Systolically free manifolds, Appendix D, pp. 531-543 of the following book: Misha Gromov, *Metric Structures for Riemannian and Non-Riemannian Spaces.* Progress in Mathematics 152, Birkhauser, Boston, 1999.

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This appendix fits in with chapter 4, section E_+, paragraph 4.45 *'On higher dimensional systoles'* (p. 264). The k-systole of a Riemannian manifold X is, roughly, the least volume of a k-dimensional non-nullhomologous submanifold of X. See paragraph 4.40 of the same section (p. 260) for a definition.

The existence of a systolic inequality to us will mean that the volume of X imposes a constraint on systoles, as in Loewner's theorem 4.1 (p. 237) and Gromov's theorem 4.41 (p. 260).

We will refer to the absence of systolic inequalities as 'systolic freedom'. To exhibit the freedom of a manifold X, we will construct families of metrics on X whose systoles are not constrained by the volume.

There are two rather different systolic problems: one involving a pair of systoles of distinct complementary dimensions, and one involving the middle-dimensional systole.

The idea for the pair of complementary dimensions is as follows. Let $k < \frac{n}{2}$ where $n = dim(X)$. Following M. Gromov (*cf.* [Ber], p. 306 and [G3], section $4.A_5$, p. 354), we use Thom's theorem (*cf.* [T]) to choose a k-dimensional submanifold with the following two properties: (a) its normal bundle is trivial; (b) its connected components generate (rationally) the homology group $H_k(X)$.

The construction takes place within a fixed trivialized tubular neighborhood of the submanifold. Gromov described an expansion-contraction procedure (*cf.* [G2], section 2) with the desired effect on the systole of the complementary dimension $n-k$. We combine the expansion-contraction with a volume-saving twist.

In more detail, Thom's theorem allows us to carry out the expansion-contraction procedure in a neighborhood of a suitable k-dimensional submanifold described above. The resulting metrics have large $(n-k)$-dimensional systole (compared to the volume). So far the $(k, n-k)$ systolic inequality is not violated. This is because the metric is essentially a product and the volume is too large. To decrease the volume, we introduce a nondiagonal term in the Riemannian metric. The key idea is due to Gromov and is described in paragraph 4.45 (p. 264) for the product of a circle by a simply connected manifold admitting a free circle action. Generalizing the construction requires a splitting

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of both the k-dimensional class and its tubular neighborhood (see below). It is at this point that the modulo 4 condition on k comes in.

Theorem 1. [BaK] Let X be an orientable $(k-1)$-connected n-dimensional manifold, where $n \geq 3$ and $k < \frac{n}{2}$. Suppose k is not a multiple of 4. Then

$$\inf_g \frac{vol(g)}{sys_k(g) sys_{n-k}(g)} = 0, \tag{1}$$

where the infimum is taken over all metrics g on X.

In the middle-dimensional case, we have the following general result.

Theorem 2. [BaKS] Let X be a manifold of dimension $n = 2m$, where $m \geq 3$. Suppose the group $H_m(X)$ is torsion free. Then

$$\inf_g \frac{vol(g)}{sys_m^2(g)} = 0, \tag{2}$$

where the infimum is taken over all metrics on X.

The case $m = 2$ is dealt with in [KS]. One may ask if systolic freedom persists if one restricts the class of competing metrics to homogeneous ones. We have the following partial result in this direction.

Theorem 3. The manifold $S^3 \times S^3$ satisfies

$$\inf_g \frac{vol_6(g)}{sys_3^2(g)} = 0, \tag{3}$$

where the infimum is taken over *homogeneous* metrics g on $S^3 \times S^3$.

A key technical result in the proof of Theorem 1 is the following lemma, inspired by Gromov's first example of free metrics on $S^1 \times S^3$ (*cf.* [Ber], p. 301).

Lemma 1 ([BeK], [BaK]). Consider the cylinder $M = T^1 \times I$, circle C, and the manifold $Y = C \times M = T^2 \times I$. We view M as a relative cycle in $H_2(Y, \partial Y)$. Then

$$\inf_g \frac{vol(g)}{sys_1(g) mass_2([M])} = 0. \tag{4}$$

Here the infimum is taken over all metrics whose restriction to each component of the boundary $\partial Y = T^2 \times \partial I$ is the standard 'unit square' torus satisfying $length(C) = length(T^1) = 1$. The 2-mass of the class $[M]$ is the infimum of areas of *rational* cycles representing it.

Proof. The metric of the universal cover $p : \mathbf{R}^3 \to N$ of the standard nilmanifold N of the Heisenberg group can be written as

$$(dz - x dy)^2 + dy^2 + dx^2. \tag{5}$$

We will use it to construct a sequence of metrics (g_j), $j \in \mathbf{N}$, on Y with the required asymptotic behavior. An alternative approach using a solvable group was used in [P].

The restriction of the above metric to the parallelepiped defined by the inequalities $0 \leq x \leq j$, $0 \leq y \leq 1$, $0 \leq z \leq 1$ defines a metric on $[0,j] \times T^1 \times C$ once we identify the opposite sides of the unit square in the y,z plane. Here $[0,j]$ is parametrized by x, the circle T^1 by y, and the circle C by z. The covering projection p induces a monomorphism

$$p_* : \pi_1([0,j] \times T^2) \to \pi_1(N) \tag{6}$$

from $\mathbf{Z}^2$ to the group of unipotent integer 3×3 matrices.

The metric on $[0,j] \times T^2$ does not satisfy the boundary condition (specified in the statement of the lemma) at $x = j$, since $length(\{j\} \times T^1) = \sqrt{1+j^2}$. We correct this by taking its "double" using a mirror placed at $x = j$. In other words, let $I = [0, 2j]$, let $X = \min(x, 2j - x)$, and consider the sequence of metrics on $Y = T^2 \times I$ defined by

$$g_j = (dz - X dy)^2 + dy^2 + dx^2. \tag{7}$$

Note that the projection p extends over Y, so that we still have the monomorphism p_* of fundamental groups.

Let us show that the growth of $mass_2[M]$ is quadratic in j. Let $\phi_j(x)$ be a partition of unity-type function with support in $[0,j]$, and such that $\phi_j(x) = 1$ for $x \in [1, j-1]$. We define a 2-form $\alpha = \alpha_j$ by setting

$$\alpha = \frac{\phi_j(x)}{\sqrt{1+x^2}} * (dz) \tag{8}$$

where $*$ is the Hodge star of the metric g_j. Let's check that α is a calibrating form, *i.e.* $\|\alpha\| = 1$ and $d\alpha = 0$.

By definition of g_j the 1-forms dx, dy, and $dz - xdy$ form an orthonormal basis. Therefore $\|dz\|_{(x,y,z)} = \|(dz - xdy) + xdy\| = \sqrt{1+x^2}$. Thus $\|\alpha\| = 1$. Furthermore,

$$*(dz) = dx \wedge dy - xdx \wedge (dz - xdy) = dx \wedge ((1+x^2)dy - xdz). \tag{9}$$

Let $a_j = \int_M \alpha = \int_M \phi_j(x)\sqrt{1+x^2}dx \wedge dy = \int_0^j \phi_j(x)\sqrt{1+x^2}dx$. Thus a_j grows as j^2. The calculation (9) shows that the 2-form α is closed. Hence by Stokes,

$$mass_2[M] \geq \int_M \alpha = a_j, \tag{10}$$

and the growth of $mass_2[M]$ is quadratic.

Meanwhile, the injectivity of p_* of (6) allows us to conclude that $sys_1(g_j) \geq \pi sys_1(N)$, where πsys_1 is the length of the shortest noncontractible curve. Hence the 1-systole is uniformly bounded from below in j. The volume of the Riemannian submersion $(Y, g_j) \to [0, 2j]$ with fiber of unit area is $2j$. Thus

$$\frac{vol(g_j)}{sys_1(g_j)mass_2[M]} = O\left(\frac{1}{j}\right), \tag{11}$$

proving the lemma.

Proof of Theorem 1. The idea was described at the beginning of the appendix. Let $e_1, \ldots, e_b$ be an integral basis for a maximal lattice in $H_k(X, \mathbf{Z}) = Z^b +$ torsion. We will exhibit a k-dimensional submanifold $A \subset X$ whose connected components $A_1, \ldots A_b$ have the following properties:

(i) each A_i is diffeomorphic to $S^{k-1} \times C$ where C is a circle;

(ii) the classes $[A_i]$ satisfy $[A_i] = 2e_i$;

(iii) the normal bundle ν of $A_i \subset X$ is trivial.

By Hurewicz's theorem, we find k-spheres $f_i : S^k \to X$, $i = 1, \ldots, b$ satisfying (ii). Since the total Stiefel-Whitney class $w(S^k) = 1$ is trivial, we have

$$w_k[f_i^*(\nu)] = w_k[f_i^*(TX)] = w_k(TX)(f_*[S^k]) = w_k(TX)(2e_i), \tag{12}$$

and so $w_k[f_i^*(\nu)] = 2w_k(TX)(e_i) = 0 \in H^0(X, \mathbf{Z}_2)$.

By real Bott periodicity, there is at most a $\mathbf{Z}_2$'s worth of inequivalent bundles of rank $\geq k+1$ over S^k for $k \neq 0 \pmod 4$, distinguished by the class w_k. Hence ν is trivial. We now perform a surgery inside X by attaching a handle to each imbedded sphere $f_i(S^k)$, turning it into a product $A_i = C \times S^{k-1}$ where C is a circle, still with trivial normal bundle. Thus a tubular neighborhood of A_i splits as a product

$$A_i \times B^{n-k}. \tag{13}$$

Let $R \subset B^{n-k}$ be any submanifold of codimension 2 with trivial normal bundle. The boundary of its tubular neighborhood is diffeomorphic to $R \times T^1$, and we obtain a hypersurface

$$\Sigma_i = A_i \times R \times T^1 = C \times S^{k-1} \times R \times T^1 = T^2 \times L \tag{14}$$

in a neighborhood of $A_i \subset X$, where $T^2 = C \times T^1$ is a 2-torus and $L = S^{k-1} \times R$.

Let $\Sigma \subset X$ be the disjoint union of such hypersurfaces $\Sigma_1, \ldots, \Sigma_b$. A tubular neighborhood of $\Sigma \subset X$ is a disjoint union of open submanifolds $\Sigma_i \times I = C \times T^1 \times L \times I$. What happens metrically can be described as follows. We choose a metric on X which is a direct sum in a neighborhood of each Σ_i, with four summands

$$C \times T^1 \times L \times I \subset X, \tag{15}$$

where C and T^1 are of unit length, L has unit volume, and I is an interval. Denote by X_0 the resulting Riemannian manifold. We now modify the direct sum metric on

$$C \times T^1 \times L \times I = Y \times L \tag{16}$$

by using the metrics g_j on $Y = C \times T^1 \times I$ of Lemma 1, while L remains a direct summand. Denote by X_j the resulting Riemannian manifold. The boundary condition of Lemma 1 ensures that the new metric on X varies continuously across the boundary $T^2 \times L \times \partial I$.

Proposition 1. The systole $sys_k(X_j)$ is uniformly bounded from below in j.

Proof. Let z be a k-cycle in X_j, $[z] \neq 0$ in $H_k(X_j)$. If the support of z lies outside $\Sigma \times I$, then whether $[z]$ has finite or infinite order, we have

$$vol_k(z \subset X_j) = vol_k(z \subset X_0) \geq sys_k(X_0), \tag{17}$$

yielding a lower bound independent of j. If z lies in $\Sigma \times I$ then its class $[z]$ lies in the span of $[A_1], \ldots, [A_b]$, so that $[z] = \lambda_1[A_1] + \ldots + \lambda_b[A_b]$. Here one of the coefficients is nonzero, say λ_l. Let

$$\pi : Y \times L \to S^{k-1} \tag{18}$$

be the projection to the first factor of $L = S^{k-1} \times R$. By the coarea inequality,

$$vol_k(z) \geq \int_{S^{k-1}} length(z \cap \pi^{-1}(t))dt \geq \int_{S^{k-1}} sys_1(g_j)dt \geq \pi sys_1(N), \tag{19}$$

also yielding a lower bound $sys_k(\Sigma \times I) \geq \pi sys_1(N)$ independent of j.

Here we must verify that the 1-cycle $z \cap \pi^{-1}(t)$ represents a nonzero class in $\pi^{-1}(t) = (Y, g_j) \times R$. This is so because the set-theoretic formula

$$(C \times S^{k-1}) \cap (Y \times R) = C \tag{20}$$

makes sense at the homological level. Indeed, inside $\Sigma_l \times I = Y \times L$ we have

$$\begin{aligned} [z] \smile [Y \times R] &= \lambda_l[A_l] \smile [Y \times R] = \lambda_l[C \times S^{k-1}] \smile [Y \times R] \\ &= \lambda_l[(C \times S^{k-1}) \cap (C \times T^1 \times I \times R)] = \lambda_l[C] \neq 0, \end{aligned} \tag{21}$$

where $\smile$ is the homological operation (dual to cup product in cohomology) pairing the groups $H_k(\Sigma_l \times I)$ and $H_{n-k+1}(\Sigma_l \times I, \Sigma_l \times \partial I)$. This operation is realized by transverse intersection of representative cycles, and hence $[z \cap \pi^{-1}(t)] \neq 0$.

For general z we argue as follows. Assume for simplicity that $b_k(X) = 1$. We will cut z at a narrow place and replace it by the sum of two cycles, each of which can be handled as above. If $b_k > 1$, we have to make several cuts, but the rest of the argument is the same.

We argue by contradiction. Suppose X_j contains a k-cycle z_j such that

$$\lim_{j \to \infty} vol_k(z_j) = 0. \tag{22}$$

Let $d : X_j \to [0, 2j]$ be the map extending the x-coordinate of Lemma 1 by collapsing the components of the complement

$$X \setminus (\Sigma \times I) \tag{23}$$

to the respective endpoints. By the coarea inequality, we find points $x_- \in [0,1] \subset I$ and $x_+ \in [2j-1, 2j] \subset I$ such that

$$vol_{k-1}(z \cap d^{-1}(x_\pm)) \to 0 \tag{24}$$

as $j \to \infty$. Let $\gamma_\pm = z \cap d^{-1}(x_\pm)$. Since $vol_{k-1}(\gamma_\pm) = o(1)$, $\gamma_\pm$ can be filled in by a k-chain $D_\pm$ with $vol_k(D_\pm) = o(1)$. This follows from the isoperimetric inequality for small cycles proved in [G1], Sublemma 3.4.B'. Since $\partial D_\pm = \gamma_\pm$, we can write $z_j = z' + z''$ where $z' = z \cap d^{-1}([x_-, x_+]) + D_+ - D_-$ is a k-cycle in $\Sigma \times I$. Now

$$\begin{aligned} vol_k(z_j) &\geq \min(vol_k(z'), vol_k(z'')) - vol_k(D_+ + D_-) \\ &\geq \min(sys_k(X_1), sys_k(\Sigma \times I)) - o(1) \\ &\geq \min(sys_k(X_1), \pi sys_1(N)) - o(1) \end{aligned} \tag{25}$$

providing a lower bound independent of j.

Proposition 2. Let X_j be the Riemannian manifold constructed above. Then the $(n-k)$ systole $sys_{n-k}(X_j)$ grows quadratically in j.

Proof. Let α_i be the 2-form of Lemma 1 with support in a neighborhood of A_i. We use the calibrating form

$$\beta_i = \alpha_i \wedge vol_R \tag{26}$$

where vol_R is the volume form of the codimension 2 submanifold $R \subset B^{n-k}$. Note that $H_{n-k}(X)$ is torsion free as $H_{k-1}(X) = 0$ by hypothesis of Theorem 1. Using Poincare duality, we choose integer $(n-k)$-cycles $M_1, \ldots, M_b$ which define a basis of $H_{n-k}(X, \mathbf{Z})$ 'dual' to the basis $e_1, \ldots, e_b$ of $H_k(X, \mathbf{Z})$, so that the *algebraic* intersections satisfy

$$[M_i] \cap e_h = \delta_{ih} \tag{27}$$

and therefore $M_i \cap A_h = 2\delta_{ih}$. We may assume that all intersections of M_i and A_h are transverse and moreover standard, so that $M_i \cap (\Sigma_h \times I)$ is a disjoint union of copies of $M \times R$ where $M = T^1 \times I$ is the cylinder of Lemma 1. Hence $\int_{M \times R} \beta_i = \pm a_j$, where $a_j = \int_M \alpha_j$ (*cf.* Lemma 1), for each connected component $M \times R$ of $M_i \cap (\Sigma_h \times I)$. In particular

$$\int_{M_i \cap (\Sigma_h \times I)} \beta_i = 0 \text{ if } \ i \neq h. \tag{28}$$

Finally, let $m \in H_{n-k}(X)$ be a nonzero class. Let $m = \epsilon_1 d_1 [M_1] + \ldots + \epsilon_b d_b [M_b]$ where $\epsilon_i = \pm 1$ and $d_i \geq 0$. We use the signs ϵ_i to specify a suitable calibrating form β by setting

$$\beta = \epsilon_1 \beta_1 + \ldots + \epsilon_b \beta_b. \tag{29}$$

Now

$$\begin{aligned} mass_{n-k}(m) &\geq \int_m \beta = \Sigma_{i=1}^b \epsilon_i d_i \int_{M_i} \beta = \Sigma_i \epsilon_i d_i \int_{M_i} (\Sigma_h \epsilon_h \beta_h) = \\ &= \Sigma_i \epsilon_i^2 d_i \int_{M_i} \beta_i \qquad \text{(by 28)} \\ &= \Sigma_i d_i \int_{M_i} \beta_i = \Sigma_i d_i 2 a_j \geq a_j \end{aligned} \tag{30}$$

since $m \neq 0$, where $a_j = \int_M \alpha_j$ grows quadratically in j. Hence the growth of $sys_{n-k}(X_j)$ is quadratic in j.

Now Propositions 1 and 2 and the linear growth of the volume of X_j yield

$$\frac{vol(X_j)}{sys_k(X_j) mass_{n-k}(X_j)} = O\left(\frac{1}{j}\right). \tag{31}$$

This completes the proof of Theorem 1.

We now prove some preliminary results for Theorem 2 as steps 1 through 4.

Step 1. A $(2m-1)$-dimensional complex admits a map to the bouquet of m-spheres which induces monomorphism in m-dimensional homology with rational coefficients.

Proof. Let K^i be the i-skeleton of the complex K. From the exact sequence of the pair (K, K^{m-1}) we see that the homomorphism $H_m(K) \to H_m(K/K^{m-1})$ is injective. This reduces the problem to the case when K is $(m-1)$-connected. Now let $b_m = b_m(K)$ and let $H_m(K) \to \mathbf{Z}^{b_m}$ be the quotient by the torsion subgroup. Let $f : X \to K(\mathbf{Z}^{b_m}, m)$ be the associated map to the Eilenberg-Maclane space. Now $K(\mathbf{Z}^{b_m}, m) = (K(\mathbf{Z}, m))^{b_m}$ and $K(\mathbf{Z}, m)$ admits a cell structure with no cells in dimension $m+1$. Hence by the cellular approximation theorem, we obtain a map from the $(m+1)$-skeleton of K to the bouquet $\vee S^m$ of b_m spheres S^m, injective on nontorsion m-dimensional homology classes. We now proceed as in [S] (page 278 and 287-288). Here an induction on skeleta uses self-maps of the bouquet $\vee S^m$ which annihilate all relevant homotopy. By Eckmann's theorem [E], a self map f_q of degree q of the sphere S^m induces multiplication by q in the (stable) finite group $\pi_i(S^m)$ for $m+1 \leq i \leq 2m-2$. By Hilton's theorem, we have a splitting of the stable homotopy groups for a bouquet. Let $g : K^i \to \vee S^m$ be a map defined on the i-skeleton. Then $f_q \circ g$ extends to the $(i+1)$ skeleton if q is a multiple of the order of the finite group $\pi_i(S^m)$.

Step 2. A bouquet of m-spheres admits a self-map which induces a homomorphism of the first unstable group whose image is contained in the subgroup generated by Whitehead products.

Proof. For a single m-sphere, we argue as follows. Let $G \subset \pi_{2m-1}(S^m)$ be the subgroup generated by the Whitehead product $[e, e]$ where e is the fundamental class of the sphere. Then G is precisely the kernel of the suspension homomorphism

$$E : \pi_{2m-1}(S^m) \to \pi_{2m}(S^{m+1}). \tag{32}$$

Since E commutes with f_q, it suffices to choose q to be the order of the group $\pi_{2m}(S^{m+1})$. For a bouquet $\vee_i S_i^m$, let e_i be the fundamental class of S_i^m. By Hilton's theorem, the group $\pi_{2m-1}(\vee_i S_i^m)$ modulo mixed Whitehead products $[e_i, e_j]$ for $i \neq j$ is just

$$\oplus_i \pi_{2m-1}(S_i^m). \tag{33}$$

Hence the same argument applies.

If we start with a $2m$-manifold represented as a CW complex with a single $2m$-cell, we can apply Steps 1 and 2 to its $(2m-1)$ skeleton K^{2m-1} to construct a suitable map to the bouquet $\vee S^m$. We can now proceed in two different ways: (a) either by eliminating all homotopy of the bouquet (at least rationally) in dimension $2m-1$, or, more economically, (b) eliminating just enough to allow the map to extend across the $2m$-cell. Described below is the first approach.

Definition 1. A $2m$-dimensional complex is *'free'* if

$$\inf_g \frac{vol_{2m}(g)}{sys_m^2(g)} = 0, \tag{34}$$

where the infimum is taken over all metrics g.

Step 3. Consider a $2m$-dimensional complex X obtained from a bouquet of m-spheres by attaching b copies of a $2m$-cell. If the closure of each $2m$-cell is free, then X is free.

Proof. One chooses the following representative X_0 in the homotopy class of X. We take the Cartesian product of the bouquet $\vee S^m$ with the bouquet of b intervals of sufficiently big length l. Here each interval corresponds to a $2m$-cell. Each $2m$-cell is then attached at the end of its interval. We choose free metrics of the same volume on each cell closure. The freedom of X reduces to obtaining a lower bound on the m-volume of a cycle z in X_0. This is done as follows. Suppose for simplicity that there are only two $2m$-cells, attached at either end of the cylinder

$$\vee S^m \times [0, l] \tag{35}$$

(for l to be chosen later). We use the coarea inequality to find a narrow place and cut z into two pieces along an $(m-1)$-cycle $\gamma = z \cap p^{-1}(x)$. Here $p : \vee S^m \times [0, l] \to [0, l]$ is the projection to the second factor. The point $x \in [0, l]$ is chosen so that

$$vol_{m-1}(\gamma) \leq \frac{1}{l} vol_m(z). \tag{36}$$

It remains to 'fill in' the cycle γ. This can be done by the isoperimetric inequality for cycles of small volume ([G1], Sublemma 3.4.B'). Here l is chosen sufficiently big so that the inequality would apply to γ.

Definition 2. Let X and Y be $2m$-dimensional manifolds. A *'meromorphic map'* from X to Y is a continuous map from X to a CW complex obtained from Y by attaching cells of dimension at most $2m-1$, inducing monomorphism in middle-dimensional homology.

Example. Let X be a complex surface and $\widehat{X} \to X$ its blow-up at a point $p \in X$. Then the classical meromorphic map $X \to \widehat{X}$ can be modified in a neighborhood of p and extended to a continuous map from X to $\widehat{X} \cup_f B^3$ where the 3-ball is attached along the exceptional curve.

Step 4. The complex obtained by 'glueing in' a single Whitehead product is free if $m \geq 3$.

Proof. For a 'mixed' Whitehead product $[e_i, e_j]$, we obtain $(S^m \vee S^m) \cup_{[e_i, e_j]} B^{2m} = S^m \times S^m$, which is free by [K].

Consider the complex P defined by

$$P = S^m \cup_{[e,e]} B^{2m}, \tag{37}$$

where e is the fundamental class of S^m. The idea is to construct a 'meromorphic map' from P to the product of spheres, using multiplicative structure in the algebra of homotopy groups. Let a and c be the generators of $H_m(S^m \times S^m)$, so that $[a, c] = 0$. We now attach an $(m+1)$-cell along the diagonal $a + c$. Let

$$W = S^m \times S^m \cup_{a+c} B^{m+1}, \tag{38}$$

so that $a = -c$ in W. Let $f : S^m \to W$, $e \mapsto a$. Then

$$f([e, e]) = [a, a] = [a, -c] = 0, \tag{39}$$

and so the map extends to $P \to W$. We pull back to P the free metrics on W. Here the cell added to $S^m \times S^m$ does not affect the $(2m)$-dimensional volume (to obtain a positive definite metric on P, we increase slightly the pullback of the quadratic form, using the convexity of the cone of positive quadratic forms). This proves the freedom of P once we know that of $S^m \times S^m$.

Proof of Theorem 2. Let X be any CW complex of dimension $2m$. We map its $(2m-1)$-skeleton K to a bouquet as in Step 1. By Step 2, we can assume that the elements of $\pi_{2m-1}(K)$ are mapped to the subgroup G generated by (finitely many) Whitehead products in the bouquet. We attach a $2m$-cell to this bouquet for each generator of G. The resulting space is free by Steps 3 and 4. Now the map from K extends to X. It follows that X is free, too.

Remark. The proof can be simplified in the case of projective planes $\mathbf{H}P^2$ and $\mathbf{Ca}P^2$. It relies on the following proposition.

Proposition 3. Let F be $\mathbf{C}$, $\mathbf{H}$ or $\mathbf{Ca}$, and let $m = dim(F)$. Consider the map of degree q from FP^1 to the first component of $S^m \times S^m$, to which we glue an $(m+1)$-ball along the diagonal sphere. Then the map extends to the projective plane FP^2 if and only if q is a multiple of

$$2\, tor(\pi_{2m-1}(FP^1)), \tag{40}$$

where tor is the order of the torsion subgroup, equal to 1, 12, or 120 respectively when F is $\mathbf{C}$, $\mathbf{H}$, or $\mathbf{Ca}$.

Proof of Theorem 3. The metrics we use are Riemannian submersions over a round 2-sphere, where the fiber is Gromov's left-invariant metric on $S^1 \times S^3$ which was the first example of systolic freedom. To describe free left-invariant metrics on $S^3 \times S^3$, it is convenient first to describe some useful coordinates on flat tori of unit area and unit 1-systole. Let $r \geq 0$, and consider the metric

$$(dz - r dy)^2 + dy^2 \quad \text{on} \quad T^2 = \mathbf{R}^2/\mathbf{Z}^2 = T^1 \times C, \tag{41}$$

and let $*$ be its Hodge star operator. Here T^1 and C are parametrized respectively by the y and z axes. Note that if r is an integer, this torus is isometric to the standard 'unit square' torus, one of whose sides is the loop C (what will appear in the sequel is the homothetic metric 'square of side 2π'). Meanwhile $length^2(T^1) = \|\partial/\partial y\|^2 = 1 + r^2$. We set $R^2 = 1 + r^2$. We have

$$*(dz) = *(dz - rdy) + r*(dy) = dy - r(dz - rdy) = (1 + r^2)dy - rdz = R^2 dy - rdz. \quad (42)$$

Now we describe Gromov's metric on $S^1 \times S^3$ in a way that lends itself to generalisation. Let $S^1 = C$ be the unit circle with the standard 1-form dz, and let b be the standard contact 1-form on S^3. Gromov's metric is then obtained simply by modifying the product metric of a circle of length 2π and 3-sphere of radius $R = \sqrt{1 + r^2}$, by adding a nondiagonal term $-2rb\,dz$ (symmetric tensor product). Here the Hopf great circles of length $2\pi R$ play the role of T^1 above. More explicitly, let us complete b to a basis (b, b', b'') of 1-forms which is orthonormal with respect to the metric of unit radius, so that $db = b' \wedge b''$. Then Gromov's metric is a modification of formula (41):

$$(dz - rb)^2 + b^2 + R^2(b'^2 + b''^2). \quad (43)$$

Note that

$$*(dz) = *(dz - rb) + r*b = (b - r(dz - rb)) \wedge R^2 db = R^4 b \wedge db - rR^2 dz \wedge db. \quad (44)$$

The form $*dz$ is closed since $d*dz = R^4 db \wedge db = 0$. We normalize $*dz$ to obtain a calibrating form $\frac{1}{R}*dz$, whose restriction to the 3-sphere $S^3 \subset S^1 \times S^3$ coincides with the volume form $R^3 b \wedge db$ of this round sphere of radius R.

Now take two 3-spheres, denoted respectively (A, a) and (B, b), with their contact forms a and b. Choose metrics on A and B as follows. The sphere A is endowed with the metric of a Riemannian submersion, whose fiber is of length 2π, over a round 2-sphere of radius $\sqrt{r}R$ (the reason for choosing $\sqrt{r}R$ instead of $R^{3/2}$ will become clear later). The sphere B is a round 3-sphere of radius R. We form the direct sum metric on $A \times B$, of volume growth on the order of R^6 when $R \to \infty$. Next we modify it by adding the nondiagonal term

$$-2rab \text{ where } r = \sqrt{R^2 - 1}, \quad (45)$$

and a and b are the contact forms. The new metric on $A \times B$ has volume growth R^5 instead of R^6, due to the appearance in the determinant of the unimodular matrix

$$\begin{bmatrix} 1 & r \\ r & R^2 \end{bmatrix} \quad (46)$$

with respect to the pair of vectors dual to (a, b). More precisely, let (a, a', a'') be a standard basis completing a, and similarly for b, as before. The metric described above can be written down as follows:

$$rR^2(a'^2 + a''^2) + (a - rb)^2 + b^2 + R^2(b'^2 + b''^2). \quad (47)$$

Here we inflate the metric of the contact plane of the first factor A to endow it with volume growth R^3, that of the second factor. If $S^1 \subset A$ is a Hopf fiber, the restriction of the metric to $S^1 \times B \subset A \times B$ coincides with Gromov's metric. Then $sys_3([A]) = vol(A) = 8\pi^2 r R^2$ by a standard argument using the coarea inequality. As far as B is concerned, we show that $sys_3([B])$ has the growth of $vol(B) = 2\pi^2 R^3$. We show this by using a calibrating form described below.

By analogy with the case of $S^1 \times S^3$, one would like to replace $*(dz)$ by $*(a \wedge da)$. The difficulty is that this form is no longer closed. There are two natural ways to introduce a corrective term, only one of which works, as we now show.

Lemma 2. With respect to the metric (47) we have:
(i) $\|da\| = r^{-1}R^{-2}$ and $\|db\| = R^{-2}$;
(ii) $\|a \wedge da\| = r^{-1}R^{-1}$;
(iii) $d * (a \wedge da) = -da \wedge db$.

Proof. (i) Indeed, $\|\sqrt{r}Ra'\| = 1$ and $\|Rb'\| = 1$. (ii) We have $\|a \wedge da\| = \|a\|\|a'\|\|a''\| = \sqrt{1+r^2}r^{-1}R^{-2} = r^{-1}R^{-1}$. (iii) We have

$$\begin{aligned} *(a \wedge da) &= *(((a-rb)+rb) \wedge da) = (b - r(a-rb))\|da\|\|db\|^{-1}db \\ &= r^{-1}((1+r^2)b - ra)db. \end{aligned} \tag{48}$$

Note that the restriction of the form $rR * (a \wedge da)$ to a coordinate sphere $B \subset A \times B$ is

$$R(1+r^2)b \wedge db = R^3 b \wedge db, \tag{49}$$

the volume form of B induced by the metric (47). We have $d * (a \wedge da) = r^{-1}(R^2 db - r da) \wedge db = -da \wedge db$. Note that we chose the metric on A so as to render the formula for $d * (a \wedge da)$ free of r and R.

Thus we need to correct $*(a \wedge da)$ by $(1-t)da \wedge b + ta \wedge db$, for some $t \in \mathbf{R}$, to make it closed.

Now $\|a \wedge db\| = \|a\|\|b'\|\|b''\| = \sqrt{1+r^2}R^{-2} = R^{-1}$ while $\|da \wedge b\| = \|a'\|\|a''\|\|b\| = r^{-1}R^{-2}$, which is much smaller. In view of (ii) above, the closed form

$$r^2(*(a \wedge da) + da \wedge b) \tag{50}$$

is of norm which tends to 1 as $r \to \infty$. Using this form, we obtain volume estimates on the order of R^3 for $[B]$ and also any class $\alpha[A] + \beta[B]$ with β in $\mathbf{Z}^*$. This completes the proof of Theorem 3.

Acknowledgments. The author is grateful to the Geometry & Dynamics Program at Tel Aviv University for hospitality during the preparation of a part of this work.

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