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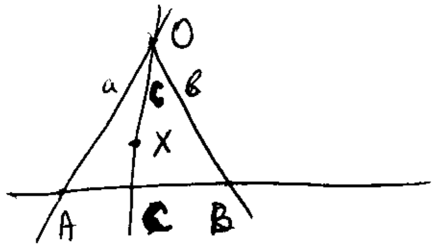
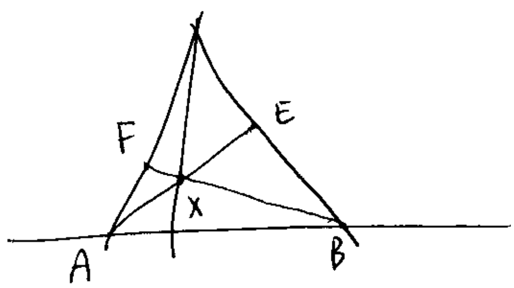
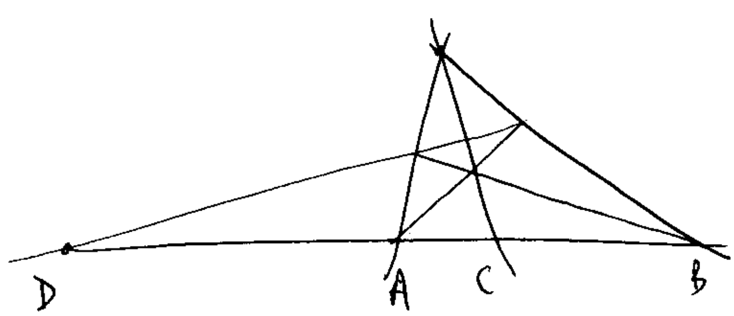
① (a) To construct a fourth harmonic line we choose a point X on c:



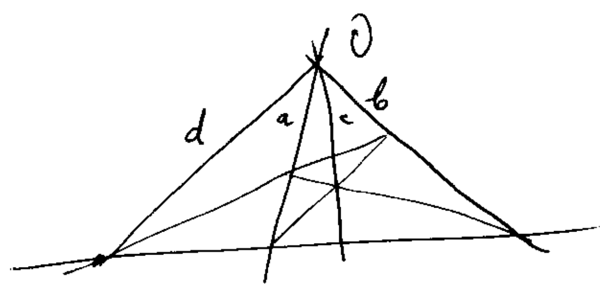
and then draw lines AX and BX:



Then the line EF meets AB in the desired point D:



We then obtain the line d as OD:



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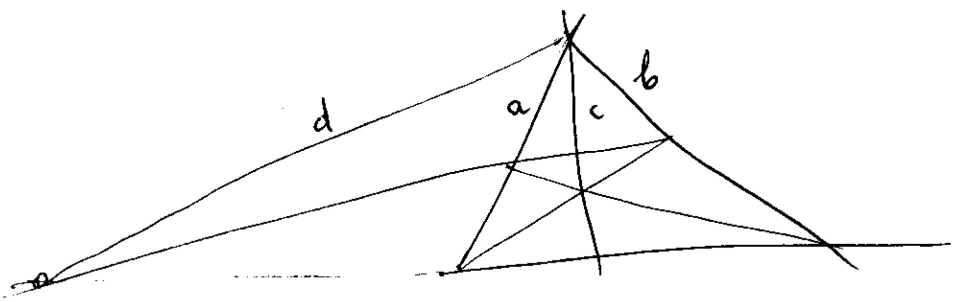
① (b) The construction produces the drawing



① (c) If $R(ABCD) = -1$ then according to the theorem proved in class we also obtain $R(BA\ CD) = \frac{1}{-1} = -1$ and $R(AB\ DC) = \frac{1}{-1} = -1$.

Also $R(CD\ AB) = \frac{CA/AD}{CB/BD} = \frac{CA/CB}{AD/BD} = \frac{AC/CB}{AD/DB} = R(ABCD) = -1$.

Permuting the first pair as before we therefore obtain

$R(DC\ AB) = -1$

and permuting the second pair we obtain

$R(CD\ BA) = -1$

Switching both pairs in $R(ABCD)$ we obtain

$R(BA\ DC) = -1$

and switching both pairs in $R(CDAB)$ we obtain

$R(DC\ BA) = -1$

resulting altogether in 8 permutations including the trivial one.

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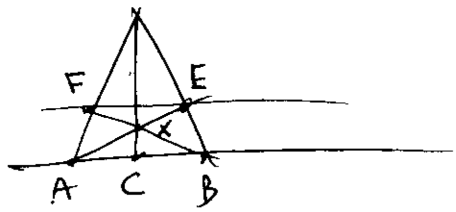
1 (d) If C is the midpoint of AB then carrying out the construction in 1 (a) we obtain a line EF parallel to AB:



This can be seen for example by choosing for X the intersection of the three medians.
Therefore the fourth harmonic point is the intersection of the parallel lines AB and EF i.e. the point at infinity of their common pencil.

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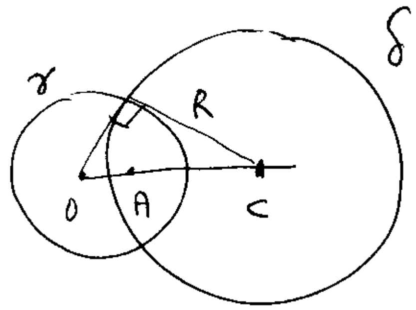
(2)



Let C be the center of δ and R the radius of δ.

By Pythagoras $1 + R^2 = (OC)^2$

and since inversion in δ satisfies $OC \cdot AC = R^2$

we obtain $1 + OC \cdot AC = OC^2$

or $1 + OC(OC - OA) = OC^2$

Thus $1 + OC^2 - OC \cdot OA = OC^2$

Hence $OC \cdot OA = 1$

Thus $OC = \frac{1}{OA}$ giving the position of the center of the circle δ.

Furthermore $R = \sqrt{OC^2 - 1} = \sqrt{\frac{1}{OA^2} - 1}$

giving the radius of the circle δ.

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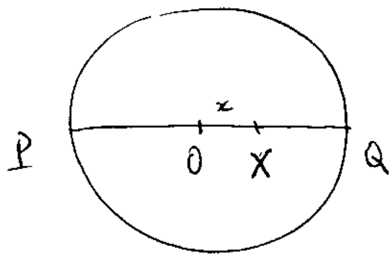
3 (a) Let $x = OX$ and assume $x > 0$ to fix ideas.



Then $d = d(O, X) = \left| \log R(OX\ PQ) \right| =$

$$= \left| \log \frac{OP/PX}{OQ/QX} \right| = \left| \log \frac{OP}{OQ}\,\frac{QX}{PX} \right| =$$

$$= \left| \log \frac{XQ}{PX} \right| = \left| \log \frac{1-x}{1+x} \right| .$$

Since $\frac{1-x}{1+x} < 1$ we obtain $d = \log \frac{1+x}{1-x}$.

3 (b) Since $d = \log \frac{1+x}{1-x}$ we obtain

$(1-x)e^d = 1+x$ hence $(e^d - 1) = x\,(e^d + 1)$.

Thus $x = \dfrac{e^d - 1}{e^d + 1}$

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(4) We first prove the following lemma.

<u>Lemma</u>. There is a 1-1 correspondence between lines through A not parallel to ℓ (where $A \notin \ell$) and points on ℓ.

<u>Proof</u>. By axiom A1 of affine geometry there is a unique line through A and a point on ℓ. Similarly, any line through A not parallel to ℓ meets ℓ in a unique point as a consequence of axioms A1 and A2.

Now suppose there are n points on ℓ. Since by axiom A3 not all points on the affine plane are collinear, we can chose a point A not on ℓ. By the Lemma, there are $n+1$ lines through A (including the line parallel to ℓ).

Now given a line m we consider two cases.

(a) If m does not pass through A, then the Lemma applies and shows that there are n points on m, corresponding to the n lines through A not parallel to m.

(b) If m passes through A, we use a construction presented in class to choose a point B not on ℓ and not on m (by completing a parallelogram). Using B, The same argument then applies to show that there are n points on m.