_{U_n}$; for other values of n let U_n be an arbitrary element of $\mathcal{U}_n$. Then $(U_n\colon n \in \mathbb{N})$ covers M.

(3) $\Rightarrow$ (1): Let $(\mathcal{B}_n\colon n \in \mathbb{N})$ be a sequence of countable Borel covers of X; enumerate each $\mathcal{B}_n$ as $(B^n_m\colon m \in \mathbb{N})$.

Since Borel sets have the property of Baire we may choose for each B^n_m an open set O^n_m and a meager set M^n_m such that

$$B^n_m = \left(O^n_m \setminus M^n_m\right) \cup \left(M^n_m \setminus O^n_m\right).$$

Then $A := \bigcup_{m,n\in\mathbb{N}} M^n_m$ is a meager set and so $A \cap X$ has property $\mathsf{S}_1(\mathcal{B},\mathcal{B})$. For each n such that $n \bmod 3 = 0$, choose a $B^n_{m_n} \in \mathcal{B}_n$ such that $A \cap X$ is covered by these.

For each n, $\mathcal{O}_n$, defined to be $\{O^n_m\colon m \in \mathbb{N}\}$, is an open cover of $X\setminus A$. Let Q be a countable dense subset of $X\setminus A$, and choose for each n with $n \bmod 3 = 1$ an $O^n_{m_n}$ such that these cover Q.

Then the set $B := X\setminus\bigcup\{O^n_{m_n}\colon n \bmod 3 = 1\}$ is meager, and so has property $\mathsf{S}_1(\mathcal{B},\mathcal{B})$. For each n such that $n \bmod 3 = 2$, choose an $O^n_{m_n} \in \mathcal{O}_n$ such that these $O^n_{m_n}$'s cover B.

Then the sequence $(B^n_{m_n}\colon n \in \mathbb{N})$ covers X. □

Combining of a result from [1,11] with one from [2] yields the following characterization:

Theorem 14. *For a set X of real numbers, the following are equivalent*:

(1) *X has property $\mathsf{S}_1(\mathcal{B},\mathcal{B})$.*

(2) *Each Borel image of X has the Rothberger property $\mathsf{S}_1(\mathcal{O},\mathcal{O})$.*

The selection property $\mathsf{S}_1(\mathcal{O},\mathcal{O})$ manifests itself in several other interesting ways: these analogues hold also for $\mathsf{S}_1(\mathcal{B},\mathcal{B})$.

Theorem 15. *For a set X of real numbers, the following are equivalent*:

(1) $\mathsf{S}_1(\mathcal{B},\mathcal{B})$ *holds.*

(2) *ONE has no winning strategy in the game $\mathsf{G}_1(\mathcal{B},\mathcal{B})$.*

Proof. We must show that (1) $\Rightarrow$ (2): Let F be a strategy for ONE of the game $\mathsf{G}_1(\mathcal{B},\mathcal{B})$. Using it, define the following array of Borel subsets of X: First, enumerate $F(\emptyset)$, ONE's first move, as $(U_n\colon n \in \mathbb{N})$. For each response U_{n_1} by TWO, enumerate ONE's corresponding move $F(U_{n_1})$ as $(U_{n_1,n}\colon n \in N)$. If TWO responds now with U_{n_1,n_2}, enumerate ONE's corresponding move $F(U_{n_1}, U_{n_1,n_2})$ as $(U_{n_1,n_2,n}\colon n \in \mathbb{N})$, and so on.

The family $(U_\tau\colon \tau \in {}^{<\omega}\mathbb{N})$ has the property that for each τ the set $\{U_{\tau\frown n}\colon n \in \mathbb{N}\}$ is a cover of X by Borel subsets of X. Moreover, for each function f in ${}^{\mathbb{N}}\mathbb{N}$, the sequence

$$F(\emptyset), U_{f(1)}, F(U_{f(1)}), U_{f(1),f(2)}, F(U_{f(1)}, U_{f(1),f(2)}), \ldots$$

is a play of $\mathsf{G}_1(\mathcal{B},\mathcal{B})$ during which ONE used the strategy F. For each such f, define $S_f := \bigcup_{n\in\mathbb{N}} U_{f(1),\ldots,f(n)}$. (Thus, S_f is the set of points covered by TWO during a play coded by f.) We must show that for some such f we have $S_f = X$.

Define the subset D of $X \times {}^{\mathbb{N}}\mathbb{N}$ by

$$D := \{(x, f): x \notin S_f\}.$$

Then D is a Borel subset of $X \times {}^{\mathbb{N}}\mathbb{N}$. Moreover, for each $x \in X$ the set $D_x = \{f: x \notin S_f\}$ is nowhere dense. (To see this, let $[(n_1,\ldots,n_k)]$ be a basic open subset of ${}^{\mathbb{N}}\mathbb{N}$. Since $\{U_{n_1,\ldots n_k,m}: m \in \mathbb{N}\}$ is a cover of X there is an n_{k+1} with $x \in U_{n_1,\ldots,n_k,n_{k+1}}$. But then $[(n_1,\ldots,n_k,n_{k+1})] \cap D_x = \emptyset$.) Now recall from [2] that as X has property $\mathsf{S}_1(\mathcal{B},\mathcal{B})$ it follows that ${}^{\mathbb{N}}\mathbb{N} \neq \bigcup_{x\in X} D_x$ (see Section 5). Let f be a function not in $\bigcup_{x\in X} D_x$. Then $X = S_f$, and we have defeated ONE's strategy F. □

We next show that $\mathsf{S}_1(\mathcal{B},\mathcal{B})$ is a Ramsey-theoretic property. First observe:

Lemma 16. *For a set X of real numbers, the following are equivalent*:
(1) *X has property $\mathsf{S}_1(\mathcal{B},\mathcal{B})$.*
(2) *X has property $\mathsf{S}_1(\mathcal{B}_\Omega,\mathcal{B})$.*

Proof. The proof for this is like that of Theorem 17 of [18]. □

The virtue of $\mathcal{B}_\Omega$ for Ramsey-theoretic purposes is that if $\mathcal{U}$ is a member of $\mathcal{B}_\Omega$, and if it is partitioned into finitely many pieces, then at least one of these pieces is a member of $\mathcal{B}_\Omega$. This statement is denoted by the abbreviation:

$$\text{for each } k,\ \mathcal{B}_\Omega \to (\mathcal{B}_\Omega)^1_k.$$

This is a special case of the more general notation

$$\text{for all } n \text{ and } k\ \mathcal{A} \to (\mathcal{C})^n_k,$$

which denotes the statement:

For each n and k, for each $A \in \mathcal{A}$, and for each $g : [A]^n \to \{1,\ldots,k\}$, there is a $C \subseteq A$ such that $C \in \mathcal{C}$ and g is constant on $[C]^n$.

Theorem 17. *For a set X of real numbers, the following are equivalent*:
(1) *X has property $\mathsf{S}_1(\mathcal{B},\mathcal{B})$.*
(2) *X has the property that for all k, $\mathcal{B}_\Omega \to (\mathcal{B})^2_k$.*

Proof. The proof of this is like that of Theorem 4 of [19]. □

The class $\mathsf{S}_1(\mathcal{B}_\Omega,\mathcal{B}_\Omega)$

It is evident that unions of countably many spaces, each having property $\mathsf{S}_1(\mathcal{B},\mathcal{B})$, have property $\mathsf{S}_1(\mathcal{B},\mathcal{B})$.

Theorem 18. *If all finite powers of X have property $\mathsf{S}_1(\mathcal{B},\mathcal{B})$, then X has property $\mathsf{S}_1(\mathcal{B},\mathcal{B})$.*

Proof. The proof of this is a minor variation on the proof of (2) ⇒ (1) of Theorem 3.9 of [8]. □

Problem 19. Is it true that if X has property $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$, then it has property $\mathsf{S}_1(\mathcal{B}, \mathcal{B})$ in all finite powers?

The class $\mathsf{S}_{fin}(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$

It is evident that unions of countably many spaces, each having property $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B})$, have property $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B})$.

Theorem 20. *If all finite powers of X have property $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B})$, then X has property $S_{fin}(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$.*

Proof. Let $Y = \sum_{k\in\mathbb{N}} X^k$. Then by the assumption, Y has property $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B})$. Assume that $\mathcal{B}_n = \{B^n_m\colon m \in \mathbb{N}\}$ are in $\mathcal{B}_\Omega$ for X. Define a Borel function Ψ from Y to ${}^{\mathbb{N}}\mathbb{N}$ so that for all k, $x_0, \ldots, x_{k-1} \in X$, and n:

$$\Psi(x_0, \ldots, x_{k-1})(n) = \min\bigl\{k\colon (\forall m \geqslant k)\ x_0, \ldots, x_{k-1} \in B^n_m\bigr\}.$$

By Theorem 6, the image of Y under Ψ is not dominating. Choose a sequence m_n witnessing this. For each n, set $\mathcal{W}_n := \{B^n_j\colon j \leqslant m_n\}$. Then each $\mathcal{W}_n$ is finite, and $\bigcup_{n\in\mathbb{N}} \mathcal{W}_n$ is in $\mathcal{B}_\Omega$ for X. □

Problem 21. Is it true that if X has property $\mathsf{S}_{fin}(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$, then it has property $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B})$ in all finite powers?

The class $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$

A standard diagonalization trick gives the following.

Lemma 22. *The following are equivalent*:

(1) *X has property $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$.*
(2) *Every Borel ω-cover of X contains a γ-cover of X.*

Proof. The proof of this is like that of the corresponding result in [5]. □

For the next characterization we need some terminology and notation. For $a, b \subseteq \mathbb{N}$, $a \subseteq^* b$ if $a\setminus b$ is finite. Let $[\mathbb{N}]^\infty$ denote the set of infinite sets of natural numbers. $X \subseteq [\mathbb{N}]^\infty$ is *centered* if every finite $F \subseteq X$ has an infinite intersection. $a \in [\mathbb{N}]^\infty$ is a pseudo-intersection of X if for all $b \in X$, $a \subseteq^* b$. $X \subseteq [\mathbb{N}]^\infty$ is a *power* if it is centered, but has no pseudo-intersection.

Every countable large Borel cover $\mathcal{U} = \{U_n\colon n \in \mathbb{N}\}$ of X is associated with a Borel function $h_{\mathcal{U}}: X \to [\mathbb{N}]^\infty$, defined by $h_{\mathcal{U}}(x) = \{n\colon x \in U_n\}$.

Lemma 23 [22]. *Assume that $\mathcal{U}$ is a cover of X. Then*:

(1) *$\mathcal{U}$ is an ω-cover of X if, and only if, $h_{\mathcal{U}}[X]$ is centered.*

(2) *$\mathcal{U}$ contains a γ-cover of X if, and only if, $h_{\mathcal{U}}[X]$ has a pseudo-intersection.*

Lemma 24. *The following are equivalent*:

(1) *Every Borel ω-cover of X contains a γ-cover of X.*

(2) *No Borel image of X in $[\mathbb{N}]^{\infty}$ is a power.*

Proof. (2) $\Rightarrow$ (1): Follows from the preceding lemma.

(1) $\Rightarrow$ (2): Assume that $f : X \to [\mathbb{N}]^{\infty}$ is Borel, such that $f[X]$ is centered. Let O_n, $n \in \mathbb{N}$, denote the clopen sets $\{a\colon n \in a\}$. As $f[X]$ is centered, $\{O_n\colon n \in \mathbb{N}\}$ is an ω-cover of $f[X]$. Thus, $\mathcal{U} = \{f^{-1}[O_n]\colon n \in \mathbb{N}\}$ is a Borel ω-cover of X. But $f = h_{\mathcal{U}}$, so we can apply the preceding lemma. □

We thus get the following characterization of $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$.

Theorem 25. *For a set X of real numbers, the following are equivalent*:

(1) *X has property $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$.*

(2) *No Borel image of X in $[\mathbb{N}]^{\infty}$ is a power.*

Corollary 26. *For a set X of real numbers, the following are equivalent*:

(1) *X has property $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$.*

(2) *Every continuous image of X has property $\mathsf{S}_1(\Omega, \Gamma)$.*

Proof. This follows from a Theorem of Recław [14], asserting that X has property $\mathsf{S}_1(\Omega, \Gamma)$ if, and only if, no continuous image of X in $[\mathbb{N}]^{\infty}$ is a power. □

Fig. 2 summarizes the equivalences proved in this section.

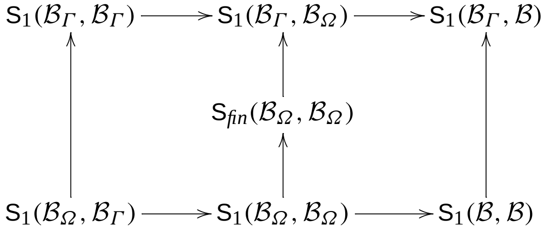

Fig. 2. The surviving Borel classes.

3. Does Fig. 2 contain all the provable information about these classes?

We now consider the question whether we have proved all the equalities that can be proved for these Borel cover classes. It will be seen that the answer is "Yes"; here is a brief outline of how this follows from the results of the present section:

(1) According to Corollary 41 it is consistent that there is a set of real numbers with property $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$, but not property $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$. This means hat none of the arrows from the left of Fig. 2 to the middle is reversible.

(2) According to Theorem 32 it is consistent that there is a set of real numbers in $\mathsf{S}_1(\mathcal{B}, \mathcal{B})$ which is not in $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Omega)$. This means that none of the arrows from the middle of Fig. 2 to the right is reversible.

(3) According to Theorem 43 it is consistent that there is a set of real numbers in $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$ and not in either of $\mathsf{S}_{fin}(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$ or $\mathsf{S}_1(\mathcal{B}, \mathcal{B})$. This implies that none of the arrows from the bottom of Fig. 2 which terminates at the top is reversible.

(4) According to Theorem 27 the minimal cardinality of a set of real numbers not having property $\mathsf{S}_{fin}(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$ is $\mathfrak{d}$, while the minimal cardinality of a set of real numbers not having property $\mathsf{S}_1(\mathcal{B}, \mathcal{B})$ is $\mathsf{cov}(\mathcal{M})$. Since it is consistent that $\mathsf{cov}(\mathcal{M}) < \mathfrak{d}$, it is consistent that none of the arrows starting at the bottom row of Fig. 2 is reversible.

For a collection $\mathcal{J}$ of separable metrizable spaces, let $\mathsf{non}(\mathcal{J})$ denote the minimal cardinality for a separable metrizable space which is not a member of $\mathcal{J}$.

We also call $\mathsf{non}(\mathcal{J})$ *the critical cardinality* for the class $\mathcal{J}$.

Theorem 27.

(1) $\mathsf{non}(\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)) = \mathfrak{p}$.

(2) $\mathsf{non}(\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)) = \mathfrak{b}$.

(3) $\mathsf{non}(\mathsf{S}_{fin}(\mathcal{B}_\Omega, \mathcal{B}_\Omega)) = \mathsf{non}(\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Omega)) = \mathsf{non}(\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B})) = \mathfrak{d}$.

(4) $\mathsf{non}(\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Omega)) = \mathsf{non}(\mathsf{S}_1(\mathcal{B}, \mathcal{B})) = \mathsf{cov}(\mathcal{M})$.

Proof. (1) and (2) follow from Theorems 25 and 1, respectively. (3) follows from Theorems 6 and 20.

For (4), we need the following lemma.

Lemma 28. *Let $\mathcal{J}$, $\mathcal{S}$ be collections of separable metrizable spaces, such that $X \in \mathcal{J}$ if, and only if, every Borel image of X is in $\mathcal{S}$. Then* $\mathsf{non}(\mathcal{J}) = \mathsf{non}(\mathcal{S})$.

Proof. Since $\mathcal{J} \subseteq \mathcal{S}$, we have $\mathsf{non}(\mathcal{J}) \leqslant \mathsf{non}(\mathcal{S})$. Now, let X witness $\mathsf{non}(\mathcal{J})$. Then there is a Borel function Ψ on X such that $\Psi[X] \notin \mathcal{S}$. As the cardinality of $\Psi[X]$ cannot be greater than the cardinality of X, we get that $\mathsf{non}(\mathcal{J}) \geqslant \mathsf{non}(\mathcal{S})$. □

Now, it is well known that $\mathsf{non}(\mathsf{S}_1(\mathcal{O}, \mathcal{O})) = \mathsf{cov}(\mathcal{M})$. Therefore, by Theorem 14, $\mathsf{non}(\mathsf{S}_1(\mathcal{B}, \mathcal{B})) = \mathsf{cov}(\mathcal{M})$. Thus, by Theorem 18, $\mathsf{non}(\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Omega)) = \mathsf{cov}(\mathcal{M})$ as well. □

Since it is consistent that $\mathfrak{p} < \mathsf{cov}(\mathcal{M})$, it is consistent that $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$ is not equal to $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$. Similarly the consistency of the inequality $\mathfrak{p} < \mathfrak{b}$ implies that $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$ is not provably equal to $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$.

It is consistent that $\mathfrak{b} < \mathsf{cov}(\mathcal{M})$, and so it is consistent that there is a set of real numbers which has property $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$ but which does not have property $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$.

Since it is consistent that $\mathsf{cov}(\mathcal{M}) < \mathfrak{d}$, it is also not provable that $\mathsf{S}_{fin}(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$ is equal to either of $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$ or $\mathsf{S}_1(\mathcal{B}, \mathcal{B})$.

What the cardinality results do not settle is whether $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$ provably coincides with $\mathsf{S}_1(\mathcal{B}, \mathcal{B})$, or whether any of the three classes associated with the cardinal number $\mathfrak{d}$ coincides with another. They also do not give any indication as to what the interrelationships among two classes might be when their critical cardinals are equal. To treat these questions we now consider specific examples which could be constructed on the basis of a variety of axioms which are consistent. All of the axioms that we use have the form of equality between certain well known cardinal invariants. Readers who are not familiar with this type of axiom may assume the Continuum Hypothesis instead (in this case, all of the cardinal invariants become equal to $\aleph_1$).

Special elements of $\mathsf{S}_1(\mathcal{B}, \mathcal{B})$

A set of real numbers is a *Lusin* set if it is uncountable, but its intersection with each meager set of real numbers is countable. More generally, for a cardinal κ an uncountable set $X \subseteq \mathbb{R}$ is said to be a κ-*Lusin* set if it has cardinality at least κ, but its intersection with each meager set is less than κ. It is evident that the smaller the value of κ, the harder it is for a set to be a κ-Lusin set. Towards the goal of using as weak hypotheses as possible, this means that we would be interested in κ-Lusin sets for as large a value of κ that would allow the conclusion we are aiming at. We now work in the group ${}^{\mathbb{N}}\mathbb{Z}$ (which *topologically* is homeomorphic to the set of irrational numbers), and construct from weak axioms special elements of $\mathsf{S}_1(\mathcal{B}, \mathcal{B})$.

Lemma 29. *If* $\mathsf{cov}(\mathcal{M}) = \mathsf{cof}(\mathcal{M})$, *and if* Y *is a subset of* ${}^{\mathbb{N}}\mathbb{Z}$ *of cardinality at most* $\mathsf{cof}(\mathcal{M})$, *then there is a* $\mathsf{cov}(\mathcal{M})$-*Lusin set* $L \subset {}^{\mathbb{N}}\mathbb{Z}$ *such that* $Y \subseteq L + L$.

Proof. Let $\{y_\alpha\colon \alpha < \mathsf{cov}(\mathcal{M})\}$ enumerate Y. Let $\{M_\alpha\colon \alpha < \mathsf{cov}(\mathcal{M})\}$ enumerate a cofinal family of meager sets, and construct L recursively as follows: At stage α set $X_\alpha = \{a_i\colon i < \alpha\} \bigcup \{b_i\colon i < \alpha\} \cup \bigcup_{i<\alpha} M_i$. Then $(y_\alpha - X_\alpha) \cup X_\alpha$ is a union of fewer than $\mathsf{cov}(\mathcal{M})$ meager sets. Choose an $a_\alpha \in {}^{\mathbb{N}}\mathbb{Z} \setminus ((y_\alpha - X_\alpha) \cup X_\alpha)$. Evidently, $a_\alpha \in (y_\alpha - {}^{\mathbb{N}}\mathbb{Z} \setminus X_\alpha) \cap ({}^{\mathbb{N}}\mathbb{Z} \setminus X_\alpha)$. Thus, choose $b_\alpha \in {}^{\mathbb{N}}\mathbb{Z} \setminus X_\alpha$ for which $y_\alpha - b_\alpha = a_\alpha$. Then we have $y_\alpha = a_\alpha + b_\alpha$.

Finally, set $L = \{a_\alpha\colon \alpha < \mathsf{cov}(\mathcal{M})\} \cup \{b_\alpha\colon \alpha < \mathsf{cov}(\mathcal{M})\}$. Then L is a $\mathsf{cov}(\mathcal{M})$-Lusin set and $L + L \supseteq Y$. $\square$

The next result is used to show that for κ small enough, κ-Lusin sets are in $\mathsf{S}_1(\mathcal{B}, \mathcal{B})$.

Corollary 30. *If* X *is a* $\mathsf{cov}(\mathcal{M})$-*Lusin set, then it has property* $\mathsf{S}_1(\mathcal{B}, \mathcal{B})$.

Proof. If M is any meager set, then $M \cap X$ has cardinality less than $\mathsf{cov}(\mathcal{M})$, and thus is in $\mathsf{S}_1(\mathcal{B}, \mathcal{B})$. Now apply Theorem 13. $\square$

The notion of a Lusin set (i.e., an $\aleph_1$-Lusin set in our current notation) was characterized as follows in [21]: For a topological space X let $\mathcal{K}$ denote the collection of $\mathcal{U}$ such that $\mathcal{U}$ is a family of open subsets of X, and $X = \bigcup\{\overline{U}\colon U \in \mathcal{U}\}$. Then X is a Lusin set if, and only if, it has property $\mathsf{S}_1(\mathcal{K}, \mathcal{K})$.

Thus we have:

Corollary 31. *If a set of real numbers has property* $\mathsf{S}_1(\mathcal{K}, \mathcal{K})$*, then it has property* $\mathsf{S}_1(\mathcal{B}, \mathcal{B})$.

Theorem 32. *If* $\mathsf{cov}(\mathcal{M}) = \mathsf{cof}(\mathcal{M})$*, then there is a* $\mathsf{cov}(\mathcal{M})$*-Lusin set in* $\mathsf{S}_1(\mathcal{B}, \mathcal{B})$ *which is not in* $\mathsf{U}_{fin}(\Gamma, \Omega)$.

Proof. From the cardinality hypothesis and the fact that $\mathsf{cov}(\mathcal{M}) \leqslant \mathfrak{d} \leqslant \mathsf{cof}(\mathcal{M})$, we see that there is in ${}^{\mathbb{N}}\mathbb{Z}$ a dominating family, say Y, of cardinality $\mathsf{cov}(\mathcal{M})$. Let L be a $\mathsf{cov}(\mathcal{M})$-Lusin set as in Lemma 29, such that $L + L \supseteq Y$. As $\max\{|f(n)|, |g(n)|\} \geqslant (|f(n)| + |g(n)|)/2$, we see that for the identity mapping Ψ, $\mathrm{maxfin}(\Psi[L])$ is dominating. Thus, by Remark 10, L does not have property $\mathsf{U}_{fin}(\Gamma, \Omega)$.

By Corollary 30 L has property $\mathsf{S}_1(\mathcal{B}, \mathcal{B})$. $\square$

This in particular implies that $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$ is not provably equivalent to $\mathsf{S}_1(\mathcal{B}, \mathcal{B})$.

Special elements of $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$

Now that we have clarified most of the interrelationships among the Borel classes, we consider how the Borel classes are related to the classes in Fig. 1. We have just seen that $\mathsf{S}_1(\mathcal{B}, \mathcal{B})$ need not be contained in $\mathsf{U}_{fin}(\Gamma, \Omega)$, even when the critical cardinalities for sets not belonging to these classes are the same.

Next we treat $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$ and $U_{fin}(\Gamma, \Gamma)$. We show how to use the Continuum Hypothesis to construct a Lusin set which has property $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$. Since it is a Lusin set, it does not satisfy $\mathsf{U}_{fin}(\Gamma, \Gamma)$.

In our construction we use the *ad hoc* concept of an *ω-fat* collection of Borel sets. A collection $\mathcal{U}$ of Borel sets is said to be *fat* if for each non-empty open interval J and for each dense G_δ-set G there is a $B \in \mathcal{U}$ such that $B \cap G \cap J \neq \emptyset$. It is said to be *ω-fat* if: for each dense G_δ-set G and for every finite family $\mathcal{F}$ of non-empty open sets there is a $B \in \mathcal{U}$ such that for each $J \in \mathcal{F}$, $B \cap J \cap G$ is nonempty.

A number of facts about these ω-fat families of Borel sets will play a crucial role in our construction. For ease of reference we state these as lemmas and give proofs where it seems necessary.

Lemma 33. *Let* $\mathcal{U}$ *be an* ω*-fat family consisting of countably many Borel sets.*

(1) *For each partition of* $\mathcal{U}$ *into two pieces, at least one of the pieces is* ω*-fat.*

(2) *If $\mathcal{U}$ is a Borel ω-cover of the set X and F is a finite subset of X, then $\{U \in \mathcal{U}\colon F \subseteq U\}$ is an ω-fat Borel ω-cover of X.*

Lemma 34. *If $\mathcal{B}$ is a countable fat Borel family, then there is a dense G_δ-set contained in $\bigcup \mathcal{B}$.*

Proof. Since $B = \bigcup \mathcal{B}$ is a Borel set, it has the property of Baire. Let U be open set such that $(U \setminus B) \cup (B \setminus U)$ is meager. Then U is dense, for let G be a dense G_δ disjoint from that meager set, and let J be a non-empty open interval. Then $J \cap G \cap B$ is non-empty. But $B = (B \setminus U) \cup (B \cap U)$, so that $(B \cap U) \cap J$ is non-empty.

Now $\mathbb{R} \setminus U$ is nowhere dense, and we may assume that G is also disjoint from this nowhere dense set. But then $G \subseteq B$. □

Lemma 35. *If $\mathcal{U}$ is a countable ω-fat family of Borel sets and $\mathcal{F}$ is a finite non-empty family of non-empty open intervals, then there are a $U \in \mathcal{U}$ and for each $J \in \mathcal{F}$ a non-empty open interval $I_J \subset J$ such that the set $U \cap I_J$ is comeager in I_J.*

Proof. Towards proving the contrapositive, take a countable ω-fat family $\mathcal{U}$ Borel sets, and a finite non-empty family $\mathcal{F}$ of non-empty open intervals such that:

For each $U \in \mathcal{U}$ there is a $J_U \in \mathcal{F}$ such that for each non-empty open interval $I \subseteq J_U$ the set $U \cap I$ is not comeager in I. Fix such a J_U for each $U \in \mathcal{U}$.

Since $U \cap J_U$ is a Borel set, it has the property of Baire. Choose an open set $V \subset J_U$ such that $(V \setminus (U \cap J_U)) \cup ((U \cap J_U) \setminus V)$ is meager. If V is non-empty, then the meagerness of $V \setminus (U \cap J_U)$ implies that $U \cap V$ is comeager in V, contradicting the choice of U and J_U. Thus, V is empty, and we find that $U \cap J_U$ is meager. Let G_U be a dense G_δ-set disjoint from $U \cap J_U$.

The set $G = \bigcap_{U \in \mathcal{U}} G_U$ is an intersection of countably many dense G_δ-sets, so is a dense G_δ-set. But then G and $\mathcal{F}$ witness that $\mathcal{U}$ is not ω-fat. □

Lemma 36. *Let S be a countably infinite set and let $(F_n\colon n \in \mathbb{N})$ be an ascending sequence of finite sets with union equal to S. If $(\mathcal{U}_n\colon n \in \mathbb{N})$ is a sequence of Borel ω-covers of S such that for each n the set $\{U \in \mathcal{U}_n\colon F_n \subseteq U\}$ is ω-fat, then there is a sequence $(U_n\colon n \in \mathbb{N})$ such that for each n $U_n \in \mathcal{U}_n$, $\{U_n\colon n \in \mathbb{N}\}$ is a Borel γ-cover of S, and $\{U_n\colon n \in \mathbb{N}\}$ is ω-fat.*

Proof. Let S, the F_n's, and the $\mathcal{U}_n$'s be as in the hypotheses. We may assume for each n that for all $U \in \mathcal{U}_n$ we have $F_n \subseteq U$. Let $(J_n\colon n \in \mathbb{N})$ be an enumeration of the non-empty open intervals with rational endpoints.

Consider n. Since $\mathcal{U}_n$ is ω-fat, choose a $U_n \in \mathcal{U}_n$ and for each $i \leqslant n$ an open non-empty interval $I_n^i \subset J_i$ such that $I_n^i \cap U_n$ is comeager in I_n^i.

Then the sequence $(U_n\colon n \in \mathbb{N})$ is as desired. To see this, let G be any dense G_δ-set and let $R_1, \ldots, R_n$ be non-empty open intervals. Choose m so large at for each $i \leqslant n$ there is a $j \leqslant m$ with $J_j \subset R_i$. When we chose U_m it was done so that for some open non-empty intervals I_j, $j \leqslant m$ we had $I_j \subset J_j$ and $U_m \cap I_j$ is comeager in I_j, whence $U_m \cap G \cap I_j$ is comeager in I_j. But then for $r \leqslant n$, $U_m \cap G \cap R_r$ is non-empty. □

Lemma 37. *If $(\mathcal{U}_n\colon n \in \mathbb{N})$ is a sequence of countable ω-fat families of Borel sets such that for each n $\mathcal{U}_{n+1} \subseteq \mathcal{U}_n$, then there is a countable ω-fat family $\{U_n\colon n \in \mathbb{N}\}$ of Borel sets such that for each n, $U_n \in \mathcal{U}_n$.*

Proof. Let $J_1, J_2, \ldots, J_n, \ldots$ be a bijective enumeration of a basis for the topology of $\mathbb{R}$. Recursively choose for each n sequences $(I_k^n\colon k \in \mathbb{N})$ of non-empty open intervals, and for each n a $U_n \in \mathcal{U}_n$ such that:

(1) For $k < n$ we have $I_k^n = J_n$.

(2) For $k \geqslant n$ we have $I_k^n \subset J_n$ and $U_k \cap I_k^n$ is comeager in I_k^n.

This is possible on account of Lemma 35. We claim that $\mathcal{U} := \{U_n\colon n \in \mathbb{N}\}$ is ω-fat.

For let G be a dense G_δ-set and let $R_1, \ldots, R_k$ be non-empty open intervals. Choose from the basis intervals $J_{n_1}, \ldots, J_{n_k}$ such that $n_1 < \cdots < n_k$ and for $1 \leqslant i \leqslant k$ we have $J_{n_i} \subset R_i$. Let m be larger than n_k. Then for $1 \leqslant i \leqslant k$ we have: $U_m \cap I_m^{n_i}$ contains a dense G_δ-subset of $I_m^{n_i}$ and so has non-empty intersection with the dense G_δ-set G. Since for each i we have $I_m^{n_i} \subset R_i$ we see that $U \cap R_i \cap G$ is non-empty. □

Lemma 38. *Let G be a dense G_δ-set and let J be a non-empty open interval. If for each n $\mathcal{U}_n$ is a countable ω-fat family of Borel sets, then there is an $x \in J \cap G$ such that for each n the set $\{U \in \mathcal{U}_n\colon x \in U\}$ is ω-fat.*

Proof. For each n let $\mathcal{U}_n$ be a countable ω-fat family of Borel sets. Let J be non-empty open interval, and let G be a dense G_δ-set.

Let $(J_n\colon n \in \mathbb{N})$ bijectively enumerate a base for the topology of $\mathbb{R}$, and write $G = \bigcap_{n\in\mathbb{N}} V_n^1$ where $V_1^1 \supseteq V_2^1 \supseteq \cdots$ are dense open sets. Also, write $R_1 := J$. We may assume that the closure of J is compact.

Recursively construct four sequences $((U_n^i\colon i \leqslant n)\colon n \in \mathbb{N})$, $((I_n^i\colon i \leqslant n)\colon n \in \mathbb{N})$, $(R_n\colon n \in \mathbb{N})$ and $((V_n^i\colon n \in \mathbb{N})\colon i \in \mathbb{N})$, such that the following requirements are satisfied for each n:

(1) For all $k \leqslant n$, $U_n^k \in \mathcal{U}_k \setminus \{U_j^i\colon i, j < n\}$.

(2) For each $i \leqslant n$, $I_n^i \subset J_i$ is a non-empty open interval such that $I_n^i \cap (\bigcap_{j\leqslant n} U_n^j)$ is comeager in I_n^i.

(3) R_{n+1} is a nonempty open interval with closure contained in $(\bigcap_{i\leqslant n} V_{n+1}^i) \cap R_n$.

(4) $R_{n+1} \cap (\bigcap_{i\leqslant n} U_n^i)$ is comeager in R_{n+1}.

(5) $V_m^n \subset V_{m+1}^n$ for all m are dense open subsets of R_n.

(6) $R_{n+1} \cap (\bigcap_{i\leqslant n} U_n^i) \subseteq \bigcap_{m\in\mathbb{N}} V_m^{n+1}$.

To see that this recursion can be carried out, first consider $n = 1$: Here we already have R_1 and each V_n^1 specified. Consider J_1 and R_1, and $\mathcal{U}_1$. Apply Lemma 35 to choose $U_1^1 \in \mathcal{U}_1$ and intervals I_1^1 and R_2 such that $\overline{R}_2 \subset R_1 \cap V_1^1$ and $U_1^1 \cap R_2$ is comeager in R_2 and $U_1^1 \cap I_1^1$ is comeager in I_1^1. Since $U_1^1 \cap R_2$ is comeager in R_2, choose a descending sequence $(V_n^2\colon n \in \mathbb{N})$ of open dense subsets of R_2 such that $R_2 \cap U_1^1 \subseteq \bigcap_{m\in\mathbb{N}} V_m^2$. Thus for $n = 1$ sets as required by the five recursion specifications have been found.

Suppose now that $n \geqslant 1$ and that the recursion has been carried through for n steps. Consider R_n, $J_1, \ldots, J_n$, and $\mathcal{U}_1, \ldots, \mathcal{U}_n$.

Choose for $i \leqslant n+1$ sets $U_{n+1}^i \in \mathcal{U}_i \setminus \{U_k^j\colon j, k \leqslant n\}$ and R_{n+1} an open non-empty interval with closure contained in $R_n \cap (\bigcap_{i\leqslant n} V_{n+1}^i)$, as well as open non-empty intervals I_{n+1}^i, $i \leqslant n+1$, such that for each i, $I_{n+1}^i \subseteq J_i$, and $\bigcap_{k\leqslant n+1} U_{n+1}^k \cap I_{n+1}^i$ is comeager in I_{n+1}^i, and $\bigcap_{k\leqslant n+1} U_n^k \cap R_{n+1}$ is comeager in R_{n+1}. This can be done on account of Lemma 35. Then let $(V_m^{n+1}\colon m \in \mathbb{N})$ be a descending sequence of sets open and dense in R_{n+1} such that $R_{n+1} \cap (\bigcap_{k\leqslant n+1} U_{n+1}^k) \supseteq \bigcap_{m\in\mathbb{N}} V_m^{n+1}$.

This shows how to continue the recursion to the next step.

With the recursive procedure completed, for each n put $\mathcal{V}_n = \{U_k^n\colon k \geqslant n\}$. By the compactness of $\overline{R}_1$, and by specification (3) of the recursion, $\bigcap_{n\in\mathbb{N}} R_n$ is non-empty. Let x be an element of this intersection.

We claim that each $\mathcal{V}_n$ is an ω-fat subset of $\mathcal{U}_n$, and that for each $V \in \mathcal{V}_n$, we have $x \in V \cap J \cap G$.

To see that $\mathcal{V}_n$ is ω-fat, let a dense G_δ-set H and a finite set $\mathcal{F}$ of non-empty open intervals be given. Choose $m > n$ so large that there is for each $F \in \mathcal{F}$ a J_i with $i \leqslant m$ such that $J_i \subseteq F$. Then U_m^n was chosen so that for each of the non-empty open intervals $I_m^i \subset J_i$, we have $U_M^n \cap I_m^i$ comeager in I_m^i. But then as H is a comeager set of reals, we have for each $i \leqslant m$ that $U_m^n \cap I_m^i \cap H$ is non-empty. This implies that for each $F \in \mathcal{F}$, $U_m^n \cap F \cap H$ is non-empty.

To see that x is a member of each element of $\mathcal{V}_n$, consider a $U_m^n \in \mathcal{V}_n$. We have $U_m^n \cap R_m \supseteq \bigcap_{j\in\mathbb{N}} V_j^m$. But for each $j \geqslant m+1$ we have $R_{j+1} \subseteq V_j^m$, and as x is in the intersection of the R_j's, it is in the intersection of the V_j^m's, so in U_m^n. □

Lemma 39. *If* $\mathsf{add}(\mathcal{M}) = \mathfrak{c}$, *then there exists a family* $(G_\alpha\colon \alpha < \mathfrak{C}_1)$ *of dense* G_δ*-sets of reals, such that*:

- *For each dense* G_δ*-set* G *there is an* α *with* $G_\alpha \subseteq G$.
- *For* $\alpha < \beta < \mathfrak{c}$ *we have* $G_\beta \subset G_\alpha$.

Proof. Let $(M_\alpha\colon \alpha < \mathfrak{c})$ be a cofinal family of meager sets. We define by induction on $\alpha < \mathfrak{c}$ a monotonically increasing sequence $(\widetilde{M}_\alpha\colon \alpha < \mathfrak{c})$ of F_σ meager sets as follows: At stage α, let $\widehat{M}_\alpha = \bigcup_{i<\alpha} \widetilde{M}_i$. As $\alpha < \mathsf{add}(\mathcal{M})$, $\widehat{M}_\alpha$ is meager, so let $\widetilde{M}_\alpha$ be an F_σ meager set containing $\widehat{M}_\alpha$.

By the Baire category Theorem, complements of meager sets in $\mathbb{R}$ are dense. Thus, setting for each $\alpha G_\alpha = \mathbb{R}\setminus\widetilde{M}_\alpha$ yields the desired sequence. □

Theorem 40 (CH). *There is a* $\mathfrak{c}$*-Lusin set which has property* $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$.

Proof. Let $(G_\alpha\colon \alpha < \mathfrak{c})$ be as in Lemma 39. Let $((\mathcal{U}_n^\alpha\colon n \in \mathbb{N})\colon \alpha < \mathfrak{c})$ list all ω-sequences where each term is an ω-fat countable family of Borel sets. We shall now recursively construct the desired Lusin set X by choosing for each α a countable dense set X_α to satisfy certain requirements, and then setting $X = \bigcup_{\alpha<\mathfrak{c}} X_\alpha \cup \mathbb{Q}$. Together with each X_α we shall choose a sequence $(U_n^\alpha\colon n \in \mathbb{N})$ of Borel sets and a sequence $(S_\gamma(\alpha)\colon \gamma < \mathfrak{c})$ of infinite subsets of $\mathbb{N}$ such that:

(1) Whenever $\gamma < \beta < \mathfrak{c}$, then $S_\gamma(\beta) = \mathbb{N}$.
(2) For each $\beta < \mathfrak{c}$, for $\gamma < \nu < \mathfrak{c}$ we have $S_\nu(\beta) \subset^* S_\gamma(\beta)$.
(3) For all β and γ, $\{U_n^\beta\colon n \in S_\gamma(\beta)\}$ is an ω-fat γ-cover of $\mathbb{Q} \cup (\bigcup_{\nu\leqslant\gamma} X_\nu\}$.
(4) For any α, if some $\mathcal{U}_n^\alpha$ is not an ω-cover of $\mathbb{Q} \cup (\bigcup_{\nu<\alpha} X_\nu)$, then for each n we have $U_n^\alpha = \mathbb{R}$.
(5) If for each n $\mathcal{U}_n^\alpha$ is an ω-cover of $\mathbb{Q} \cup (\bigcup_{\nu<\alpha} X_\nu\}$, then for each n we have $U_n^\alpha \in \mathcal{U}_n^\alpha$, and $\{U_n^\alpha\colon n \in \mathbb{N}\}$ is an ω-fat γ-cover of $\mathbb{Q} \cup (\bigcup_{\nu<\alpha} X_\nu)$.
(6) For each α, $X_\alpha \subset G_\alpha \setminus (\mathbb{Q} \cup (\bigcup_{\nu<\alpha} X_\nu))$ is dense in $\mathbb{R}$.

Before showing that this can be accomplished, we show that constructing X to satisfy these requirements is sufficient. Thus, let X be obtained like this. Let $(\mathcal{U}_n\colon n \in \mathbb{N})$ be a sequence of countable Borel ω-covers of X. Since each X_α is dense and contained in G_α it follows that for each n $\mathcal{U}_n$ is ω-fat. Thus, for some β we have $(\mathcal{U}_n\colon n \in \mathbb{N}) = (\mathcal{U}_n^\beta\colon n \in \mathbb{N})$. Since each $\mathcal{U}_n^\beta$ is an ω-cover of X, it is an ω-cover of $\mathbb{Q} \cup (\bigcup_{\gamma<\beta} X_\gamma)$, and thus is as in (5). Let F be a finite subset of X and choose a $\beta > \alpha$ such that $F \subset \mathbb{Q} \cup (\bigcup_{\gamma\leqslant\beta} X_\gamma)$. By (3) $\{U_n^\alpha\colon n \in S_\beta(\alpha)\}$ is a γ-cover of $\mathbb{Q} \cup (\bigcup_{\gamma\leqslant\beta} X_\gamma)$, whence for some n $F \subset U_n^\alpha$. It follows that $\{U_n^\alpha\colon n \in \mathbb{N}\}$ is an ω-cover of X, as desired.

Now the recursive construction: Fix $\mathbb{Q}$, the set of rational numbers, and ask: Is $(\mathcal{U}_n^0\colon n \in \mathbb{N})$ a sequence of ω-covers of $\mathbb{Q}$?

No: Then for each n set $U_n^0 = \mathbb{R}$, choose $X_0 \subset G_0 \setminus \mathbb{Q}$ countable and dense, and put $S_0(0) = \mathbb{N}$.

Yes: For each n choose a $U_n^0 \in \mathcal{U}_n^0$ such that $\{U_n^0\colon n \in \mathbb{N})$ is an ω-fat γ-cover of $\mathbb{Q}$. Repeatedly apply Lemma 38 to recursively choose numbers $x_1 \in J_1 \cap G_0 \setminus \mathbb{Q}$ and $x_{n+1} \in J_{n+1} \cap G_0 \setminus (\mathbb{Q} \cup \{x_1, \ldots, x_n\})$ such that: $\mathcal{V}_1 := \{U_n^0\colon x_1 \in U_n^0\}$ is an ω-fat family of Borel sets, and for each n $V_{n+1} := \{U_m^0 \in \mathcal{V}_n\colon x_{n+1} \in U_m^0\}$ is an ω-fat family of Borel sets. In the end put $X_0 = \{x_n\colon n \in \mathbb{N}\}$, and choose by Lemma 37 a $\mathcal{V} \subset \mathcal{V}_1$ such that $\mathcal{V}$ is ω-fat, and for each n also $\mathcal{V} \subseteq^* \mathcal{V}_n$. Finally set $S_0(0) = \{n\colon U_n^0 \in \mathcal{V}\}$. Observe that $\{U_n^0\colon n \in S_0(0)\}$ is a γ-cover of $\mathbb{Q} \cup X_0$.

This shows that the six recursive requirements are satisfiable for $\alpha = 0$. Assume now that $\alpha > 0$ is given, and for each $\beta < \alpha$ we already have X_β as well as the sequence $(U_n^\beta\colon n \in \mathbb{N})$ and $(S_\gamma(\beta)\colon \gamma < \alpha)$ such that the six recursive requirements are satisfied. To verify that stage α can then be carried out, do the following. First, for all $\beta < \alpha$ define $S_\beta(\alpha) = \mathbb{N}$. Also, using Lemma 37, choose for each $\beta < \alpha$ an infinite set $S_\beta \subset \mathbb{N}$ such that for all $\gamma < \alpha$ we have $S_\beta \subset^* S_\gamma(\beta)$, and such that $\{U_n^\beta\colon n \in S_\beta\}$ is an ω-fat γ-cover of $\bigcup_{\gamma<\alpha} X_\gamma \cup \mathbb{Q}$.

Consider $(\mathcal{U}_n^\alpha\colon n\in\mathbb{N})$ and ask: Is each $\mathcal{U}_n^\alpha$ an ω-cover of $\bigcup_{\gamma<\alpha} X_\gamma\cup\mathbb{Q}$?

No: Then for each n put $\mathcal{U}_n^\alpha=\mathbb{R}$, and declare $S_\alpha(\alpha)=\mathbb{N}$. Next we choose X_α recursively as follows from $H_\alpha := G_\alpha\setminus(\bigcup_{\gamma<\alpha} X_\gamma\cup\mathbb{Q})$: By Lemma 38 choose an $x_1\in J\cap H_\alpha$ such that for each $\beta<\alpha$ the set $\mathcal{V}_1^\beta=\{U_n^\beta\colon n\in S_\beta \text{ and } x_1\in U_n^\beta\}$ is an ω-fat family. For each n choose $x_{n+1}\in J_{n+1}\cap H_\alpha\setminus\{x_1,\dots,x_n\}$ such that $\mathcal{V}_{n+1}^\beta := \{U_m^\beta\in\mathcal{V}_n^\beta\colon x_{n+1}\in U_m^\beta\}$ is an ω-fat family. Finally apply Lemma 37 to choose for each $\beta<\alpha$ an ω-fat family $\mathcal{V}_\beta\subseteq\mathcal{V}_1^\beta$ such that for each n $\mathcal{V}^\beta\subset^*\mathcal{V}_n^\beta$, and set $X_\alpha=\{x_n\colon n\in\mathbb{N}\}$. Observe that each $\mathcal{V}^\beta$ is a γ-cover of $\bigcup_{\gamma\leqslant\alpha} X_\gamma\cup\mathbb{Q}$, and X_α is a dense subset of $\mathbb{R}$. For each $\beta<\alpha$ define $S_\alpha(\beta) := \{m\colon U_m^\beta\in\mathcal{V}^\beta\}$.

Yes: Then first choose for each n a $U_n^\alpha\in\mathcal{U}_n^\alpha$ such that $\{U_n^\alpha\colon n\in\mathbb{N}\}$ is a γ-cover of $\bigcup_{\gamma<\alpha} X_\gamma\cup\mathbb{Q}$. For each $\beta<\alpha$ set $S_\beta(\alpha)=\mathbb{N}$. Next we construct X_α. For convenience, put $H_\alpha=G_\alpha\setminus(\bigcup_{\gamma<\alpha} X_\gamma\cup\mathbb{Q})$. Applying Lemma 38 choose $x_1\in J_1\cap H_\alpha$ such that for each $\beta<\alpha$ the set $\mathcal{U}_1^\beta := \{U_n^\beta\colon n\in S_\beta \text{ and } x_1\in U_n^\beta\}$ is ω-fat, and $\mathcal{U}_1^\alpha=\{U_n^\alpha\colon x_1\in U_n^\alpha\}$ is ω-fat. For each n choose $x_{n+1}\in J_{n+1}\cap H_\alpha\setminus\{x_1,\dots,x_n\}$ such that for $\beta\leqslant\alpha$ we have $\mathcal{V}_{n+1}^\beta=\{U_m^\beta\in\mathcal{V}_n^\beta\colon x_{n+1}\in U_m^\beta\}$ is an ω-fat family. Finally, by Lemma 37 choose for each β an ω-fat family $\mathcal{V}^\beta$ such that for all n $\mathcal{V}^\beta\subseteq^*\mathcal{V}_n^\beta$. Observe that each $\mathcal{V}^\beta$ is a γ-cover of $\bigcup_{\beta\leqslant\alpha} X_\beta\cup\mathbb{Q}$. For $\beta\leqslant\alpha$ define: $S_\alpha(\beta)=\{n\colon U_n^\beta\in\mathcal{V}^\beta\}$.

In either case we succeeded in extending the satisfiability of the recursive requirements before stage α, to stage α. □

Corollary 41. (CH) *There is a set of real numbers with property* $\mathsf{S}_1(\mathcal{B}_\Omega,\mathcal{B}_\Omega)$ *which does not have property* $\mathsf{U}_{fin}(\Gamma,\Gamma)$.

Proof. We may think of having carried out the preceding construction in ${}^{\mathbb{N}}\mathbb{N}$; here, every set with property $\mathsf{U}_{fin}(\Gamma,\Gamma)$ is bounded, and so meager. But a Lusin set is non-meager. □

Special elements of $\mathsf{S}_1(\mathcal{B}_\Gamma,\mathcal{B}_\Gamma)$

Our next task is to determine the relationship of the top row of Fig. 2 to the bottom rest of Fig. 1. For this we compare $\mathsf{S}_1(\mathcal{B}_\Gamma,\mathcal{B}_\Gamma)$ with $\mathsf{S}_1(\mathcal{O},\mathcal{O})$ and with $S_{fin}(\Omega,\Omega)$. A set X of real numbers is said to be a *Sierpiński set* if it is uncountable, and its intersection with each Lebesgue measure zero set is countable. More generally, for an uncountable cardinal number κ a set of real numbers is a κ-Sierpiński set if it has cardinality at least κ, but its intersection with each set of Lebesgue measure zero is less than κ.

In Theorem 2.9 of [8] it was shown that all Sierpiński sets have the property $\mathsf{U}_{fin}(\mathcal{B}_\Gamma,\mathcal{B}_\Gamma)$. This also follows easily from our characterization of $\mathsf{S}_1(\mathcal{B}_\Gamma,\mathcal{B}_\Gamma)$ (Theorem 3), since each countable set has this property. Indeed, our characterization and the fact that every set of real numbers of cardinality less than $\mathfrak{b}$ has property $\mathsf{S}_1(\mathcal{B}_\Gamma,\mathcal{B}_\Gamma)$ gives that every $\mathfrak{b}$-Sierpiński set has property $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$. Since sets of real numbers having property $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$ have measure zero, no $\mathfrak{b}$-Sierpiński set has property $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$.

Let $\mathbb{P}$ denote the set of irrational numbers.

Lemma 42. *If* $\mathsf{cov}(\mathcal{N}) = \mathsf{cof}(\mathcal{N})$, *and if* $Y \subseteq \mathbb{P}$ *has cardinality at most* $\mathsf{cof}(\mathcal{N})$, *then there is a* $\mathsf{cov}(\mathcal{N})$*-Sierpiński set* $S \subseteq \mathbb{P}$ *such that* $Y \subseteq S + S \subseteq \mathbb{P}$.

Proof. Let $\{y_\alpha\colon \alpha < \mathsf{cov}(\mathcal{N})\}$ enumerate Y. Let $\{N_\alpha\colon \alpha < \mathsf{cov}(\mathcal{N})\}$ enumerate a cofinal family of measure zero sets, and construct S recursively as follows: At stage α set

$$X_\alpha = \bigcup_{i<\alpha} \bigl(\{a_i, b_i\} \cup (\mathbb{Q} - a_i) \cup (\mathbb{Q} - b_i) \cup N_i\bigr).$$

Note that for each $x \in \mathbb{P} \setminus X_\alpha$ and $i < \alpha$, $x + a_i$ and $x + b_i$ are irrational.

X_α is a union of fewer than $\mathsf{cov}(\mathcal{N})$ measure zero sets. As in Lemma 29, we can choose $a_\alpha, b_\alpha \in \mathbb{P}\setminus X_\alpha$ such that $a_\alpha + b_\alpha = y_\alpha$. (Note that $y_\alpha \in \mathbb{P}$.)

Finally, set $S = \{a_\alpha\colon \alpha < \mathsf{cov}(\mathcal{N})\} \cup \{b_\alpha\colon \alpha < \mathsf{cov}(\mathcal{N})\}$. Then S is a $\mathsf{cov}(\mathcal{N})$-Sierpiński set and $Y \subseteq S + S \subseteq \mathbb{P}$. □

Theorem 43. *If* $\mathfrak{b} = \mathsf{cov}(\mathcal{N}) = \mathsf{cof}(\mathcal{N})$, *then there is a* $\mathfrak{b}$*-Sierpiński set of real numbers* S *such that*:

(1) *S has property* $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$.
(2) *S does not have property* $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$.
(3) *$S \times S$ does not have property* $\mathsf{U}_{fin}(\Gamma, \mathcal{O})$.
(4) *S does not have property* $\mathsf{S}_{fin}(\Omega, \Omega)$.

Proof. Note that the hypothesis $\mathfrak{b} = \mathsf{cof}(\mathcal{N})$ implies that $\mathfrak{b} = \mathfrak{d}$. Let Ψ be a homeomorphism from the irrationals onto ${}^{\mathbb{N}}\mathbb{N}$. Let $D \subseteq {}^{\mathbb{N}}\mathbb{N}$ be a dominating family of size $\mathfrak{d}$, and set $Y = \Psi^{-1}[D]$. Use Lemma 42 to construct a $\mathfrak{b}$-Sierpiński set $S \subseteq \mathbb{P}$ such that $Y \subseteq S + S \subseteq \mathbb{P}$. Now, define $f : S \times S :\to {}^{\mathbb{N}}\mathbb{N}$ by $f(x, y) = \Psi(x + y)$. Then f is continuous, and $f[S \times S] = \Psi[S + S] \supseteq \Psi[X] = D$ is dominating. This makes (1), (2), and (3).

Now, in [8] it is proved that $S_{fin}(\Omega, \Omega)$ is closed under taking finite powers. Thus, (4) follows from (3). □

Thus, we have that $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$ is not provably contained in $S_{fin}(\Omega, \Omega)$. It follows that Fig. 2 gives all the provable relations among the Borel covering classes.

In light of Theorem 6, the following Theorem of Recław [15] implies that none of the properties involving open classes implies any of the properties involving Borel classes. Recław's proof assumes Martin's axiom, but the partial order used is σ-centered so that in fact $\mathfrak{p} = \mathfrak{c}$ is enough.

Theorem 44. ($\mathfrak{p} = \mathfrak{c}$) *There is a set having the* $\mathsf{S}_1(\Omega, \Gamma)$ *property which can be mapped onto* ${}^{\mathbb{N}}\mathbb{N}$ *by a Borel function.*

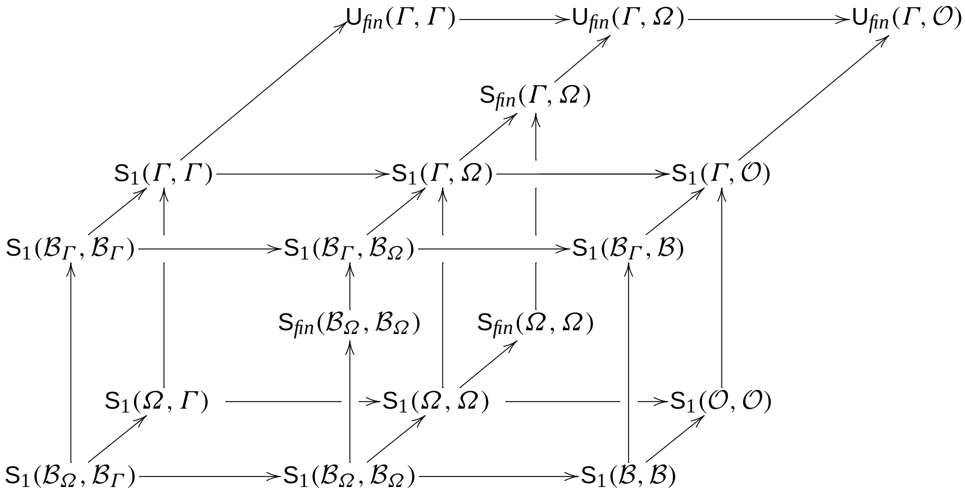

Fig. 3. The combined diagram.

Fig. 3 summarizes the relationships among the various classes considered so far in this paper and in [8], including the Borel classes. In this diagram there must also be a vector pointing from $\mathsf{S}_{fin}(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$ to $\mathsf{S}_{fin}(\Omega, \Omega)$; we omitted this one for “aesthetic” reasons.

With this we have now shown that in Fig. 3, no arrows can be added to, or removed from, the layer of Borel classes.

At present it is not known if there always is an uncountable set of real numbers which belongs to some class in Fig. 2. In light of what we know about this diagram, the most modest form of this question is

Problem 45. Is there always an uncountable set of reals with property $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B})$?

While the boldest form would be:

Problem 46. Is there always an uncountable set of real numbers with property $\mathsf{S}_{fin}(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$?

Special elements of $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$

It might be wondered whether any of our Borel notions trivializes to contain only sets of size smaller than the critical cardinality of that notion. With the knowledge obtained thus far, the only candidate to trivialize is $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$. A Theorem of Brendle [4] shows that this is not the case.

Theorem 47. (CH) *There is a set of reals X of size $\mathfrak{c}$ $(= \aleph_1)$ which has property* $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$.

4. Preservation of properties

The selection properties for open covers are preserved when taking continuous images or closed subsets [8]. We have the following analogue.

Theorem 48. *Let Π be one of S_1, S_{fin}, or U_{fin} and let $\mathcal{U}$ and $\mathcal{V}$ range over the set $\{\mathcal{B}, \mathcal{B}_\Omega, \mathcal{B}_\Lambda, \mathcal{B}_\Gamma\}$. Assume that X has property $\Pi(\mathcal{U}, \mathcal{V})$. Then*:

(1) *If Y is a Borel subset of X, then Y has property $\Pi(\mathcal{U}, \mathcal{V})$.*
(2) *If $f : X \to Y$ is Borel and onto, then Y has property $\Pi(\mathcal{U}, \mathcal{V})$.*

Proof. This proof is similar to the proof of Theorem 3.1 in [8]. □

In particular, if $\mathcal{U}$ and $\mathcal{V}$ are among $\{\mathcal{O}, \Omega, \Lambda, \Gamma\}$ for X, and X has property $\Pi(\mathcal{B}_\mathcal{U}, \mathcal{B}_\mathcal{V})$ for some Π, then every Borel image of X has property $\Pi(\mathcal{U}, \mathcal{V})$. This gives rise to the following question: Using the above notation, assume that every Borel image of X has property $\Pi(\mathcal{U}, \mathcal{V})$. Does X necessarily have the $\Pi(\mathcal{B}_\mathcal{U}, \mathcal{B}_\mathcal{V})$ property? For the following classes, a positive answer was given:

- $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$—Theorem 14.
- $\mathsf{U}_{fin}(\Gamma, \Gamma)$—Theorem 2.
- $\mathsf{S}_1(\Gamma, \Gamma)$—this one follows from the preceding one, since $\mathsf{S}_1(\Gamma, \Gamma)$ implies $\mathsf{U}_{fin}(\Gamma, \Gamma)$, and $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$ is equivalent to $\mathsf{U}_{fin}(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$ (Theorem 1).
- $\mathsf{U}_{fin}(\Gamma, \mathcal{O})$—Theorem 7.
- $\mathsf{S}_1(\Gamma, \mathcal{O})$—this one too follows from the preceding one, since $\mathsf{S}_1(\Gamma, \mathcal{O})$ implies $\mathsf{U}_{fin}(\Gamma, \mathcal{O})$, and $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B})$ is equivalent to $\mathsf{U}_{fin}(\mathcal{B}_\Gamma, \mathcal{B})$ (Theorem 6).
- $\mathsf{S}_1(\Omega, \Gamma)$—Theorem 26.

For the following classes, the problem remains open:

- $\mathsf{S}_1(\Gamma, \Omega)$, $\mathsf{S}_{fin}(\Gamma, \Omega)$, and $\mathsf{U}_{fin}(\Gamma, \Omega)$—if (4) implies (3) were true in Remark 10, we could have added these classes to the positive list.
- $\mathsf{S}_1(\Omega, \Omega)$.
- $\mathsf{S}_{fin}(\Omega, \Omega)$.

Finite powers

$\mathsf{S}_1(\mathcal{B}, \mathcal{B})$ is not provably closed under taking finite powers.

Theorem 49. *If $\mathsf{cov}(\mathcal{M}) = \mathsf{cof}(\mathcal{M})$, then there exists a set of reals X such that X has property $\mathsf{S}_1(\mathcal{B}, \mathcal{B})$, and $X \times X$ does not have the property $\mathsf{U}_{fin}(\Gamma, \mathcal{O})$.*

Proof. The $\mathsf{cov}(\mathcal{M})$-Lusin set L from Theorem 32 has the property that $L + L$, a continuous image of $L \times L$, is dominating. Thus, $L \times L$ does not have the property $\mathsf{U}_{fin}(\Gamma, \mathcal{O})$. □

Dually, Theorem 43 shows that $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$ is not provably closed under taking finite powers.

Problem 50. Is any of the classes $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$, $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$, and $\mathsf{S}_{fin}(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$ closed under taking finite powers?

Note that a positive answer to Problem 19 would imply that $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$ is closed under taking finite powers. Similarly, a positive answer to Problem 21 would imply that $\mathsf{S}_{fin}(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$ is closed under taking finite powers.

5. Connections with other approaches to smallness properties

Three schemas for describing smallness of sets of real numbers have been developed over recent years. These have their roots in classical literature and can be described, broadly speaking, by:

- properties of the vertical sections of a sufficiently describable planar set;
- properties of the image in ${}^{\mathbb{N}}\mathbb{N}$ under a sufficiently describable function;
- selection properties for sequences of sufficiently describable topologically significant families of subsets.

The vertical sections schema has been inspired by the papers [12–14], and is as follows:

Let H be a subset of $\mathbb{R} \times \mathbb{R}$ and let $\mathcal{J}$ be a collection of subsets of $\mathbb{R}$. For x and y real numbers, define

$$H_x = \{y \in \mathbb{R}\colon (x, y) \in H\};$$
$$H^y = \{x \in \mathbb{R}\colon (x, y) \in H\}.$$

A Borel set H is said to be a $\mathcal{J}$-set if for each x $H_x \in \mathcal{J}$.

The following three collections of subsets of the real line have been defined in terms of properties of vertical sections, see [11]:

- $\mathsf{ADD}(\mathcal{J})$: The set of $X \subseteq \mathbb{R}$ such that for each $\mathcal{J}$-set H, $\bigcup_{x \in X} H_x \in \mathcal{J}$.
- $\mathsf{COV}(\mathcal{J})$: The set of $X \subseteq \mathbb{R}$ such that for each $\mathcal{J}$-set H, $\bigcup_{x \in X} H_x \neq \mathbb{R}$.
- $\mathsf{COF}(\mathcal{J})$: The set of $X \subseteq \mathbb{R}$ such that $\{H_x\colon x \in X\}$ is not a cofinal subset of $\mathcal{J}$.

The sets in $\mathsf{COV}(\mathcal{M})$ have also been called $R^{\mathcal{M}}$-sets in [1]; in that paper it was shown that X is an $R^{\mathcal{M}}$-set if, and only if, every Borel image of X in ${}^{\mathbb{N}}\mathbb{N}$ has property $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$. It was shown in [2] that this class is also characterized by $\mathsf{S}_1(\mathcal{B}, \mathcal{B})$.

The sets in $\mathsf{ADD}(\mathcal{M})$ have also been called $SR^{\mathcal{M}}$-sets, and it has been shown in [1] that X is in $\mathsf{ADD}(\mathcal{M})$ if, and only if, every Borel image of X in ${}^{\mathbb{N}}\mathbb{N}$ has both properties $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$ and $U_{fin}(\Gamma, \Gamma)$. Due to a result in [10], a set X of real numbers has both properties $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$ and $\mathsf{U}_{fin}(\Gamma, \Gamma)$ if, and only if, it has the property $(*)$ which was introduced in [5]. Using our results here and results of [10] one can show that a set of reals has property $\mathsf{ADD}(\mathcal{M})$ if, and only if, it is a member of $\mathsf{S}_1(\mathcal{B}, \mathcal{B})$ and $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$.

The "properties of the image" schema takes inspiration from three papers [7,14] and [17, Lemma 3]. In each of these papers it is proven that a set of real numbers has a certain property of interest if, and only if, each of its continuous images (in some cases into a specific range space) has another property of interest.

The following four classes of sets were introduced in [11]:

- $\mathsf{NON}(\mathcal{J})$: The set of $X \subseteq \mathbb{R}$ such that for every Borel function f from $\mathbb{R}$ to $\mathbb{R}$, $f[X]$ is a member of $\mathcal{J}$.
- P: The set of $X \subseteq \mathbb{R}$ such that for no Borel function f from $\mathbb{R}$ to $[\mathbb{N}]^\infty$, $f[X]$ is a power.
- B: The set of $X \subseteq \mathbb{R}$ such that for every Borel function f from $\mathbb{R}$ to ${}^{\mathbb{N}}\mathbb{N}$, $f[X]$ is bounded under eventual domination.
- D: The set of $X \subseteq \mathbb{R}$ such that for every Borel function f from $\mathbb{R}$ to ${}^{\mathbb{N}}\mathbb{N}$, $f[X]$ is not a dominating family.

The classes of sets defined by these two schemas are related for the special case where $\mathcal{J}$ is $\mathcal{M}$, the collection of meager sets of real numbers, or $\mathcal{N}$, the collection of measure zero subsets of the real line. The results from [11] regarding the interrelationships of these classes of sets are summarized in Fig. 4.

The relationship between Fig. 4 and the well-known Cichoń diagram that expresses provable relationships among certain cardinal numbers is that a cardinal number in a particular position in Cichoń's diagram is actually the minimal cardinality for a set of real numbers not belonging to the class in the corresponding position in Fig. 4.

Our results imply the following.

Corollary 51. $\mathsf{COF}(\mathcal{M})$ *contains a set of reals whose size is* $\mathsf{cov}(\mathcal{M})$.

Proof. If $\mathsf{cov}(\mathcal{M}) < \mathsf{cof}(\mathcal{M}) (= \mathsf{non}(\mathsf{COF}(\mathcal{M})))$, then any set of size $\mathsf{cov}(\mathcal{M})$ will do. Otherwise by Theorem 32 there exists a $\mathsf{cov}(\mathcal{M})$-Lusin set in $\mathsf{S}_1(\mathcal{B}, \mathcal{B})$, which is in $\mathsf{COV}(\mathcal{M})$. □

In [7] Hurewicz characterized the covering properties $\mathsf{U}_{fin}(\Gamma, \Gamma)$ and $\mathsf{S}_{fin}(\mathcal{O}, \mathcal{O})$ in terms of properties of the continuous images in ${}^{\mathbb{N}}\mathbb{N}$. In particular, Hurewicz showed that X has property $U_{fin}(\Gamma, \Gamma)$ if, and only if, each continuous image of X in ${}^{\mathbb{N}}\mathbb{N}$ is bounded. He also showed that X has property $\mathsf{S}_{fin}(\mathcal{O}, \mathcal{O})$ if, and only if, each continuous image of X into ${}^{\mathbb{N}}\mathbb{N}$ is not a dominating family. The sets in B have also been called *A-sets* in [2]; where they show that that $\mathsf{B} = \mathsf{U}_{fin}(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$, and $\mathsf{D} = \mathsf{S}_{fin}(\mathcal{B}, \mathcal{B})$. By our results here we know $\mathsf{B} = \mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$, and $\mathsf{D} = \mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B})$.

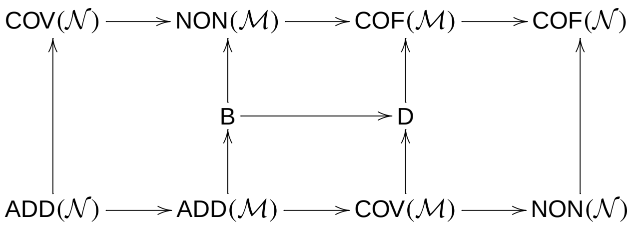

Fig. 4. Cichoń-like diagram.

Note added in proof

As stated, item (2) of Lemma 33 is wrong: Let $\mathcal{U} = \{\mathbb{R} \setminus \mathbb{Z}\} \cup [\mathbb{Z}]^{<\omega}$. Then $\mathcal{U}$ is an ω-fat ω-cover of $\mathbb{Z}$. But for any nonempty finite subset F of $\mathbb{Z}$, the collection $\{U \in \mathcal{U}\colon\ F \subset U\}$ is not ω-fat. However, if X is a Lusin set such that for each nonempty basic open set G, $X \cap G$ is uncountable, then item (2) of this lemma holds. As the special set X which we will construct is a Lusin set, we can easily make sure that it has the required property and the proof works. This idea is extended and explained further in: T. Bartoszynski, S. Shelah and B. Tsaban, *Additivity properties of topological diagonalizations* (preprint).

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