February 26, 2019

DIFFERENTIAL GEOMETRY   88-826-01   HOMEWORK SET 1

1. Consider the plane $\mathbb{R}^2$ with standard basis e_1, e_2. Consider the unit circle $S^1 \subseteq \mathbb{R}^2$. In the lecture we constructed an atlas for the manifold S^1 consisting of four coordinate neighborhoods, and specified the transition functions. This exercise seeks to use the stereographic projection to construct an atlas for the manifold S^1 consisting of only two coordinate neighborhoods, (A, u) and (B, v).

   (a) Let $A = S^1 \setminus \{e_2\}$. Given a point $x \in A$, consider the line $\ell_x^+ \subseteq \mathbb{R}^2$ through x and e_2. Let $u : A \to \mathbb{R}$ map each point $x \in A$ to the intersection of the line ℓ_x with the x-axis in $\mathbb{R}^2$. Find an explicit formula for u.

   (b) Let $B = S^1 \setminus \{-e_2\}$. Consider the line $\ell_x^- \subseteq \mathbb{R}^2$ through x and $-e_2$. Let $v : B \to \mathbb{R}$ map each point $x \in B$ to the intersection of the line ℓ_x^- with the x-axis in $\mathbb{R}^2$. Find an explicit formula for v.

   (c) Determine the transition function in the overlap $A \cap B$.

   (d) With respect to the new atlas, is S^1 a manifold of class C^1? Is it of class C^∞? Of class C^{an}?

2. Let $Mat_{n,n}(\mathbb{R})$ be the set of square matrices with real coefficients. Consider the subset $S \subseteq Mat_{n,n}(\mathbb{R})$ consisting of all matrices X such that $\text{Tr}(X) \neq 0$. Is S an open submanifold?

3. Let $X = \mathbb{C}^{n+1} \setminus \{0\}$ be the collection of $(n + 1)$-tuples $x = (x^0, \ldots, x^n)$ distinct from the origin. Define an equivalence relation $\sim$ between $x, y \in X$ by setting $x \sim y$ if and only if there is a complex number $t \neq 0$ such that $y = tx$, i.e.,

$$y^i = tx^i, \quad i = 0, \ldots, n.$$

Denote by $[x]$ the equivalence class of $x \in X$. Define the complex projective space, $\mathbb{CP}^n$, as the collection of equivalence classes $[x]$, i.e., $\mathbb{CP}^n = \{[x] : x \in X\}$. Prove that $\mathbb{CP}^n$ is a smooth manifold and determine its real dimension.