Infinitesimal analysis 88-503 Moed B 1 sept '22

All answers must be fully justified by giving complete proofs.
Each problem is worth 22 points.

1. Let $s : \mathbb{N} \to \mathbb{R}^+$ be a sequence such that the extended hypersequence ${}^*s : \mathbb{N} \to {}^*\mathbb{R}^+$ never takes infinitesimal values. Prove that s is bounded away from zero in $\mathbb{R}$.

2. Suppose that $a_i \geq 0$ for all $i \in \mathbb{N}$. Prove that the series $\sum_1^\infty a_i$ converges iff $\sum_1^n a_i$ is finite for *all* infinite n, and that this holds iff $\sum_1^n a_i$ is finite for *some* infinite n.

3. Let f be a real function that is defined on some open neighbourhood of $c \in \mathbb{R}$. Show that if f is constant on $hal(c)$, then it is constant on some interval $(c - \varepsilon, c + \varepsilon) \subseteq \mathbb{R}$.

4. Prove that a set $A \subseteq \mathbb{R}$ is open if and only if for every point $x \in A$ one has $hal(x) \subseteq {}^*A$.

5. Show that the overflow principle is equivalent to the following statement: If an internal subset $X \subseteq {}^*\mathbb{N}$ contains arbitrarily small infinite members, then it is unbounded in $\mathbb{N}$, i.e., contains arbitrarily large finite members.

Good Luck!