PROJECTIVE GEOMETRY

Abstract. Projective geometry

Contents

1. Ceva, Menelaus 1
2. Desargues' theorem, Pappus's theorem, Pascal's theorem, Brianchon's theorem 1
3. Axioms of affine planes and projective planes 2
4. Cross-ratio 2
5. Geometric constructions, projective transformations, transitivity on triples, projective pl
6. Duality, self-dual axiom systems 4
7. Cross-ratio in homogeneous coordinates 4
8. Conic sections 4
9. Polarity, reciprocity 6
10. Construction of a generic point on a conic passing through 5 given points 7
11. Mobius transformations 7
12. Poncelet's porism 8
References 11

1. Ceva, Menelaus

The material in this section is from the book by H. Perfect [8] (see the bibliography).

Collinearity and concurrency.

Ceva and Menelaus Theorems.

Explain "signed length" at great length.

Concurrency of altitudes, angle bisectors, and medians in a triangle.

2. Desargues' theorem, Pappus's theorem, Pascal's theorem, Brianchon's theorem

There are two points of view on Desargues' theorem: the slick statement: "triangles in pespective from a point, are in perspective from a line", and a detailed statement in terms of specific intersections, etc.

On must insist on the explicit version, for otherwise students come away without a true understanding of Desargues' theorem.

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Pappus's theorem.

Pascal's theorem (six points on a circle).

Brianchon's theorem.

Note that the usual statement of Brianchon's theorem in terms of the sides and diagonals of a hexagon, at least on the surface of it, is *less general* than the polar dual of Pascal.

Students need to be able to state this duality precisely in terms of *labeled points*. Here again the connection between the dual theorems needs to be explained in detail, otherwise the students don't learn to translate theorems to their duals/polars.

Internal and external bisectors.

Harmonic 4-tuple.

3. Axioms of affine planes and projective planes

The material is in Hartshorne [4].

Axiomatisation of affine planes, including proofs of certain basic results derived from the axioms.

Note that a line by definition is parallel to itself (in previous years students protested, citing Margolis).

The real case in detail, including the homogeneous coordinates, and the equivalence of the two approaches:

(1) adding points at infinity, and (2) homogeneous coordinates.

The 4 axioms of projective planes.

The model obtained by completing the affine line by adding points at infinity defined by pencils of parallel lines.

Proof in detail that this model satisfies the four axioms.

Definition of homogeneous coordinates.

Proof of the theorem that completion at infinity and homogeneous coordinates give isomorphic models of real projective plane.

Affine neighborhoods

4. Cross-ratio

To complete the material from last week: Formulas for numbers of points in finite planes.

The material on cross-ratios is in Adler [1].

Definition 4.1. Cross-ratio (yachas hakaful).

Definition 4.2. Perspectivity from a point.

Prove invariance under perspectivity, using areas.

Definition 4.3. Cross ratio of a pencil of lines.

The 6 cross-ratios: $\lambda, 1-\lambda, \frac{1}{\lambda}$, etc.

Role of the symmetric group on 4 letters and of the Klein 4-group.

Exceptional case: the 3 cross-ratios.

Remark 4.4. Over the complex numbers: an additional exceptional case of only 2 distinct values, when

$$\lambda = e^{\pm i\pi/3}.$$

The cross-ratio of 4 points on a circle.

Relation to polarity (which has not been treated formally yet): work with tangent lines to a circle instead of points on a circle.

Then a variable tangent line t meets a 4-tuple of fixed tangents in a 4-tuple of points whose cross-ratio is independent of t.

5. Geometric constructions, projective transformations, transitivity on triples, projective plane over an arbitrary field

Exceptional values $0, 1, \infty$ of the cross-ratio when some of the points collide.

Theorem 5.1. $R(\infty, 0, 1, \lambda) = \lambda$.

Constructions in projective geometry.

An explicit geometric construction of the 4th harmonic point, using Ceva and Menelaus.

Recall the notion of a perspectivity.

Definition 5.2. A projectivity is a transformation preserving the cross-ratio.

The notation: a wedge under the equality sign.

Theorem 5.3. *Thansitivity of projective transformations on triples of collinear points.*

Proof by composition of suitable perspectivities.

Corollary 5.4. *On every line in projective plane, given a triple of points, a fourth point is uniquely determined by the cross-ratio.*

More axiomatics: prove from the 4 axioms the following:

Theorem 5.5. *There is a 1-1 correspondence between points on a line ℓ and lines through a point A not on ℓ.*

Construction of the projective plane over an arbitrary field.

6. Duality, self-dual axiom systems

Duality.

The four axioms of projective geometry give rise to a self-dual system, i.e. the dual of each axiom can be proved from the original list of four.

Discussion the construction in homogeneous coordinates over any field, using a generalisation of the vector product.

Counting points in a projective plane, discuss in a bit more detail the notion of an affine neighborhood (to break the idea that the affine plane is "special").

Detailed discussion of the case over the field F_2, writing out the homogeneous coordinates of all the points, and explicit equations of some of the lines.

7. Cross-ratio in homogeneous coordinates

The definition of cross-ratio in homogeneous coordinates follows the book by Kaplansky [7].

Here if $A, B, C, D \in \mathbb{RP}^1$ we view A, B, C, D as 1-dimensional subspaces in $\mathbb{R}^2$. We choose representative nonzero vectors $\alpha \in A$, $\beta \in B$, $\gamma \in C$, and $\delta \in D$. We show that the vectors can be picked in such a way as to satisfy the relations

$$\gamma = \alpha + \beta$$

and

$$\delta = k\alpha + \beta, \tag{7.1}$$

where $k \in \mathbb{R}$ is suitably chosen. Then the coefficent k in (7.1) is the cross-ratio of A, B, C, D:

Theorem 7.1. *The coefficient k is independent of choices made and satisfies $R(A, B, C, D) = k$.*

8. Conic sections

Conic sections: intersection of cone in $\mathbb{R}^3$ and plane.

Ellipse, parabola, hyperbola and number of points at infinity: 0, 1, 2.

Theorem 8.1. *Every nondegenerate nonempty real conic section is projectively equivalent to the circle.*

Example 8.2. To transform a circle into a parabola by a projective transformation, consider the equation of the circle

$$+x_1^2 + x_2^2 - x_3^2 = 0. \tag{8.1}$$

Here in the affine neighborhood $x_3 = 1$ we obtain the usual circle equation

$$x^2 + y^2 = 1 \tag{8.2}$$

where $x = \frac{x_1}{x_3}$ and $y = \frac{x_2}{x_3}$. We would like to tranform this into a parabola

$$X_2 X_3 = X_1^2. \tag{8.3}$$

Here in the affine neighborhood $X_3 = 1$ this becomes the usual equation of a parabola $Y = X^2$, where $X = \frac{X_1}{X_3}$ and $Y = \frac{X_2}{X_3}$. We rewrite (8.3) as

$$(X_2 + X_3)^2 - (X_2 - X_3)^2 = (2X_1)^2,$$

or

$$+(2X_1)^2 + (X_2 - X_3)^2 - (X_2 + X_3)^2 = 0. \tag{8.4}$$

Note that the signs $+, +, -$ in equations (8.1) and (8.4) are compatible. Therefore we exploit the transformation

$$x_1 = 2X_1, \; x_2 = X_2 - X_3, \; x_3 = X_2 + X_3.$$

This is a linear transformation in homogeneous coordinates and therefore defines a projective transformation on the projective planes.

Next, this can be expressed in an affine neighborhood by noting that

$$\frac{x_1}{x_3} = \frac{2X_1}{X_2 + X_3} = \frac{2\frac{X_1}{X_3}}{\frac{X_2}{X_3} + 1}$$

and

$$\frac{x_2}{x_3} = \frac{X_2 - X_3}{X_2 + X_3} = \frac{\frac{X_2}{X_3} - 1}{\frac{X_2}{X_3} + 1}.$$

In affine coordinates, we obtain

$$x = \frac{X}{Y + 1}, \quad y = \frac{Y - 1}{Y + 1}. \tag{8.5}$$

Substituting (8.5) into the circle equation (8.2) we obtain the equation of parabola $Y = X^2$.

Example 8.3. Transform parabola into hyperbola.

Example 8.4. Transform ellipse $x^2 + xy + y^2 = 1$ into parabola $Y = X^2$.

9. Polarity, reciprocity

Definition of polar line.

Metric characterisation of polar lines.

Axioms of Fano, Desargues, and Pappus.

Discussion of relation between algebraic properties and geometric axioms:

Theorem 9.1. *Suppose a projective plane π satisfies the axioms P1, ... P4 as well as Desargues' axiom. Then there exists a division ring D such that $\pi = DP^2$.*

Theorem 9.2. *Suppose in addition to the hypotheses above, π satisfies Fano's axiom (the diagonal points of a complete quadrilateral are not collinear). Then char $D \neq 2$.*

Theorem 9.3. *Suppose in addition to the hypotheses above, π satisfies Pappus' axiom. Then $\pi = DP^2$ where D is a field.*

This point of view may be found in the book by Kadison and Kromann [6, chapter 8]. It originates with Hilbert's book [5], see chapter 5 there, particularly paragraph 24: "Introduction of an algebra of segments based upon Desargues's theorem and independence of the axioms of congruence", starting on page 79. Hilbert mentions that this was also discussed by Moore.

1. proof of the reciprocity theorem: if Q is on p, then P is on q.

2. proof of the fact that polarity is a projective transformation, in two stages. First one proves it for 4 points lying on a tangent to the conic. Then one proves it for an arbitrary collinear 4-tuple.

3. A nice application is the theorem that every conic defines a projective transformation from points on a tangent, to points on another tangent. Namely, a point B on a tangent t is sent to a point B' on tangent t' if and only if the line BB' is tangent to the conic.

4. Present another example of a construction in projective geometry. So far the only construction we had is the construction of the fourth harmonic point, using Menelaus theorem.

5. Using the result that polarity is a projective transformation, construct a conic from 5 pieces of data. The 5 pieces are points L and L', the corresponding tangent lines l and l' through them, and an additional tangent line a". One constructs the map as in item 3 above, as the composition of two perspectivities.

Geometric constructions using projective theorems is an important topic in projective geometry that we have barely touched upon.

Polarity is closely related to inversion in circles. An application of inversions is the Peaucellier–Lipkin linkage, see `https://en.wikipedia.org/wiki/Peaucellier%`

10. Construction of a generic point on a conic passing through 5 given points

Construction of a generic point on a conic passing through 5 given points, using Pascal's theorem.

Translating it to a polar statement, so as to construct the polar pencil of parallel lines to the conic.

Finding a projective map between a pair of pencils of lines through a pair of points on a conic.

11. Mobius transformations

Every projective map from P^1 to itself is of the form

$$x \to \frac{ax+b}{cx+d}.$$

I already mentioned the fact that projective transformations correspond to linear maps when you write them in homogeneous coordinates. The fractional-linear presentation is a consequence of this.

More material on axioms of Fano, Desargues, Pappus, related material on the polar line, perhaps a proof of Desargues assuming existence of imbedding in projective 3-space.

12. Poncelet's porism

An elementary proof was found in '15 by Halbeisen [3].

(draw illustration of Poncelet's porism for $n = 3$, a triangle that is inscribed in one circle and circumscribes another.)

In geometry, Poncelet's porism, named after French engineer and mathematician Jean-Victor Poncelet, states the following.

Theorem 12.1 (Poncelet's porism)**.** *Let C and D be two plane conics. If it is possible to find, for a given $n > 2$, one n-sided polygon that is simultaneously inscribed in C and circumscribed around D, then it is possible to find infinitely many of them.*

Definition 12.2. By an *elliptic curve* is a meant a 2-torus.

Remark 12.3. Poncelet's porism can be proved via elliptic curves; geometrically this depends on the representation of an elliptic curve as the double cover of C with four ramification points. (Note that C is isomorphic to the complex projective line.) The relevant ramification is over the four points of C where the conics intersect. (There are four such points by Bézout's theorem.) One can also describe the elliptic curve as a double cover of D; in this case, the ramification is over the contact points of the four bitangents.

Sketch of proof of Theorem 12.1. Let $C \subset \mathbb{CP}^2$, $D \subset \mathbb{CP}^2$ be the two conics. Let p be a point of the projective plane

$$P = \mathbb{CP}^2$$

and ℓ a line of the dual projective plane, denoted

$$P^*.$$

The key tool is the curve X given by the set of pairs $p \in \ell$ where p is on the conic C and ℓ is tangent to the conic D, namely

$$X = \{(p, \ell) \in P \times P^* : \quad (p \in C) \wedge (\ell \text{ tangent to } D) \wedge (p \in \ell)\}.$$

This can be reformulated by stating that the D-polar point ℓ^* to ℓ lies on D, or $\ell^* \in D$.

Then X is smooth; more specifically X is an elliptic curve. There is an involution σ of X mapping (p, ℓ) to (p', ℓ) where p' is the other point of intersection of ℓ with C, so that

$$\ell \cap C = \{p, p'\}.$$

Thus,

$$\sigma : X \to X, \quad (p, \ell) \mapsto (p', \ell).$$

There is another involution τ that sends (p, ℓ) to (p, ℓ') where ℓ' is the other tangent from p to D, so that

$$\tau : X \to X, \quad (p, \ell) \mapsto (p, \ell').$$

Now an elliptic curve has natural addition on it, inherited from addition on $\mathbb{C}$. With respect to the natural addition on X, it turns out that the composition $\tau \circ \sigma$ is a translation. If $(\tau \circ \sigma)^n$ has one fixed point, then $(\tau \circ \sigma)^n$ must be the identity translation, i.e., every point is a fixed point of $(\tau \circ \sigma)^n$.

Translated back into the language of the conics C and D, this means that if one point on C gives rise to an orbit that closes up (i.e. gives an n-gon), then every point does, as well. □

We will now present a more detailed version of the argument though some crucial details will be left for a course in algebraic geometry.

Definition 12.4. An *involution* of a set S is a map

$$\sigma : S \to S$$

such that the composition of σ with itself is the identity map of S: $\sigma \circ \sigma = Id_S$.

Lemma 12.5. *The number of points of intersection of a line not tangent to C with a conic C in $\mathbb{CP}^2$ equals* 2.

Proof. The hypotheses reduce to solving a quadratic equation over C. Therefore there are two solutions. □

Lemma 12.6. *The number of tangent lines from a point $p \in P \setminus C$ to the conic C equals* 2.

Proof. The proof is similar. □

Definition 12.7. A lattice

$$L \subset \mathbb{C}$$

is a subgroup generated by two nonzero numbers z_1 and z_2 which are linearly independent over $\mathbb{R}$. Equivalently, z_1 and z_2 have different arguments, namely $\theta_1 \neq \theta_2$, where $e_1 = r_1 e^{i\theta_1}$ and $e_2 = r_2 e^{i\theta_2}$.

Definition 12.8. A torus T^2 is quotient $\mathbb{C}/L$. Thus, a point $[z] \in T^2$ is an equivalence class of a point $z \in \mathbb{C}$:

$$[z] = z + L,$$

namely the coset for the usual addition in $\mathbb{C}$.

Such a torus is sometimes called an elliptic curve.

Lemma 12.9. *The torus comes equipped with a natural addition on torus inherited from $\mathbb{C}$. Namely, if $[z_1], [z_2] \in T^2$, then*

$$[z_1] + [z_2] = [z_1 + z_2]$$

Proof. The addition is at the level of the representative complex numbers. □

Viewing $\mathbb{C}$ as $\mathbb{R}^2$, we can speak of orientation of a selfmap. A nonsingular linear selfmap of $\mathbb{R}^2$ is called orientation-preserving if the determinant of the matrix representing it is positive, and orientation-reversing otherwise. An affine transformation f of the form $f(x) = Ax + v$ is called orientation-preserving if A is.

Example 12.10. Complex conjugation is orientation reversing; translation by $z_0 \in \mathbb{C}$ is orientation preserving; multiplication by $e^{i\theta}$ (rotation) is orientation preserving.

Definition 12.11. Orientation of a selfmap $\sigma : T^2 \to T^2$ is inherited from orientation of selfmap of $\mathbb{C}$.

Example 12.12. A translation on $\mathbb{C}$ induces a self-map of the torus $\mathbb{C}/L$.

Definition 12.13. Gaussian lattice L_G is the lattice of points with integer coordinates:

$$L_G = \{z \in \mathbb{C} : z = n + im,\ n, m \in \mathbb{N}\}.$$

Definition 12.14. Translation by z_0 in $\mathbb{C}/L_G$ is called rational if there is an $n \in \mathbb{N}$ such that

$$nz_0 \in L_G.$$

A more detailed proof of Poncelet's porism. Let C and D be the conics of Poncelet's porism. Let p be a point of the projective plane $P = \mathbb{CP}^2$ and ℓ a line of the dual projective plane, denoted P^*. The key tool is the curve $X \subset P \times P^*$ defined as follows. We set

$$X = \{(p, \ell) \in P \times P^* : p \in C,\ \ell \text{ tangent to } D,\ p \in \ell\}.$$

Then X is a smooth elliptic curve. We define an involution

$$\sigma : X \to X \quad \sigma(p, \ell) = (p', \ell),$$

where p' is the other point of intersection of ℓ with C. Such a point exists by Lemma 12.5.

We define a second involution $\tau : X \to X$, by setting

$$\sigma : X \to X \quad \sigma(p, \ell) = (p, \ell'),$$

where ℓ' is the other tangent from p to D. Such a tangent exists by Lemma 12.6. We have a natural addition on X as in Lemma 12.9. It turns out that the composition

$$\tau \circ \sigma : X \to X$$

is a translation. The hypothesis of Poncelet's porism asserts that the map $(\tau \circ \sigma)^n$ has a fixed point. But a translation having a fixed point, is necessarily the identity. Thus, $\tau\sigma$ is a rational translation of Definition 12.14.

Translated back into the language of the conics C and D, this means that if one point on C gives rise to an orbit that closes up (i.e. gives an n-gon), then every point does, as well. $\square$

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