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Algebra, selections, and additive Ramsey theory

by

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Abstract. Hindman's celebrated Finite Sums Theorem, and its high-dimensional version due to Milliken and Taylor, can be viewed as coloring theorems concerning countable covers of countable, discrete sets. These theorems are extended to covers of arbitrary topological spaces with Menger's classical covering property. The methods include, in addition to Hurewicz's game-theoretic characterization of Menger's property, extensions of the classical idempotent theory in the Stone–Čech compactification of semigroups, and of the more recent theory of selection principles. This provides strong versions of the mentioned celebrated theorems, where the monochromatic substructures are large, beyond infinitude, in an analytical sense. Reducing the main theorems to the purely combinatorial setting, we obtain nontrivial consequences concerning uncountable cardinal characteristics of the continuum.

The main results, modulo technical adjustments, are of the following type (definitions provided in the main text): Let X be a Menger space, and $\mathcal{U}$ be an infinite open cover of X. Consider the complete graph whose vertices are the open sets in X. For each finite coloring of the vertices and edges of this graph there are disjoint finite subsets $\mathcal{F}_1, \mathcal{F}_2, \dots$ of the cover $\mathcal{U}$ whose unions $V_1 := \bigcup \mathcal{F}_1$, $V_2 := \bigcup \mathcal{F}_2, \dots$ have the following properties:

(1) The sets $\bigcup_{n\in F} V_n$ and $\bigcup_{n\in H} V_n$ are distinct for all nonempty finite sets $F < H$.
(2) All vertices $\bigcup_{n\in F} V_n$ for nonempty finite sets F have the same color.
(3) All edges $\{\bigcup_{n\in F} V_n, \bigcup_{n\in H} V_n\}$ for nonempty finite sets $F < H$ have the same color.
(4) The family $\{V_1, V_2, \dots\}$ forms a cover of X.

A self-contained introduction to the necessary parts of the needed theories is included.

1. Background. For a wider accessibility of this paper, the following is a brief, self-contained introduction to the Stone–Čech compactification of a semigroup and its necessary algebraic and combinatorial properties [11, 17], including all results employed later. The reader may skip familiar parts.

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1.1. The Stone–Čech compactification and Hindman's Theorem. Almost throughout, S denotes an infinite semigroup. We do not assume that S is commutative; however, with the applications in mind, we use additive notation. The *Stone–Čech compactification* of S, βS, is the set of all ultrafilters on S. We identify each element $s \in S$ with the principal ultrafilter associated to it. Thus, we view the set S as a subset of βS. A filter $\mathcal{F}$ on S is *free* if the intersection $\bigcap \mathcal{F}$ of all elements of $\mathcal{F}$ is empty. An *ultrafilter* is free if and only if it is nonprincipal.

A topology on the set βS is defined by taking the sets $[A] := \{p \in \beta S : A \in p\}$, for $A \subseteq S$, as a basis for the topology. The function $A \mapsto [A]$ respects finite unions, finite intersections, and complements. For an element $s \in S$ and a set $A \subseteq S$, we have $s \in [A]$ if and only if $s \in A$. In particular, the set S is dense in βS. The topological space βS is compact: if $\beta S = \bigcup_{\alpha \in I} [A_\alpha]$ and no finite union of sets A_α is S, then the family $\{A_\alpha^{\mathsf{c}} : \alpha \in I\}$ extends to an ultrafilter $p \in \beta S$, so p is in some set $[A_\alpha]$, a contradiction.

Define the sum of elements $p, q \in \beta S$ by

$$A \in p + q \quad \text{if and only if} \quad \{b \in S : \exists C \in q,\, b + C \subseteq A\} \in p.$$

Then $p + q \in \beta S$. We obtain an extension of the addition operator from S to βS, with the following continuity properties:

(1) For each $x \in S$, the function $q \mapsto x + q$ is continuous.
(2) For each $q \in \beta S$, the function $p \mapsto p + q$ is continuous.
(3) The addition function $+$ on βS is associative, that is, $(\beta S, +)$ is a semigroup.

Associativity follows from associativity in the dense subset S of βS: Consider the equality $(x + y) + z = x + (y + z)$. If we fix x and y in S, the equality is true for all $z \in S$. By (1), it is true for all $z \in \beta S$. If we fix $z \in \beta S$, the equality holds for all $y \in S$. By (1) and (2), it holds for all $y \in \beta S$. If we fix $y, z \in \beta S$, the equality holds for all $x \in S$. By (2), it holds for all $x \in \beta S$.

If $e \in \beta S$ is an idempotent element, that is, if $e + e = e$, then for each set $A \in e$ there are a set $B \in e$, and for each $b \in B$, a set $C \in e$ such that $b + C \subseteq A$. Conversely, the latter property of e implies that $e \subseteq e + e$ and thus $e = e + e$. In this characterization, by intersecting C with A, we may assume that $C \subseteq A$.

The Ellis–Numakura Lemma asserts that every closed subsemigroup T of βS has idempotent elements. Indeed, Zorn's Lemma provides us with a minimal closed subsemigroup E of T, and it follows by minimality that $E = \{e\}$ for some (necessarily, idempotent) element $e \in T$ ([1]).

([1]) To see that a minimal closed subsemigroup E of βS must be of the form $\{e\}$, fix an element $e \in E$. By the continuity of the functions $p \mapsto p + e$, the set $E + e$ is a closed subsemigroup of E, and thus $E + e = E$. Thus, the stabilizer of e, $\{t \in E : t + e = e\}$, is a (closed) subsemigroup of E, and is therefore equal to E, so that $e + e = e$.

Definition 1.1. For elements $a_1, a_2, \ldots$ in a semigroup S, and a nonempty finite set $F = \{i_1, \ldots, i_k\} \subseteq \mathbb{N}$ with $k \geq 1$ and $i_1 < \cdots < i_k$, define $a_F := a_{i_1} + \cdots + a_{i_k}$. Let

$$\mathrm{FS}(a_1, a_2, \ldots) := \{a_F : F \subseteq \mathbb{N},\ F \text{ finite nonempty}\},$$

the set of all finite sums, in increasing order of indices, of elements a_i. Similarly, for elements $a_1, \ldots, a_n \in S$, the set $\mathrm{FS}(a_1, \ldots, a_n)$ consists of the elements a_F for F a nonempty subset of $\{1, \ldots, n\}$.

A *finite coloring* of a set A is a function $f\colon A \to \{1, \ldots, k\}$ for $k \in \mathbb{N}$. Given a finite coloring f of a set A, a set $B \subseteq A$ is *monochromatic* if there is a color i with $f(b) = i$ for all $b \in B$.

Theorem 1.2 (Hindman [10]). *For each finite coloring of* $\mathbb{N}$, *there are elements* $a_1, a_2, \ldots \in \mathbb{N}$ *such that the set* $\mathrm{FS}(a_1, a_2, \ldots)$ *is monochromatic.*

The following strikingly elegant proof of Hindman's Theorem is due to Galvin and Glazer. Fix an idempotent element $e \in \beta\mathbb{N}$. Let a k-coloring of $\mathbb{N}$ be given. If C_i is the set of elements of color i, then $C_1 \cup \cdots \cup C_k = \mathbb{N} \in e$, and thus there is a color i with $A_1 := C_i \in e$. For $n = 1, 2, \ldots$, since e is an idempotent ultrafilter, there are an element $a_n \in A_n$ and a set $A_{n+1} \subseteq A_n$ in e such that $a_n + A_{n+1} \subseteq A_n$. It then follows, considering the sums from right to left, that every finite sum $a_{i_1} + \cdots + a_{i_k}$, for $i_1 < \cdots < i_k$, is in A_{i_1}. Thus, the set $\mathrm{FS}(a_1, a_2, \ldots)$ is a subset of the monochromatic set A_1.

1.2. The Milliken–Taylor Theorem and proper sumsequences. For a set S, let $[S]^2$ be the set of all 2-element subsets of S, or equivalently, the edge set of the complete graph with vertex set S.

Definition 1.3. Let S be a semigroup. For nonempty finite sets F and H of natural numbers, we write $F < H$ if all elements of F are smaller than all elements of H. A *sumsequence* (or sum subsystem) of a sequence $a_1, a_2, \ldots \in S$ is a sequence of the form $a_{F_1}, a_{F_2}, \ldots$ for nonempty finite sets $F_1 < F_2 < \cdots$ of natural numbers.

A sequence $b_1, b_2, \ldots \in S$ is *proper* if $b_F \neq b_H$ for all nonempty finite sets $F < H$ of natural numbers. The *sum graph* of a proper sequence $b_1, b_2, \ldots \in S$ is the subset of $[\mathrm{FS}(b_1, b_2, \ldots)]^2$ consisting of the edges $\{b_F, b_H\}$ for nonempty finite sets $F < H$ of natural numbers.

If $b_1, b_2, \ldots$ is a sumsequence of $a_1, a_2, \ldots$, then we have $\mathrm{FS}(b_1, b_2, \ldots) \subseteq \mathrm{FS}(a_1, a_2, \ldots)$. The relation of being a sumsequence is transitive.

Ramsey's Theorem [18] asserts that, for each finite coloring of an infinite complete graph $[V]^2$ with vertex set V, there is an infinite complete monochromatic subgraph, that is, an infinite set $I \subseteq V$ such that the set $[I]^2$ is monochromatic. The Milliken–Taylor Theorem unifies Hindman's and Ramsey's Theorems.

THEOREM 1.4 (Milliken–Taylor [16, 28]). *Let* $a_1, a_2, \dots$ *be a sequence in* $\mathbb{N}$. *For each finite coloring of the set* $[\mathbb{N}]^2$, *there is a proper sumsequence* $b_1, b_2, \dots$ *of* $a_1, a_2, \dots$ *such that the sum graph of* $b_1, b_2, \dots$ *is monochromatic.*

The Milliken–Taylor Theorem can be proved by combining the proofs of Ramsey's and Hindman's Theorems, as can be gleaned from the proof of the forthcoming Theorem 3.6.

Our applications are in a setting where all elements of the semigroup S are idempotents. In this case, stating Hindman's Theorem for the semigroup S instead of $\mathbb{N}$ yields a trivial statement: for an idempotent element $e \in S$, the set $\mathrm{FS}(e, e, \dots) = \{e\}$ is obviously monochromatic. The sequence $e, e, \dots$ is improper, and so are all of its sumsequences. Thus, the Milliken–Taylor Theorem cannot be extended to such cases. An example of a semigroup with all elements idempotent is $\mathrm{Fin}(\mathbb{N})$, the set of nonempty finite subsets of $\mathbb{N}$, with the operation $\cup$. For this semigroup, we have the following theorem.

THEOREM 1.5 (Milliken–Taylor). *For each finite coloring of the set* $[\mathrm{Fin}(\mathbb{N})]^2$, *there are elements* $F_1 < F_2 < \cdots$ *in* $\mathrm{Fin}(\mathbb{N})$ *such that the sum graph of* $F_1, F_2, \dots$ *is monochromatic.*

As every sequence of natural numbers has a proper sumsequence, and the sequence $\{1\}, \{2\}, \dots$ is proper, Theorems 1.4 and 1.5 are special cases of the following one.

THEOREM 1.6. *Let* S *be a semigroup, and* $a_1, a_2, \dots \in S$. *If the sequence* $a_1, a_2, \dots$ *has a proper sumsequence, then for each finite coloring of the set* $[S]^2$, *there is a proper sumsequence of* $a_1, a_2, \dots$ *whose sum graph is monochromatic.*

Proof. By moving to a sumsequence, we may assume that the sequence $a_1, a_2, \dots$ is proper. Let χ be a finite coloring of $[S]^2$. Define a coloring κ of $[\mathrm{Fin}(\mathbb{N})]^2$ by $\kappa(\{F, H\}) = \chi(\{a_F, a_H\})$ for $F < H$, and $\kappa(\{F, H\})$ arbitrary otherwise, and apply Theorem 1.5, using the fact that sumsequences of proper sequences are proper. ∎

The hypothesis of having a proper sumsequence fails only in degenerate cases.

PROPOSITION 1.7. *Let* S *be a semigroup, and* $a_1, a_2, \dots \in S$. *If the sequence* $a_1, a_2, \dots$ *has no proper sumsequence, then every sumsequence of* $a_1, a_2, \dots$ *has a sumsequence of the form* $e, e, \dots$, *where* e *is an idempotent element of* S, *or equivalently, a sumsequence whose set of finite sums is a singleton.*

Proof. We use Theorem 1.5. Define a coloring of the set $[\mathrm{Fin}(\mathbb{N})]^2$ by

$$\{F, H\} \mapsto |\{a_F, a_H\}|.$$

Let $F_1 < F_2 < \cdots$ be elements of $\mathrm{Fin}(\mathbb{N})$ such that the sum graph of the sequence $F_1, F_2, \ldots$ is monochromatic.

Consider the sumsequence $b_1 := a_{F_1}$, $b_2 := a_{F_2}$, $\ldots$. Assume that the color is 2. Then the sumsequence $b_1, b_2, \ldots$ is proper, a contradiction. Thus, the color must be 1. Then $b_{H_1} = b_{H_2}$ for all $H_1 < H_2$ in $\mathrm{Fin}(\mathbb{N})$. Let $e := b_1$. Then $b_n = b_1 = e$ for all $n > 1$. For each set $H \in \mathrm{Fin}(\mathbb{N})$, take $n > H$. Then $b_H = b_n = e$. In particular, $e + e = b_1 + b_2 = b_{\{1,2\}} = e$. Therefore, $\mathrm{FS}(b_1, b_2, \ldots) = \{e\}$. ■

2. Idempotent filters and superfilters. Superfilters provide a convenient way to identify closed subsets of βS ([2]).

DEFINITION 2.1. A family $\mathcal{A}$ of subsets of a set S is a *superfilter* on S if:

(1) All sets in $\mathcal{A}$ are infinite.
(2) For each set $A \in \mathcal{A}$, all subsets of S that contain A are in $\mathcal{A}$.
(3) Whenever $A_1 \cup A_2 \in \mathcal{A}$, then A_1 or A_2 is in $\mathcal{A}$; equivalently, for each set $A \in \mathcal{A}$ and each finite coloring of A, there is in $\mathcal{A}$ a monochromatic subset of A.

The simplest example of a superfilter on a set S is the family $[S]^\infty$ consisting of all infinite subsets of S. Many examples of superfilters are provided by Ramsey-theoretic theorems. For example, van der Waerden's Theorem asserts that monochromatic arithmetic progressions of any prescribed finite length will be found in any long enough, finitely-colored arithmetic progression. By van der Waerden's Theorem, the family of all sets of natural numbers containing arbitrarily long finite arithmetic progressions is a superfilter on $\mathbb{N}$.

The notions of free filter and superfilter are dual. For a family $\mathcal{F}$ of subsets of a set S, define $\mathcal{F}^+ := \{A \subseteq S : A^{\mathsf{c}} \notin \mathcal{F}\}$. The following assertions are easy to verify.

LEMMA 2.2 (Folklore). *Let S be a set.*

(1) *For all families $\mathcal{F}_1$ and $\mathcal{F}_2$ of subsets of S, $\mathcal{F}_1 \subseteq \mathcal{F}_2$ implies that $\mathcal{F}_1^+ \supseteq \mathcal{F}_2^+$.*
(2) *For each family $\mathcal{F}$ of subsets of S, $\mathcal{F}^{++} = \mathcal{F}$.*
(3) *For each free filter $\mathcal{F}$ on S, the set $\mathcal{F}^+$ is a superfilter containing $\mathcal{F}$.*
(4) *For each superfilter $\mathcal{A}$ on S, the set $\mathcal{A}^+$ is a free filter contained in $\mathcal{A}$.*
(5) *For each filter $\mathcal{F}$, if $A \in \mathcal{F}^+$ and $B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}^+$.*
(6) *For each ultrafilter p on S, $p^+ = p$.*

([2]) Superfilters have various names in the classical literature, including coideals, grilles, and partition-regular families, depending on the context where they are used. Some of the definitions in the literature are not equivalent to the one given here, but they are always conceptually similar. The present term is adopted from [21].

Proof of (5). Since $A \subseteq B^{\mathsf{c}} \cup (A \cap B)$, the latter set is in $\mathcal{F}^+$. Since $B^{\mathsf{c}} \notin \mathcal{F}^+$, we have $A \cap B \in \mathcal{F}^+$. ∎

Every free ultrafilter on S is a superfilter on S, and so is any union of free ultrafilters on S. Since elements of superfilters are infinite, the filter of cofinite subsets of S is contained in all superfilters on S. By the following lemma, every superfilter $\mathcal{A}$ is a union of a closed set of free ultrafilters. Indeed, taking $\mathcal{F} = \{\mathbb{N}\}$ we see by the lemma that the set $C := \{p \in \beta S : p \subseteq \mathcal{A}\}$ is closed, and for each set $A \in \mathcal{A}$, letting $\mathcal{F}$ be the filter generated by A we deduce, again by the lemma, that there is an ultrafilter $p \in C$ with $A \in p$. Thus, $\bigcup C = \mathcal{A}$.

Lemma 2.3. *Let S be an infinite set. For each superfilter $\mathcal{A}$ on S, and each filter $\mathcal{F} \subseteq \mathcal{A}$, the set $\{p \in \beta S : \mathcal{F} \subseteq p \subseteq \mathcal{A}\}$ is a nonempty closed subset of $\beta S \setminus S$.*

Proof. It is straightforward to verify that the set is closed. We prove that it is nonempty. By Lemma 2.2(4), the set $\mathcal{A}^+$ is a filter. By Lemma 2.2(2, 5) applied to the filter $\mathcal{A}^+$, we have $A \cap B \in \mathcal{A}$ for all $A \in \mathcal{A}$ and $B \in \mathcal{A}^+$. In particular, the set $A \cap B$ is infinite for all $A \in \mathcal{F}$ and $B \in \mathcal{A}^+$. The family $\{A \cap B : A \in \mathcal{F}, B \in \mathcal{A}^+\}$ is closed under finite intersections. Since its elements are infinite, it extends to a free ultrafilter p. Necessarily, $\mathcal{F} \subseteq p$. If there were an element $B \in p \setminus \mathcal{A}$, then $B^{\mathsf{c}} \in \mathcal{A}^+ \subseteq p$, a contradiction. ∎

Definition 2.4. Let S be a semigroup.

(1) For a set $A \subseteq S$ and a family $\mathcal{F}$ of subsets of S, let
$$A^\star(\mathcal{F}) := \{b \in S : \exists C \in \mathcal{F},\, b + C \subseteq A\}.$$
(2) A filter $\mathcal{F}$ on S is an *idempotent filter* if for each set $A \in \mathcal{F}$, the set $A^\star(\mathcal{F})$ is in $\mathcal{F}$.
(3) A superfilter $\mathcal{A}$ on S is an *idempotent superfilter* if for each $A \subseteq S$ such that $A^\star(\mathcal{A})$ is in $\mathcal{A}$, we have $A \in \mathcal{A}$.

Thus, for ultrafilters p, q on S, $A \in p + q$ if and only if $A^\star(q) \in p$.

Let S be a semigroup. A superfilter $\mathcal{A}$ on S is *translation-invariant* if $s + A \in \mathcal{A}$ for all $s \in S$ and $A \in \mathcal{A}$. Every translation-invariant superfilter on a semigroup S is an idempotent superfilter.

Since ultrafilters are maximal filters, we find that, for an ultrafilter p on a semigroup S, being an idempotent ultrafilter, idempotent filter, and idempotent superfilter is the same.

Lemma 2.5. *Let S be a semigroup.*

(1) *For each free idempotent filter $\mathcal{F}$ on S, the superfilter $\mathcal{F}^+$ is idempotent.*
(2) *For each idempotent superfilter $\mathcal{A}$ on S, the free filter $\mathcal{A}^+$ is idempotent.*

Proof. (1) Let $A \subseteq S$, and assume that the set $B_1 := A^\star(\mathcal{F}^+)$ is in $\mathcal{F}^+$. Assume that $A \notin \mathcal{F}^+$. Then $A^\mathsf{c} \in \mathcal{F}$, and thus the set $B_2 := (A^\mathsf{c})^\star(\mathcal{F})$ is in $\mathcal{F}$. By Lemma 2.2(5), there is an element $b \in B_1 \cap B_2$. Then there are sets $C_1 \in \mathcal{F}^+$ and $C_2 \in \mathcal{F}$ such that $b + C_1 \subseteq A$ and $b + C_2 \subseteq A^\mathsf{c}$. Pick $c \in C_1 \cap C_2$. Then $b + c \in A \cap A^\mathsf{c}$, a contradiction.

(2) Similar. ∎

LEMMA 2.6. *Let S be a semigroup, and $\mathcal{F}$ be a free idempotent filter on S. Then the set $T := \{p \in \beta S : \mathcal{F} \subseteq p\}$ is a closed subsemigroup of βS disjoint from S.*

Proof. By Lemma 2.3, with $\mathcal{A} = [S]^\infty$, the set T is a closed subset of βS. Since the filter $\mathcal{F}$ is free, we have $T \subseteq \beta S \setminus S$. Let $p, q \in T$, and $A \in \mathcal{F}$. Since the filter $\mathcal{F}$ is idempotent, $A^\star(\mathcal{F}) \in \mathcal{F} \subseteq p$. Since $\mathcal{F} \subseteq q$, we have $A^\star(\mathcal{F}) \subseteq A^\star(q)$, and therefore $A^\star(q) \in p$. By the definition of sum of ultrafilters, $A \in p + q$. ∎

THEOREM 2.7. *Let S be a semigroup, and assume that $\mathcal{F}$ is a free idempotent filter on S contained in an idempotent superfilter $\mathcal{A}$ on S. Then there is a free idempotent ultrafilter e with $\mathcal{F} \subseteq e \subseteq \mathcal{A}$.*

Proof. Let $T_1 = \{p \in \beta S : \mathcal{F} \subseteq p\}$ and $T_2 := \{p \in \beta S : p \subseteq \mathcal{A}\}$. By Lemma 2.6, the set T_1 is a closed subsemigroup of βS, and so is the set $\{p \in \beta S : \mathcal{A}^+ \subseteq p^+ = p\} = T_2$.

By Lemma 2.3, the intersection $T := T_1 \cap T_2$ is nonempty, and is therefore a closed subsemigroup of βS. Pick an idempotent element in T. ∎

3. Selection principles and an abstract partition theorem.

We use the following notions from Scheepers's seminal paper [22]. Let $\mathcal{A}$ and $\mathcal{B}$ be families of sets. Then $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ is the property that, for each sequence $A_1, A_2, \ldots \in \mathcal{A}$, one can select one element from each set, $b_1 \in A_1, b_2 \in A_2, \ldots$, such that $\{b_1, b_2, \ldots\} \in \mathcal{B}$. Furthermore, $\mathsf{G}_1(\mathcal{A}, \mathcal{B})$ is a game associated to $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$. This game is played by two players, *Alice* and *Bob*, and has an inning per each natural number. In the nth inning, Alice plays a set $A_n \in \mathcal{A}$, and Bob selects an element $b_n \in A_n$. Bob wins if $\{b_1, b_2, \ldots\} \in \mathcal{B}$. Otherwise, Alice wins.

If Alice does not have a winning strategy in the game $\mathsf{G}_1(\mathcal{A}, \mathcal{B})$, then $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ holds. The converse implication holds in some important cases, including the ones in our main applications. A survey of known results of this type is provided, e.g., in [24, Section 11].

EXAMPLE 3.1. Let S be a set, and $\mathcal{F}$ be a filter on S generated by countably many sets. Then Alice does not have a winning strategy in the game $\mathsf{G}_1(\mathcal{F}^+, \mathcal{F}^+)$; moreover, Bob has one: Fix sets $B_1, B_2, \ldots \in \mathcal{F}$ such that every member of $\mathcal{F}$ contains one of these sets. In each inning, by Lemma 2.2(5), Bob can pick an element $b_n \in A_n \cap B_n$. Then $\{b_1, b_2, \dots\} \in \mathcal{F}^+$.

For the filter $\mathcal{F}$ of cofinite sets, this reproduces the simple observation that Bob has a winning strategy in the game $\mathsf{G}_1([S]^\infty, [S]^\infty)$.

In general, the game $\mathsf{G}_1(\mathcal{A}, \mathcal{B})$ is not determined, and the property that Alice does not have a winning strategy is strictly weaker than Bob's having one. This will be the case in our main applications [24, Section 11].

Definition 3.2. A *free idempotent chain* in a semigroup S is a descending sequence $A_1 \supseteq A_2 \supseteq \cdots$ of infinite subsets of S such that:

(1) $\bigcap_n A_n = \emptyset$.
(2) For each n, the set $A^\star_n(\{A_1, A_2, \dots\})$ contains one of the sets A_m, or equivalently, there is $m > n$ such that, for each $a \in A_m$, there is $k > m$ with $a + A_k \subseteq A_m$.

For a family $\mathcal{A}$ of subsets of S, a *free idempotent chain in* $\mathcal{A}$ is a free idempotent chain of elements of $\mathcal{A}$.

Example 3.3. For each proper sequence $a_1, a_2, \dots$ in a semigroup, the sets $\mathrm{FS}(a_n, a_{n+1}, \dots)$ for $n \in \mathbb{N}$ form a free idempotent chain. Thus, if a sequence $a_1, a_2, \dots$ has a *proper sumsequence*, then there is a free idempotent chain $A_1 \supseteq A_2 \supseteq \cdots$ with $A_n \subseteq \mathrm{FS}(a_n, a_{n+1}, \dots)$ for all n.

Lemma 3.4. *Let S be a semigroup, and $\mathcal{A}$ be a superfilter on S. Every filter generated by a free idempotent chain in $\mathcal{A}$ is a free idempotent filter contained in $\mathcal{A}$.*

Proof. Let $A_1 \supseteq A_2 \supseteq \cdots$ be a free idempotent chain in $\mathcal{A}$, and let $\mathcal{F}$ be the filter generated by the sets $A_1, A_2, \dots$. Since $\bigcap_n A_n = \emptyset$, the filter $\mathcal{F}$ is free. Since $A_n \in \mathcal{A}$ for each n, we have $\mathcal{F} \subseteq \mathcal{A}$. The filter $\mathcal{F}$ is idempotent: For $A \in \mathcal{F}$, let A_n be a subset of A. By the definition, there is m such that $A^\star(\mathcal{F}) \supseteq A^\star_n(\{A_1, A_2, \dots\}) \supseteq A_m$. Since $A_m \in \mathcal{F}$, we have $A^\star(\mathcal{F}) \in \mathcal{F}$. ∎

Our theorems can be stated for any finite dimension. For clarity, we state them in the one-dimensional case, which extends Hindman's Theorem, and in the two-dimensional case, which extends the Milliken–Taylor Theorem. The one-dimensional case always follows from the two-dimensional, for the following reason.

Proposition 3.5. *Let S be a semigroup, and χ be a finite coloring of the sets S and $[S]^2$. There is a finite coloring η of $[S]^2$ such that, for each proper sequence $b_1, b_2, \dots$ with η-monochromatic sum graph, the set* $\mathrm{FS}(b_1, b_2, \dots)$ *and the sum graph of $b_1, b_2, \dots$ are both χ-monochromatic.*

Proof. By enumerating the elements of the countable set $\mathrm{FS}(b_1, b_2, \dots)$, we obtain an order on this set such that every element has only finitely many smaller elements. Define a coloring κ of $[\mathrm{FS}(b_1, b_2, \ldots)]^2$ by

$$\kappa(\{s,t\}) := \chi(\min\{s,t\}).$$

Extend κ to a coloring of $[S]^2$ in an arbitrary manner.

Assume that the set $\mathrm{FS}(b_1, b_2, \ldots)$ is monochromatic for κ, say green. Being proper, the sequence $b_1, b_2, \ldots$ is bijective. For each nonempty finite set F of natural numbers, since there are at most finitely many elements in $\mathrm{FS}(b_1, b_2, \ldots)$ smaller than b_F, there is $n > F$ such that $b_F < b_n$. Then $\kappa(\{b_F, b_n\}) = \chi(b_F)$. Thus, the element b_F is green.

The finite coloring η of the set $[S]^2$ defined by

$$\eta(\{s,t\}) := \big(\kappa(\{s,t\}), \chi(\{s,t\})\big)$$

is as required. If χ is a k-coloring, we may represent the range set of η in the form $\{1, \ldots, k^2\}$. ∎

The two monochromatic sets in Proposition 3.5 may be of different colors. Moreover, this can be forced by adding a coordinate to $\chi(x)$ that is 1 if $x \in S$ and 2 if $x \in [S]^2$.

The proof of the following theorem is a natural combination of a standard proof of the Milliken–Taylor Theorem (which, in turn, is an application of Hindman's Theorem along a standard argument for proving Ramsey's Theorem) with the concept of an infinite game. The importance of this theorem lies in its identifying important notions needed in the proofs of our main theorems.

Theorem 3.6. *Let S be a semigroup. Let $\mathcal{A}$ be an idempotent superfilter on S, and $\mathcal{B}$ be a family of subsets of S such that Alice does not have a winning strategy in the game $\mathsf{G}_1(\mathcal{A}, \mathcal{B})$. Let $a_1, a_2, \ldots$ be a sequence in S, and $A_1 \supseteq A_2 \supseteq \cdots$ be a free idempotent chain in $\mathcal{A}$ with $A_n \subseteq \mathrm{FS}(a_n, a_{n+1}, \ldots)$ for all n. For each finite coloring of the sets S and $[S]^2$, there are elements $b_1 \in A_1, b_2 \in A_2, \ldots$ such that:*

(1) *The set $\{b_1, b_2, \ldots\}$ is in $\mathcal{B}$.*
(2) *The sequence $b_1, b_2, \ldots$ is a proper sumsequence of $a_1, a_2, \ldots$.*
(3) *The set $\mathrm{FS}(b_1, b_2, \ldots)$ is monochromatic.*
(4) *The sum graph of $b_1, b_2, \ldots$ is monochromatic.*

Proof. By Proposition 3.5, it suffices to prove the two-dimensional assertion, that is, item (3) follows from (4).

By Lemma 3.4, there is a free idempotent filter $\mathcal{F}$ with $\{A_1, A_2, \ldots\} \subseteq \mathcal{F} \subseteq \mathcal{A}$. By Theorem 2.7, there is a free idempotent ultrafilter e on S such that $\mathcal{F} \subseteq e \subseteq \mathcal{A}$.

Let a finite coloring $\chi\colon [S]^2 \to \{1, \ldots, k\}$ be given. For each element $s \in S$, let

$$C_i(s) := \{t \in S \setminus \{s\} : \chi(\{s,t\}) = i\}.$$

As $C_1(s) \cup \cdots \cup C_k(s) = S \setminus \{s\} \in e$, there is a unique i with $C_i(s) \in e$. Define a finite coloring $\kappa\colon S \to \{1, \dots, k\}$ by letting $\kappa(s)$ be this unique i with $C_i(s) \in e$. Since e is an ultrafilter, there is in e a set $M \subseteq S$ that is monochromatic for the coloring κ. Assume that the color is *green*. Then, for each finite set $F \subseteq M$, we have

$$G(F) := \bigcap_{s\in F} \bigl\{t \in S \setminus \{s\} : \{s, t\} \text{ is green}\bigr\} \in e,$$

and for each $s \in F$ and each $t \in G(F)$, we have $s \neq t$ and the edge $\{s, t\}$ is green.

For a set $D \in e$, define

$$D^\star := \{b \in D : \exists B \subseteq D \in e,\ b + B \subseteq D\} = D^\star(e) \cap D.$$

Then $D^\star \subseteq D$ and, since e is an idempotent ultrafilter, $D^\star \in e$.

We define a strategy for Alice. In this strategy, Alice makes choices from certain nonempty sets. Formally, she does that by applying prescribed choice functions to the given nonempty sets.

- In the first inning, Alice sets $D_1 := M \cap A_1$, and plays the set $D_1^\star$.
- Assume that Bob plays an element $b_1 \in D_1^\star$. Then Alice chooses a set $B \subseteq D_1$ in e such that $b_1 + B \subseteq D_1$ and a set F_1 with $a_{F_1} = b_1$. She then chooses a natural number $m_1 > F_1$, and sets $D_2 := B \cap G(\{b_1\}) \cap A_{m_1}$. Having done that, Alice plays the set $D_2^\star$.
- Assume that Bob plays an element $b_2 \in D_2^\star$. Then $b_1 + b_2 \in D_1 \subseteq M$. Alice chooses a set $B \subseteq D_2$ in e such that $b_2 + B \subseteq D_2$, a set $F_2 > m_1$ with $a_{F_2} = b_2$, and a natural number $m_2 > F_2$. She sets $D_3 := B \cap G(\mathrm{FS}(b_1, b_2)) \cap A_{m_2}$, and plays $D_3^\star$.
- In the $(n+1)$st inning, Bob has picked elements $b_1 \in D_1^\star, \dots, b_n \in D_n^\star$. As in the Galvin–Glazer proof of Hindman's Theorem, by computing sums from right to left, we see that $\mathrm{FS}(b_1, \dots, b_n) \subseteq D_1 \subseteq M$. Alice chooses a set $B \subseteq D_n$ in e such that $b_n + B \subseteq D_n$, a set $F_n > m_{n-1}$ with $a_{F_n} = b_n$, and a natural number $m_n > F_n$. She then sets $D_{n+1} := B \cap G(\mathrm{FS}(b_1, \dots, b_n)) \cap A_{m_n}$, and plays the set $D_{n+1}^\star$.

Since Alice has no winning strategy, there is a play $(D_1^\star, b_1, D_2^\star, b_2, \dots)$, according to Alice's strategy, won by Bob. By the construction, the sequence $b_1, b_2, \dots$ is a sumsequence of $a_1, a_2, \dots$. The set $\{b_1, b_2, \dots\}$ is in $\mathcal{B}$, since Bob has won this play.

Let $i_1 < \cdots < i_k < j_1 < \cdots < j_l$, $F = \{i_1, \dots, i_k\}$, and $H = \{j_1, \dots, j_l\}$. Then

$$b_F \in \mathrm{FS}(b_1, \dots, b_{i_k}) \quad \text{and} \quad b_H = b_{j_1} + \cdots + b_{j_l}.$$

Computing the latter sum from right to left, we see that

$$b_H \in D_{j_1} \subseteq D_{i_k+1} \subseteq G(\mathrm{FS}(b_1, \dots, b_{i_k})).$$

It follows that the elements b_F and b_H are distinct, and the edge $\{b_F, b_H\}$ is green. ∎

To gain some intuition on Theorem 3.6, we provide several simple examples. They can also be established via somewhat more direct arguments.

EXAMPLE 3.7. Let S be a semigroup. Let $a_1, a_2, \dots$ be a sequence in S, and $A_1 \supseteq A_2 \supseteq \cdots$ be a free idempotent chain with $A_n \subseteq \mathrm{FS}(a_n, a_{n+1}, \dots)$ for all n. For each finite coloring of the sets S and $[S]^2$, there are elements $b_1 \in A_1, b_2 \in A_2, \dots$ such that:

(1) The sequence $b_1, b_2, \dots$ is a proper sumsequence of $a_1, a_2, \dots$.
(2) The set $\mathrm{FS}(b_1, b_2, \dots)$ is monochromatic.
(3) The sum graph of $b_1, b_2, \dots$ is monochromatic.

Proof. By Lemma 3.4, with the trivial superfilter $\mathcal{A} = [S]^\infty$, the filter $\mathcal{F}$ on S generated by the sets $A_1, A_2, \dots$ is a free idempotent filter. By Lemma 2.5, the superfilter $\mathcal{F}^+$ is also idempotent. By Example 3.1, Bob has a winning strategy in the game $\mathsf{G}_1(\mathcal{F}^+, \mathcal{F}^+)$. Since $\mathcal{F} \subseteq \mathcal{F}^+$, Theorem 3.6 applies with $\mathcal{A} = \mathcal{B} = \mathcal{F}^+$. ∎

In most semigroups S one encounters, left addition is at most finite-to-one. In this case, the superfilter $[S]^\infty$ is translation-invariant; in particular, idempotent. In this case, the proof of Example 3.7 reduces to one short sentence: Apply Theorem 3.6 with $\mathcal{A} = \mathcal{B} = [S]^\infty$.

The Milliken–Taylor Theorem in arbitrary semigroups (Theorem 1.6) follows from Example 3.7, by Example 3.3.

EXAMPLE 3.8. Let $\mathcal{F}_1, \mathcal{F}_2, \dots \subseteq \mathrm{Fin}(\mathbb{N})$, and $A \subseteq \mathbb{N}$. Assume that every cofinite subset of A contains a member from each family $\mathcal{F}_n$. For each finite coloring of the sets $\mathrm{Fin}(\mathbb{N})$ and $[\mathrm{Fin}(\mathbb{N})]^2$, there are nonempty finite subsets $F_1 < F_2 < \cdots$ of A such that:

(1) Each set F_n contains some element of $\mathcal{F}_n$.
(2) All nonempty finite unions H of sets F_n have the same color.
(3) All sets $\{H_1, H_2\}$, for $H_1 < H_2$ nonempty finite unions of sets F_n, have the same color.

Proof. We work with the semigroup $\mathrm{Fin}(A)$ of all nonempty finite subsets of A. Enumerate $A = \{a_1, a_2, \dots\}$. For each n, let

$$\begin{aligned} A_n &:= \bigl\{F \in \mathrm{Fin}(\{a_n, a_{n+1}, \dots\}) : \exists H \in \mathcal{F}_n, H \subseteq F\bigr\} \\ &\subseteq \mathrm{FS}(\{a_n\}, \{a_{n+1}\}, \dots). \end{aligned}$$

Then $\bigcap_n A_n = \emptyset$. Every set A_n is a subsemigroup of S. Thus, the sequence $A_1 \supseteq A_2 \supseteq \cdots$ is a free idempotent chain. Apply Example 3.7. ∎

Example 3.9. Let $A \subseteq \mathbb{N}$ be a set containing arbitrarily long arithmetic progressions. For each finite coloring of the sets $\mathrm{Fin}(\mathbb{N})$ and $[\mathrm{Fin}(\mathbb{N})]^2$, there are nonempty finite subsets $F_1 < F_2 < \cdots$ of A such that:

(1) The set $\bigcup_n F_n$ contains arbitrarily long arithmetic progressions.
(2) All nonempty finite unions H of sets F_n have the same color.
(3) All sets $\{H_1, H_2\}$, for $H_1 < H_2$ nonempty finite unions of sets F_n, have the same color.

Additional examples are provided by any notion that is captured by finite sets, e.g., entries of solutions of homogeneous systems of equations, and entries of image vectors of matrices. The *upper density* of a set $A \subseteq \mathbb{N}$ is the real number $\limsup_n |A \cap \{1, \dots, n\}|/n$.

Example 3.10. Let $A \subseteq \mathbb{N}$ be a set of upper density δ. For each finite coloring of the sets $\mathrm{Fin}(\mathbb{N})$ and $[\mathrm{Fin}(\mathbb{N})]^2$, there are nonempty finite subsets $F_1 < F_2 < \cdots$ of A such that:

(1) The set $\bigcup_n F_n$ has upper density δ.
(2) All nonempty finite unions H of sets F_n have the same color.
(3) All sets $\{H_1, H_2\}$, for $H_1 < H_2$ nonempty finite unions of sets F_n, have the same color.

Proof. The upper density of a set does not change on removing finitely many elements from that set. Take a sequence $\delta_1, \delta_2, \dots$ increasing to δ. For each n, let $\mathcal{F}_n := \{F \in \mathrm{Fin}(A) : |F|/\max F > \delta_n\}$. Apply Example 3.8. ∎

An analogous assertion also holds for the so-called *Banach density*.

4. Menger spaces. A topological space X is a *Menger space* if, for each sequence $\mathcal{U}_1, \mathcal{U}_2, \dots$ of open covers of X, there are finite subsets $\mathcal{F}_1 \subseteq \mathcal{U}_1$, $\mathcal{F}_2 \subseteq \mathcal{U}_2, \dots$ such that the sets $\bigcup \mathcal{F}_1, \bigcup \mathcal{F}_2, \dots$ form an open cover of X. A property introduced by Menger [15] was proved equivalent to this covering property by Hurewicz [12]. Thus, every compact space has Menger's property, and every space with Menger's property is a Lindelöf space, that is, one where every open cover has a countable subcover.

Every compact space is a Menger space, and every countable union of Menger spaces is Menger. However, even among subsets of the real line there are large families of Menger spaces that are substantially different from countable unions of compact spaces (e.g., [30, 31]). Menger's property, which is central in the recent theory of *selection principles* (see [19] and references therein), found applications to seemingly unrelated notions in set-theoretic and general topology and in real analysis.

Remark 4.1. We mention two examples illustrating the importance of Menger's property in general and set-theoretic topology. This remark is independent of the remainder of the present paper, and we refer the interested reader to the cited references for definitions.

One of the major problems in set-theoretic topology asks whether every regular Lindelöf space is a D-space. In the realm of Hausdorff spaces, the problem was answered in the negative [27]. It turned out that all *Menger* spaces are D-spaces [1]. Menger's property still yields the most general natural class of spaces for which a positive answer to the D-space problem is known.

In a series of papers (see [8, 9] and references therein), a number of authors have studied an important type of filters with a property introduced by Canjar. This property is related to the theory of forcing: A filter has *Canjar's property* if the Mathias forcing notion associated to the filter does not add dominating reals. It turned out that a filter has Canjar's property if and only if it is Menger in the standard Cantor space topology [4]. This made a wide body of knowledge on Menger's property applicable to Canjar filters. In particular, a number of earlier results follow immediately from this characterization.

Following Hurewicz [12], we restrict Menger's property to *countable* open covers. For Lindelöf spaces, the two variations of Menger's property coincide, but otherwise the results obtained are more general. This will be of importance to some applications at the end of this paper.

DEFINITION 4.2. Let X be a topological space. A countable family $\mathcal{U}$ of subsets of X is an *ascending cover* of X if it is a cover of X and there is an enumeration $\mathcal{U} = \{V_1, V_2, \ldots\}$ such that $V_1 \subsetneq V_2 \subsetneq \cdots$. Let $\mathrm{Asc}(X)$ be the family of open covers of X that *contain* an ascending cover of X.

We consider the family $P(X)$ of subsets of a set X as a semigroup with the addition operator $\cup$. Thus, for a family $\mathcal{U} \subseteq P(X)$, the set $\mathrm{FS}(\mathcal{U})$ consists of all finite unions of members of $\mathcal{U}$. Only covers with no finite subcover constitute a challenge to Menger's property.

LEMMA 4.3. *Let X be a topological space. For each countable open cover $\mathcal{U}$ with no finite subcover, we have* $\mathrm{FS}(\mathcal{U}) \in \mathrm{Asc}(X)$. ∎

For a topological space X, let $\mathrm{O}(X)$ be the family of countable open covers of X. A cover of X is *point-infinite* if every point of X is contained in infinitely many members of the cover. Let $\Lambda(X)$ be the family of countable open point-infinite covers of X. The proof of [22, Corollary 6] establishes, in fact, that $\mathsf{S}_1(\mathrm{Asc}(X), \Lambda(X))$ holds whenever $\mathsf{S}_1(\mathrm{Asc}(X), \mathrm{O}(X))$ does.

COROLLARY 4.4 (Folklore). *A topological space X is Menger if and only if $\mathsf{S}_1(\mathrm{Asc}(X), \Lambda(X))$ holds.* ∎

Using a game-theoretic theorem of Hurewicz, Scheepers proved in [23] that a space X is Menger if and only if Alice does not have a winning strategy in the game $\mathsf{G}_{\mathrm{fin}}(\Lambda(X), \Lambda(X))$, a variation of $\mathsf{G}_1(\Lambda(X), \Lambda(X))$ where Bob is allowed to choose any finite number of elements in each turn. Scheepers's Theorem is used in the following proof.

Proposition 4.5. *A topological space X is Menger if and only if Alice does not have a winning strategy in the game $\mathsf{G}_1(\mathrm{Asc}(X), \Lambda(X))$.*

Proof. ($\Leftarrow$) If Alice does not have a winning strategy in $\mathsf{G}_1(\mathrm{Asc}(X), \Lambda(X))$, then $\mathsf{S}_1(\mathrm{Asc}(X), \Lambda(X))$ holds. Then X is a Menger space.

($\Rightarrow$) Assume that Alice has a winning strategy in $\mathsf{G}_1(\mathrm{Asc}(X), \Lambda(X))$. Using this strategy, define a strategy for Alice in $\mathsf{G}_{\mathrm{fin}}(\mathrm{Asc}(X), \Lambda(X))$, as follows. In the nth inning, Alice's strategy proposes a cover containing an ascending cover. Alice thins out this cover to make it ascending, and then removes from it the finitely many elements chosen by Bob in the earlier innings. This can only make Bob's task harder. If Bob picks a finite subset $\mathcal{F}_n$ of this ascending cover, Alice takes the largest set chosen by Bob, B_n, and applies her original strategy, pretending that Bob chose only this set.

Assume that Bob won a play $(\mathcal{U}_1, \mathcal{F}_1, \mathcal{U}_2, \mathcal{F}_2, \dots)$ of $\mathsf{G}_{\mathrm{fin}}(\mathrm{Asc}(X), \Lambda(X))$. Then $\bigcup_n \mathcal{F}_n$ is a point-infinite cover of X. Since the sets $\mathcal{F}_n$ are disjoint, the set $\{B_1, B_2, \dots\}$ is also a point-infinite cover of X, and we obtain a play of $\mathsf{G}_1(\mathrm{Asc}(X), \Lambda(X))$ that is won by Bob, a contradiction. Thus, Alice has a winning strategy in $\mathsf{G}_{\mathrm{fin}}(\mathrm{Asc}(X), \Lambda(X))$. Since $\mathrm{Asc}(X) \subseteq \Lambda(X)$, Alice has a winning strategy in $\mathsf{G}_{\mathrm{fin}}(\Lambda(X), \Lambda(X))$. By Scheepers's Theorem, the space X is not Menger. ■

With results proved thus far, we are ready to prove our main theorem. For σ-compact spaces, that is, spaces that are countable unions of compact sets, this theorem can be deduced directly from the Milliken–Taylor Theorem. The case of general Menger spaces, however, cannot, and forms the core of the proof.

Theorem 4.6. *Let (X, τ) be a Menger space, and $\mathcal{U}_1 \supseteq \mathcal{U}_2 \supseteq \cdots$ be countable point-infinite open covers of X with no finite subcover. For each finite coloring of the sets τ and $[\tau]^2$, there are nonempty disjoint finite sets $\mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \dots$ such that the sets $V_n := \bigcup \mathcal{F}_n$ for $n \in \mathbb{N}$ have the following properties:*

1. *The family $\{V_1, V_2, \dots\}$ is a point-infinite cover of X.*
2. *The sets $\bigcup_{n\in F} V_n$ and $\bigcup_{n\in H} V_n$ for nonempty finite sets $F < H$ are distinct.*
3. *All sets $\bigcup_{n\in F} V_n$ for nonempty finite sets $F \subseteq \mathbb{N}$ have the same color.*
4. *All sets $\{\bigcup_{n\in F} V_n, \bigcup_{n\in H} V_n\}$ for nonempty finite sets $F < H$ have the same color.*

Moreover, if $\mathcal{U}_1 = \{U_1, U_2, \dots\}$, we may require that the sets $F_n := \{m : U_m \in \mathcal{F}_n\}$ satisfy $F_1 < F_2 < \cdots$.

Proof. Enumerate $\mathcal{U}_1 = \{U_1, U_2, \dots\}$. Consider the semigroup $(\tau, \cup)$. We will work inside its subsemigroup $S := \mathrm{FS}(U_1, U_2, \dots)$. Let

$$\mathcal{A} := \{A \subseteq S : A \in \mathrm{Asc}(X)\}.$$

The family $\mathcal{A}$ is a superfilter: Since $\mathcal{U}_1$ has no finite subcover, the sequence $U_1, U_1 \cup U_2, \dots$ has an ascending subsequence. Thus, $\{U_1, U_1 \cup U_2, \dots\} \in \mathcal{A}$. If $A \cup B \in \mathcal{A}$, then $A \cup B$ contains an ascending cover $V_1 \subsetneq V_2 \subsetneq \cdots$, and A or B must contain a subsequence of $V_1, V_2, \dots$. Thus, $A \in \mathcal{A}$ or $B \in \mathcal{A}$. The superfilter $\mathcal{A}$ is translation-invariant. In particular, $\mathcal{A}$ is idempotent.

For each n, using the fact that $\mathcal{U}_1$ has no finite subcover, fix an element $x_n \in X \setminus \bigcup_{i=1}^{n} U_i$. For each n, let

$$\mathcal{V}_n := \bigl\{V \in \mathrm{FS}(\{U_m \in \mathcal{U}_n : m \geq n\}) : x_1, \dots, x_{n-1} \in V\bigr\}.$$

Note that $\mathcal{V}_1 = S$, and $\mathcal{V}_n \subseteq \mathrm{FS}(U_n, U_{n+1}, \dots)$ for all n. For each n, the set $\{U_m \in \mathcal{U}_n : m \geq n\}$, being a cofinite subset of the point-infinite cover $\mathcal{U}_n$, is a (point-infinite) cover of X. Since $\mathcal{U}_n$ has no finite subcover, we see that $\mathcal{V}_n \in \mathrm{Asc}(X)$. In particular, the sets $\mathcal{V}_n$ are infinite. We deduce that $\mathcal{V}_1 \supseteq \mathcal{V}_2 \supseteq \cdots$, and $\bigcap_n \mathcal{V}_n = \emptyset$. For each n, $\mathcal{V}_n$ is a subsemigroup of S. Thus, $\mathcal{V}_1, \mathcal{V}_2, \dots$ is a free idempotent chain in $\mathcal{A}$.

By Proposition 4.5, Alice does not have a winning strategy in the game $\mathsf{G}_1(\mathcal{A}, \Lambda(X))$. By Theorem 3.6, for each finite coloring of the sets S and $[S]^2$, there are elements $V_1 \in \mathcal{V}_1, V_2 \in \mathcal{V}_2, \dots$ such that:

- The set $\{V_1, V_2, \dots\}$ is in $\Lambda(X)$.
- The sequence $V_1, V_2, \dots$ is a proper sumsequence of $U_1, U_2, \dots$.
- The set $\mathrm{FS}(V_1, V_2, \dots)$ is monochromatic.
- The sum graph of $V_1, V_2, \dots$ is monochromatic.

The last assertion in the theorem is clear from the proof of Theorem 3.6. ∎

The assumption in Theorem 4.6 that the space is Menger is necessary. It is proved in [22] that being a Menger space is equivalent to the following property: For each descending sequence $\mathcal{U}_1 \supseteq \mathcal{U}_2 \supseteq \cdots$ of countable point-infinite open covers of X with no finite subcover, there are nonempty finite sets $\mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \dots$ such that the family $\{\bigcup \mathcal{F}_n : n \in \mathbb{N}\}$ is a cover of X.

The following example shows that the Milliken–Taylor Theorem, and thus Hindman's Theorem, is an instance of Theorem 4.6 where Menger's property is trivial: a countable, discrete space.

Example 4.7. Consider Theorem 1.5. Let X be the set $\mathrm{Fin}(\mathbb{N})$ with the discrete topology. Since the space X is countable, it is a Menger space.

For each n, let $O_n := \{F \in X : n \notin F\}$. The family $\{O_1, O_2, \dots\}$ is a point-infinite open cover of X with no finite subcover. According to our conventions, for a set $F \in \mathrm{Fin}(\mathbb{N})$ we have $O_F = \bigcup_{n \in F} O_n$.

Let $S := \mathrm{FS}(O_1, O_2, \ldots)$. Then S is a semigroup, and the map $\mathrm{Fin}(\mathbb{N}) \to S$ defined by $F \mapsto O_F$ is a semigroup isomorphism. Thus, a finite coloring of the set $[\mathrm{Fin}(\mathbb{N})]^2$ may be viewed as a finite coloring of the set $[S]^2$. Let $F_1 < F_2 < \cdots$ be nonempty finite sets such that the sets $V_n := O_{F_n}$ satisfy Theorem 4.6(4), and the sets F_n are as requested in Theorem 1.5.

The deduction of the classical theorems in Example 4.7 uses a twist: It would have been more natural to consider the cover of $\mathbb{N}$ by singletons, but there are 2-colorings of $\mathrm{Fin}(\mathbb{N})$ with no monochromatic cover of $\mathbb{N}$ by disjoint finite sets.

According to Example 4.7, the Milliken–Taylor (or Hindman) Theorem may be viewed as a theorem about countable open covers of countable sets, and Theorem 4.6 may be viewed as an extension of these theorems from countable spaces to Menger spaces of arbitrary cardinality. It is illustrative to compare this interpretation with Fernández Bretón's impossibility result [6]: for every set S, there is a 2-coloring of the semigroup $\mathrm{Fin}(S)$ of finite subsets of S such that no uncountable subsemigroup of $\mathrm{Fin}(S)$ is monochromatic. This demonstrates that any improvement over Hindman's Theorem must be on the qualitative side. In our case, we color a countable object induced by a countable cover; it is the covered space that is uncountable.

5. Richer covers. Let X be a topological space, and $\mathcal{A}$ and $\mathcal{B}$ be families of covers of X. Let $\mathsf{U}_{\mathrm{fin}}(\mathcal{A}, \mathcal{B})$ be the property that, for covers $\mathcal{U}_1, \mathcal{U}_2, \ldots \in \mathcal{A}$ with no finite subcover, there are finite sets $\mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \ldots$ such that $\{\bigcup \mathcal{F}_1, \bigcup \mathcal{F}_2, \ldots\} \in \mathcal{B}$.

Menger's covering property is the same as $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}(X), \mathrm{O}(X))$. A number of important covering properties are of the form $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}(X), \mathcal{B})$. Some examples are provided in the survey [19] and in the references therein. By Lemma 4.3, we have the following observation.

Proposition 5.1. *Let X be a topological space, and $\mathcal{B}$ be a family of covers of X. The assertions $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}(X), \mathcal{B})$ and $\mathsf{S}_1(\mathrm{Asc}(X), \mathcal{B})$ are equivalent.* ∎

Let $\Omega(X)$ be the family of open covers $\mathcal{U}$ of X such that $X \notin \mathcal{U}$ and every finite subset of X is contained in some member of the cover. This family, introduced by Gerlits and Nagy [7], is central to the study of local properties in function spaces. The property $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}(X), \Omega(X))$ was first considered by Scheepers [22]. By the requirement that X does not belong to any member of $\Omega(X)$, the members of $\Omega(X)$ are infinite. Moreover, $\Omega(X)$ is a superfilter on the topology τ of X. If a cover $\mathcal{U} \in \Omega(X)$ is finer than another open cover $\mathcal{V}$ (in the sense that every member of $\mathcal{U}$ is contained in some member of $\mathcal{V}$) with $X \notin \mathcal{V}$, then $\mathcal{V} \in \Omega(X)$. To cover additional important cases, we generalize these properties.

DEFINITION 5.2. Let (X,τ) be a topological space. A family $\mathcal{B}$ of open covers of X is *regular* if it has the following properties:

(1) Whenever $\mathcal{U}\cup\mathcal{V}\in\mathcal{B}$, we have $\mathcal{U}\in\mathcal{B}$ or $\mathcal{V}\in\mathcal{B}$.
(2) For each cover $\mathcal{U}\in\mathcal{B}$ and each finite-to-one function $f\colon\mathcal{U}\to\tau\setminus\{X\}$ with $U\subseteq f(U)$ for all $U\in\mathcal{U}$, the image of f is in $\mathcal{B}$.

Most of the important families of rich covers are regular.

EXAMPLE 5.3. Let X be a topological space. The family $\Omega(X)$ is regular. The family $\Lambda(X)$ satisfies the second, but not the first, regularity condition. Let $\Gamma(X)$ be the family of infinite open covers of X such that each point in X is contained in all but finitely many members of the cover. The family $\Gamma(X)$ is regular. The property $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}(X),\Gamma(X))$ was introduced by Hurewicz [12]. Another well-studied regular family, denoted $\mathrm{T}^*(X)$, was introduced in [29].

In the next proof, we use the following observation. It extends, by induction, to any finite number of ascending covers.

LEMMA 5.4. *Let $\{U_1,U_2,\dots\}$ and $\{V_1,V_2,\dots\}$ be ascending covers of a set X, enumerated as such. Then the set $\{U_1\cap V_1,U_2\cap V_2,\dots\}$ is an ascending cover of X.* ∎

THEOREM 5.5. *Let (X,τ) be a topological space, and $\mathcal{B}$ be a regular family of open covers of X. The following assertions are equivalent:*

(1) $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}(X),\mathcal{B})$.
(2) $\mathsf{S}_1(\mathrm{Asc}(X),\mathcal{B})$.
(3) *Alice does not have a winning strategy in the game* $\mathsf{G}_1(\mathrm{Asc}(X),\mathcal{B})$.
(4) *Alice does not have a winning strategy in the game associated to* $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}(X),\mathcal{B})$.

Proof. Proposition 5.1 asserts the equivalence of (1) and (2). It is immediate that (4) implies (1).

(3)⇒(4). Assume that Alice has a winning strategy in the game associated to $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}(X),\mathcal{B})$. By the definition of the selection principle $\mathsf{U}_{\mathrm{fin}}(\mathcal{A},\mathcal{B})$, Alice's covers must not have finite subcovers. By taking finite unions, turn every cover in Alice's strategy into an ascending one. This only restricts the possible moves of Bob, and turns them into moves in the game $\mathsf{G}_1(\mathrm{Asc}(X),\mathcal{B})$. Thus, we obtain a winning strategy for Alice in the latter game.

(2)⇒(3). Assume that Alice has a winning strategy in $\mathsf{G}_1(\mathrm{Asc}(X),\mathcal{B})$. We encode this strategy as follows. Let $\mathcal{U}=\{U_1,U_2,\dots\}$ be Alice's first move. For each choice U_{m_1} of Bob, let $\mathcal{U}_{m_1}=\{U_1^{m_1},U_2^{m_1},\dots\}$ be Alice's next move. For each choice $U_{m_2}^{m_1}$ of Bob, let $\mathcal{U}_{m_1,m_2}=\{U_1^{m_1,m_2},U_2^{m_1,m_2},\dots\}$ be Alice's next move, etc. Thus, for each sequence $m_1,\dots,m_k\in\mathbb{N}$ we have a cover $\mathcal{U}_{m_1,\dots,m_k}=\{U_1^{m_1,\dots,m_k},U_2^{m_1,\dots,m_k},\dots\}\in\mathrm{Asc}(X)$.

Thinning out the covers Alice plays will only restrict Bob's moves. Thus, we may assume that Alice plays ascending covers, and that no cover played by Alice contains any of the finitely many elements played by Bob in the earlier innings. For a natural number n, let $\{1,\dots,n\}^{\le n} := \bigcup_{i=0}^{n}\{1,\dots,n\}^i$, the set of all sequences of length at most n taking values in $\{1,\dots,n\}$, where the only sequence in $\{1,\dots,n\}^0$ is the empty sequence ε. We define $U^{\varepsilon}_m := U_m$ for all m. For each n, set

$$\mathcal{V}_n := \Big\{ \bigcap_{\sigma\in\{1,\dots,n\}^{\le n}} U^{\sigma}_1, \bigcap_{\sigma\in\{1,\dots,n\}^{\le n}} U^{\sigma}_2, \dots \Big\}.$$

Then $\mathcal{V}_n$ is an ascending cover of X. By the property $\mathsf{S}_1(\mathrm{Asc}(X),\mathcal{B})$, there are elements $V_1\in\mathcal{V}_1, V_2\in\mathcal{V}_2,\dots$ such that $\{V_1,V_2,\dots\}\in\mathcal{B}$.

The cover $\{V_1,V_2,\dots\}$ refines $\mathcal{U}$. Since $\mathcal{U}$ has no finite subcover, the set $\{V_1,V_2,\dots\}$ is infinite. We construct two parallel plays,

$$(\mathcal{U}, U_{m_1}, \mathcal{U}_{m_1}, U^{m_1}_{m_3}, \mathcal{U}_{m_1,m_3},\dots) \quad\text{and}\quad (\mathcal{U}, U_{m_2}, \mathcal{U}_{m_2}, U^{m_2}_{m_4}, \mathcal{U}_{m_2,m_4},\dots),$$

according to Alice's strategy. We use the fact that Alice's covers are ascending.

- Pick a natural number $m_1 > 1$ such that
$$V_1 \subseteq U_{m_1}\in\mathcal{U}, \qquad \{V_2,\dots,V_{m_1}\}\setminus\{V_1\}\neq\emptyset.$$
- Each of the sets $V_2,\dots,V_{m_1}$ is contained in some member of the cover $\mathcal{U}$. Pick a natural number $m_2 > m_1$ such that $U_{m_2}\neq U_{m_1}$ and
$$V_2\cup\dots\cup V_{m_1}\subseteq U_{m_2}\in\mathcal{U}, \qquad \{V_{m_1+1},\dots,V_{m_2}\}\setminus\{V_1,\dots,V_{m_1}\}\neq\emptyset.$$
- For $n=3,4,\dots$:
  - If n is odd: Each of the sets $V_{m_{n-2}+1},\dots,V_{m_{n-1}}$ is contained in some member of the cover $\mathcal{U}_{m_1,m_3,\dots,m_{n-2}}$. Pick a natural number $m_n > m_{n-1}$ such that the set $U := U^{m_1,m_3,\dots,m_{n-2}}_{m_n}$ is distinct from all sets picked earlier and
$$V_{m_{n-2}+1}\cup\dots\cup V_{m_{n-1}} \subseteq U \in \mathcal{U}_{m_1,m_3,\dots,m_{n-2}},$$
$$\{V_{m_{n-1}+1},\dots,V_{m_n}\}\setminus\{V_1,\dots,V_{m_{n-1}}\}\neq\emptyset.$$
  - If n is even: Each of the sets $V_{m_{n-2}+1},\dots,V_{m_{n-1}}$ is contained in some member of the cover $\mathcal{U}_{m_2,m_4,\dots,m_{n-2}}$. Pick a natural number $m_n > m_{n-1}$ such that the set $U := U^{m_2,m_4,\dots,m_{n-2}}_{m_n}$ is distinct from all sets picked earlier and
$$V_{m_{n-2}+1}\cup\dots\cup V_{m_{n-1}} \subseteq U \in \mathcal{U}_{m_2,m_4,\dots,m_{n-2}},$$
$$\{V_{m_{n-1}+1},\dots,V_{m_n}\}\setminus\{V_1,\dots,V_{m_{n-1}}\}\neq\emptyset.$$

Define a function

$$f\colon \{V_1,V_2,\dots\}\to\{U_{m_1},U_{m_2},U^{m_1}_{m_3},U^{m_2}_{m_4},\dots\}$$

as follows:

- Map V_1 to U_{m_1}.
- Map each element of $\{V_2, \dots, V_{m_1}\} \setminus \{V_1\}$ to U_{m_2}.
- For $n = 3, 4, \dots$ map each element of the set $\{V_{m_{n-2}+1}, \dots, V_{m_{n-1}}\} \setminus \{V_1, \dots, V_{m_{n-2}}\}$ to $U_{m_n}^{m_1,m_3,\dots,m_{n-2}}$ if n is odd, and to $U_{m_n}^{m_2,m_4,\dots,m_{n-2}}$ if n is even.

The function f is as needed in Definition 5.2(2), and is surjective. Since the family $\{V_1, V_2, \dots\}$ is in $\mathcal{B}$ and $\mathcal{B}$ is regular, the set $\{U_{m_1}, U_{m_2}, U_{m_3}^{m_1}, U_{m_4}^{m_2}, \dots\}$ is in $\mathcal{B}$. By Definition 5.2(1), one of the families $\{U_{m_1}, U_{m_3}^{m_1}, \dots\}$ or $\{U_{m_2}, U_{m_4}^{m_2}, \dots\}$ is in $\mathcal{B}$. It follows that Bob wins one of these two games against Alice's winning strategy, a contradiction. ∎

THEOREM 5.6. *Let (X, τ) be a topological space, and $\mathcal{B}$ be a regular family of open covers of X (e.g., $\Omega(X)$, $\mathrm{T}^*(X)$, or $\Gamma(X)$). Assume that $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}(X), \mathcal{B})$ holds. Let $\mathcal{U}_1 \supseteq \mathcal{U}_2 \supseteq \cdots$ be countable point-infinite open covers of X with no finite subcover. For each finite coloring of the sets τ and $[\tau]^2$, there are nonempty disjoint finite sets $\mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \dots$ such that the sets $V_n := \bigcup \mathcal{F}_n$ for $n \in \mathbb{N}$ have the following properties:*

(1) *The family $\{V_1, V_2, \dots\}$ is in $\mathcal{B}$.*
(2) *The sets $\bigcup_{n\in F} V_n$ and $\bigcup_{n\in H} V_n$ for nonempty finite sets $F < H$ are distinct.*
(3) *All sets $\bigcup_{n\in F} V_n$ for nonempty finite sets $F \subseteq \mathbb{N}$ have the same color.*
(4) *All sets $\{\bigcup_{n\in F} V_n, \bigcup_{n\in H} V_n\}$ for nonempty finite sets $F < H$ have the same color.*

Moreover, if $\mathcal{U}_1 = \{U_1, U_2, \dots\}$, we may require that the sets $F_n := \{m : U_m \in \mathcal{F}_n\}$ satisfy $F_1 < F_2 < \cdots$.

Proof. The proof is identical to that of Theorem 4.6, upon replacing $\Lambda(X)$ by $\mathcal{B}$ and using Theorem 5.5 instead of Proposition 4.5. ∎

In all of our theorems, the converse implications also hold.

PROPOSITION 5.7. *Let X be a topological space, and $\mathcal{B}$ be a regular family of open covers of X. Assume that, for each descending sequence $\mathcal{U}_1 \supseteq \mathcal{U}_2 \supseteq \cdots$ of countable point-infinite open covers of X with no finite subcover, there are nonempty disjoint finite sets $\mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \dots$ with $\{\bigcup \mathcal{F}_1, \bigcup \mathcal{F}_2, \dots\} \in \mathcal{B}$. Then $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}(X), \mathcal{B})$ holds.*

Proof. Let $\beth(\mathcal{B})$ be the family of open covers $\mathcal{U}$ of X with no finite subcover, such that there are disjoint finite sets $\mathcal{F}_1, \mathcal{F}_2, \dots \subseteq \mathcal{U}$ for which $\{\bigcup \mathcal{F}_1, \bigcup \mathcal{F}_2, \dots\} \in \mathcal{B}$. By the first regularity property of $\mathcal{B}$, we have $\Lambda(X) \supseteq \beth(\mathcal{B})$, and by the premise of the proposition, $\Lambda(X) \subseteq \beth(\mathcal{B})$. By Scheepers's Theorem, quoted after the proof of Theorem 4.6, the space X is Menger. By [20, Corollary 10 and Lemma 11], $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}(X), \mathcal{B})$ holds. ∎

6. Covers by more general sets

6.1. Borel covers. Consider the variation of Menger's property where covers by *Borel sets* are considered. Here, the restriction to countable covers is necessary to make the property nontrivial ([3]). This property has its own history and applications (see, e.g., [26] and the papers citing it). As a rule, the results known for Menger's property extend to its Borel version [26], and thus Theorem 4.6 and its consequences also hold with "open" replaced by "Borel". The same assertion holds for the Borel versions of the other covering properties considered above.

In addition to open or Borel, one may consider other types of sets. As long as these types are preserved by the basic operations used in the proof (mainly, finite intersections), the results obtained here apply to countable covers by sets of the type considered.

6.2. A combinatorial theorem. Order the set $\mathbb{N}^{\mathbb{N}}$ by coordinatewise comparison: $f \le g$ if $f(n) \le g(n)$ for all n. Let $\mathfrak{d}$ be the minimal cardinality of a *dominating family* $D \subseteq \mathbb{N}^{\mathbb{N}}$, that is, such that for each function $f \in \mathbb{N}^{\mathbb{N}}$ there is a function $g \in D$ with $f \le g$. It is known that $\aleph_1 \le \mathfrak{d} \le 2^{\aleph_0}$, but it is consistent that the cardinal $\mathfrak{d}$ is strictly greater than $\aleph_1$ (more details are available in [3]). Let $D \subseteq \mathbb{N}^{\mathbb{N}}$ be a dominating family of cardinality $\mathfrak{d}$. Then the property $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}(D), \mathrm{O}(D))$ fails [13]. On the other hand, since we consider countable covers only, the property $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}(X), \mathrm{O}(X))$ holds for spaces X of cardinality smaller than $\mathfrak{d}$ [13]. Thus, thinking of a cardinal number κ as a discrete space of cardinality κ, the following assertions are equivalent:

(1) $\kappa < \mathfrak{d}$.
(2) $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}(\kappa), \Omega(\kappa))$ holds.
(3) $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}(\kappa), \mathrm{O}(\kappa))$ holds.

By Theorem 5.6, we have the following purely combinatorial result. In the case $\kappa = \aleph_0$, this is a straightforward consequence of the Milliken–Taylor Theorem. Uncountable cardinals necessitate the application of the main theorems of the present paper.

THEOREM 6.1. *Let κ be a cardinal smaller than $\mathfrak{d}$. Let $\mathcal{U}_1 \supseteq \mathcal{U}_2 \supseteq \cdots$ be countable point-infinite covers of κ with no finite subcover. For each finite coloring of the sets $P(\kappa)$ and $[P(\kappa)]^2$, there are nonempty disjoint finite sets $\mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \ldots$ such that the sets $A_n := \bigcup \mathcal{F}_n$ for $n \in \mathbb{N}$ have the following properties:*

(1) *Every finite subset of κ is contained in some A_n.*

([3]) Otherwise, the space could be covered by singletons, and then being Menger would be the same as being countable.

(2) *The sets* $\bigcup_{n\in F} A_n$ *and* $\bigcup_{n\in H} A_n$ *for nonempty finite sets* $F < H$ *are distinct.*
(3) *All sets* $\bigcup_{n\in F} A_n$ *for nonempty finite sets* $F \subseteq \mathbb{N}$ *have the same color.*
(4) *All sets* $\{\bigcup_{n\in F} A_n, \bigcup_{n\in H} A_n\}$ *for nonempty finite sets* $F < H$ *have the same color.*

Moreover, if $\mathcal{U}_1 = \{B_1, B_2, \ldots\}$, *we may require that the sets* $F_n := \{m : B_m \in \mathcal{F}_n\}$ *satisfy* $F_1 < F_2 < \cdots$. ■

7. Comments

7.1. Higher dimensions. Our theorems also hold in dimensions larger than 2, with minor modifications in the proofs. For a natural number d, let $[S]^d$ be the family of all d-element subsets of S. We state the d-dimensional versions of Theorems 3.6 and 4.6. For brevity, the last part of Theorem 7.2 is omitted.

THEOREM 7.1. *Let* S *be a semigroup, and* d *be a natural number. Let* $\mathcal{A}$ *be an idempotent superfilter on* S, *and* $\mathcal{B}$ *be a family of subsets of* S *such that Alice does not have a winning strategy in the game* $\mathsf{G}_1(\mathcal{A}, \mathcal{B})$. *Let* $a_1, a_2, \ldots$ *be a sequence in* S, *and* $A_1 \supseteq A_2 \supseteq \cdots$ *be a free idempotent chain in* $\mathcal{A}$ *with* $A_n \subseteq \mathrm{FS}(a_n, a_{n+1}, \ldots)$ *for all* n. *For each finite coloring of the set* $[S]^d$, *there are elements* $b_1 \in A_1, b_2 \in A_2, \ldots$ *such that:*

(1) *The set* $\{b_1, b_2, \ldots\}$ *is in* $\mathcal{B}$.
(2) *The sequence* $b_1, b_2, \ldots$ *is a proper sumsequence of* $a_1, a_2, \ldots$.
(3) *The set* $\{\{b_{F_1}, \ldots, b_{F_d}\} : F_1, \ldots, F_d \in \mathrm{Fin}(\mathbb{N}),\ F_1 < \cdots < F_d\}$ *is monochromatic.*

THEOREM 7.2. *Let* (X, τ) *be a Menger space, and* d *be a natural number. For each descending sequence* $\mathcal{U}_1 \supseteq \mathcal{U}_2 \supseteq \cdots$ *of countable point-infinite open covers of* X *with no finite subcover, and each finite coloring of the set* $[\tau]^d$, *there are nonempty disjoint finite sets* $\mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \ldots$ *such that the sets* $V_n := \bigcup \mathcal{F}_n$ *for* $n \in \mathbb{N}$ *have the following properties:*

(1) *The family* $\{V_1, V_2, \ldots\}$ *is a point-infinite cover of* X.
(2) *The sets* $\bigcup_{n\in F} V_n$ *and* $\bigcup_{n\in H} V_n$ *for nonempty finite sets* $F < H$ *are distinct.*
(3) *All sets* $\{\bigcup_{n\in F_1} V_n, \ldots, \bigcup_{n\in F_d} V_n\}$ *for nonempty finite sets* $F_1 < \cdots < F_d$ *have the same color.*

The d-dimensional versions of Theorems 5.6 and 6.1 are similar.

7.2. Proper sequences. We have taken the approach of proper sequences, or having proper sumsequences, to avoid pathological cases in theorems of Milliken–Taylor type. Hindman and Strauss propose an unconditional approach in [11]. Corollary 18.9 in [11] allows loops in the sum graph and considers colorings of the set $[S]^1 \cup [S]^2$. In a manner similar to the proof of Proposition 1.7, we obtain the following observation.

PROPOSITION 7.3. *Let S be a semigroup, and consider the coloring χ of $[S]^1 \cup [S]^2$ defined by $\chi(\{a,b\}) := |\{a,b\}|$. If a sequence $a_1, a_2, \ldots \in S$ has no proper sumsequence, then every monochromatic sum graph of a sumsequence of $a_1, a_2, \ldots$ is a singleton.* ■

Since we may assume that any given finite coloring of the set $[S]^1 \cup [S]^2$ is finer than the one of Proposition 7.3, there is no advantage in this approach over that of Theorem 1.6.

7.3. New covering properties. Our results suggest a number of new covering properties that were not considered thus far, and it remains unclear how exactly these relate to the classical ones. For example, the property in Theorem 4.6 in the case where $\mathcal{U}_n = \mathcal{U}$ for all n is formally weaker than Menger's property. Is it equivalent to it?

7.4. Additional directions. Using the selection principle $\mathsf{S}_{\mathrm{fin}}$ and its corresponding game, one obtains an abstract version of a theorem of Deuber and Hindman [5], and stronger forms of this theorem, in the spirit of the main theorem in Bergelson and Hindman [2]. This direction may be pursued further.

Acknowledgments. Marion Scheepers [22] was the first to realize the connection between Ramsey theory and selection principles, by proving the following beautiful qualitative extension of Ramsey's Theorem: Let X be a topological space. If $\mathsf{S}_1(\Omega(X), \Omega(X))$ holds, then for each cover $\mathcal{U} \in \Omega(X)$ and each finite coloring of the set $[\mathcal{U}]^2$, there is in $\Omega(X)$ a cover $\mathcal{V} \subseteq \mathcal{U}$ such that the graph $[\mathcal{V}]^2$ is monochromatic. Scheepers proved a good number of results of this type, including ones jointly with Ljubiša Kočinac and others (e.g., [14, 25, 20]).

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