Due Date: 29 march '23

1. Consider the plane $\mathbb{R}^2$ with standard basis (e_1, e_2). Consider the unit circle $S^1 \subseteq \mathbb{R}^2$. In Section 1.7 of the lecture notes (see `http://u.math.biu.ac.il/~katzmik/88-826.html`) we constructed an atlas for the manifold S^1 consisting of four coordinate neighborhoods, and specified the transition functions ϕ. This exercise seeks to use the stereographic projection to construct a different atlas for the manifold S^1 consisting of only two coordinate neighborhoods, (A, u) and (B, v).

(a) Let $A = S^1 \setminus \{e_2\}$. Given a point $x \in A$, consider the line $\ell_x^+ \subseteq \mathbb{R}^2$ through x and e_2. Let $u \colon A \to \mathbb{R}$ map each point $x \in A$ to the intersection of the line ℓ_x with the x-axis in $\mathbb{R}^2$. Find an explicit formula for u.

(b) Let $B = S^1 \setminus \{-e_2\}$. Consider the line $\ell_x^- \subseteq \mathbb{R}^2$ through x and $-e_2$. Let $v \colon B \to \mathbb{R}$ map each point $x \in B$ to the intersection of the line ℓ_x^- with the x-axis in $\mathbb{R}^2$. Find an explicit formula for v.

(c) Determine the transition function $v = \phi(u)$ associated with the overlap $A \cap B$.

(d) With respect to the new atlas, is S^1 a manifold of class C^1? Is it of class C^∞? Of class C^{an}?

2. Let $\mathrm{Mat}_{n,n}(\mathbb{R})$ be the set of square matrices with real coefficients. Consider the subset $S \subseteq \mathrm{Mat}_{n,n}(\mathbb{R})$ consisting of all matrices X such that $\mathrm{Tr}(X) \neq 0$ (matrices with nonzero trace). Determine whether S is an open submanifold, with explanation.

3. Let $X = \mathbb{C}^2 \setminus \{0\}$ be the collection of pairs $x = (x^0, x^1)$ distinct from the origin. Define an equivalence relation $\sim$ between $x, y \in X$ by setting $x \sim y$ if and only if there is a complex number $t \neq 0$ such that $y = tx$, i.e.,

$$y^i = tx^i, \quad i = 0, 1 \quad \text{where} \quad t \in \mathbb{C} \setminus \{0\}.$$

Denote by $[x]$ the equivalence class of $x \in X$. Define the complex projective line, $\mathbb{CP}^1$, as the collection of equivalence classes $[x]$, i.e., $\mathbb{CP}^1 = \{[x] \colon x \in X\}$.

(1) Prove that $\mathbb{CP}^1$ is a smooth manifold by exhibiting charts and the transition function ϕ;
(2) check the metrizability condition;
(3) determine the real dimension of $\mathbb{CP}^1$.