March 10, 2014

Differential geometry 88-826-01 homework set 2

1. Let $x(u^1, u^2)$ be a parametrized surface in $\mathbb{R}^3$. Consider indices i, j, k, ℓ. Set $x_{ij} = \frac{\partial^2 x}{\partial u^i \partial u^j}$. Find an expression for the scalar product $\langle x_{ij}, x_{k\ell} \rangle$ in terms of a combination of the following data: the Γ_{ij}^k symbols, the coefficients of the first fundamental form, and the coefficients of the second fundamental form.

2. This problem concerns the calculation of Gaussian curvature K, and relies on the material of the course 88-201, as well.

(a) Describe *four* possible ways of calculating K.

(b) Which of the approaches in (a) are applicable if the data one is given is that the metric is defined in coordinates (u^1, u^2) by the metric coefficients $g_{ij}(u^1, u^2) = \frac{1}{(u^2)^2}\delta_{ij}$ but one is *not* given any explicit imbedding in Euclidean space?

(c) Calculate K for the metric in (b).

3. Let $x(u^1, u^2)$ be a parametrized surface in 3-space, and $n = n(u^1, u^2)$ its unit normal vector. Express the following quantities in terms of the coefficents g_{ij} of the first fundamental form; the inverse matrix $g^{k\ell}$; the symbols Γ_{ij}^k; the coefficents $L^i_{\ j}$ of the Weingarten map; and the coefficients L_{ij} of the second fundamental form, simplifying the expression as much as possible. Here the Einstein summation convention implies summation over every index occurring both in a lower position and in an upper position.

Expand the scalar product and simplify as much as possible:

(a) $\langle x_{\ell j}, x_k \rangle \left(\delta^k_{\ m}\right) g^{m\ell}$.

(b) $\langle n_j, x_{pq} \rangle \left(\delta^j_{\ r}\right)$.

(c) $\langle x_{stu}, n \rangle$.

(d) $g_{pq} \left(\delta^q_{\ s}\right) g^{su} \delta^p_{\ u}$.