March 5, 2019

Differential geometry 88-826 homework set 2

1. Let $M = T^2$ be the 2-torus. Prove that the tangent bundle of M can be naturally identified with the product $T^2 \times \mathbb{R}^2$.

2. Consider the unit sphere S^2 in spherical coordinates (θ, φ). Consider the vector fields $\frac{\partial}{\partial\theta}$ and $\frac{\partial}{\partial\varphi}$ defined everywhere on S^2.

(a) Find the zeros of the vector field $\frac{\partial}{\partial\theta}$ if any;
(b) compute the length of the vector field $\frac{\partial}{\partial\theta}$ at an arbitrary point with coordinates (θ, φ);
(c) Find the zeros of the vector field $\frac{\partial}{\partial\varphi}$ if any;
(d) compute the length of the vector field $\frac{\partial}{\partial\varphi}$ at an arbitrary point with coordinates (θ, φ).

3. Consider the real projective plane $\mathbb{RP}^2$ defined in the lecture as the collection of equivalence classes $[x]$ where $x \in \mathbb{R}^3 \setminus \{0\}$ (see choveret of the course, section 1.5 on pages 15–16). Prove that the following two definitions are naturally equivalent to the one given in the lecture:

(1) Let S^2 be the unit 2-sphere. Then $\mathbb{RP}^2$ is the set of unordered pairs $\{p, -p\}$ where $p \in S^2$.
(2) Let $U \subseteq S^2$ be the upper hemisphere, namely the set $U = \{(x, y, z) \in S^2 \colon z \geq 0\}$. Then $\mathbb{RP}^2$ is obtained from U by identifying antipodal points on the equator by an equivalence relation $\sim$ where by definition $(x, y, 0) \sim (-x, -y, 0)$ whenever $x^2 + y^2 = 1$.