88826 Differential geom., moed B, 10 sep '15

Duration of the exam: 3 hours.

All answers must be justified by providing complete proofs.

1. Let $\mathbb{R}$ act on the manifold $M = \mathbb{R}^2$ by means of the flow $\theta_t(x, y)$ acting according to the formulas

$$x \mapsto x\cos t + y\sin t, \quad y \mapsto -x\sin t + y\cos t,$$

i.e., $\theta_t(x, y) = (x\cos t + y\sin t, -x\sin t + y\cos t)$.

(a) Show that this is a globally defined action of $\mathbb{R}$ on M.
(b) find the infinitesimal generator X of this flow.
(c) Describe the orbits of this flow.

2. Let X be the infinitesimal generator of a flow $\theta = \theta(t, p)$ on a manifold M.

(a) Give a definition of a vector field X on a smooth manifold M.
(b) Define what it means for X to be invariant under a flow on M.
(c) Prove that X is invariant under θ.

3. Let F a prevector field on a manifold.

(a) Give a definition of a D^1 prevector field.
(b) Show how using transfer one defines hyperfinite iteration of F.
(c) Define the hyperreal flow F_t and the real flow f_t on M.
(d) Prove that F is invariant under the hyperreal flow defined by F.

4. If c is an upper bound for a set $A \subset \mathbb{R}$ we will write $A \leq c$. The completeness property of $\mathbb{R}$ asserts that if A is bounded from above, then there is a least upper bound $d \in \mathbb{R}$ for A, or in formulas

$$(\forall A \subset \mathbb{R})\big[(\exists c \in \mathbb{R})[A \leq c] \Rightarrow (\exists d \in \mathbb{R})[A \leq d] \wedge (\forall e \in \mathbb{R})[A \leq e \Rightarrow d \leq e]\big]$$

(a) Express the condition $A \leq c$ by an explicit first-order formula with quantification only over numbers.
(b) Reformulate the completeness property given by the formula above in a way amenable to an application of the transfer principle.
(c) Apply the transfer principle to the resulting formula so as to obtain a correct statement over ${}^*\mathbb{R}$.
(d) Give an example of the failure of the naive application of transfer to the formula above.

GOOD LUCK!