88201 Analytic and Differential Geometry, Prof. Katz, moed A, 1 august 2025. Each problem is worth 27 points.

You must write legibly and provide explanation and justification for all answers.

1. For each of the following curves, determine whether it is degenerate and determine its type.

(a) $x^2 - 4xy + 5y^2 + 1 = 0$.
(b) $x^2 - 4x + 3y^2 + 4 = 0$.
(c) $x^2 - 4xy + 3y^2 = 0$.
(d) $x^2 - 6xy + y + 9y^2 = 0$.

2. Let $a > 0$, and let $z = \sqrt{a^2 - x^2}$ be a curve in the (x, z) plane.

(a) Find a parametrisation of the corresponding surface of revolution $M \subseteq \mathbb{R}^3$.
(b) Give two definitions of the mean curvature H of an arbitrary surface in $\mathbb{R}^3$.
(c) Calculate the mean curvature of the surface M of part (a).

3. The following expressions use the Einstein summation convention. All of the expressions need to be expressed in terms of the coefficients of the first and second fundamental forms, and the Gamma coefficients.

(a) Which indices in expression $\langle x_{\ell j}, n_k \rangle \, \delta^k{}_m \, g^{m\ell}$ are free? Simplify the expression as much as possible, and determine which indices in the final expression are free.
(b) Which indices in expression $\langle x_j, x_{pq} \rangle \, \delta^j{}_r$ are free? Simplify the expression as much as possible, and determine which indices in the final expression are free.
(c) Which indices in expression $\langle x_{pqr}, x_m \rangle$ are free? Simplify the expression as much as possible, and determine which indices in the final expression are free.
(d) Indices a, b, c, d run from 1 to 2. Which indices in expression $\delta^a{}_b \, g_{ca} \, g^{bd} \, \delta^c{}_d$ are free? Simplify the expression as much as possible, and determine which indices in the final expression are free.

4. Find point or points of maximum (if they exist) of curvature for the following curves in the (x, y)-plane:

(a) $x + y^2 = 1$.
(b) $xy + 1 = 0, \; x > 0$.
(c) $x + \ln y = 0$.

List of formulas:

$$D_B(F) = F_{xx}F_y^2 - 2F_{xy}F_xF_y + F_{yy}F_x^2$$

$$k_C = \frac{|D_B(F)|}{|\nabla F|^3}$$

$$\Gamma_{ij}^k = \frac{1}{2}(g_{i\ell,j} - g_{ij,\ell} + g_{j\ell,i})g^{\ell k}$$

$$K = \frac{2}{g_{11}}\left(\Gamma^2_{1[1,2]} + \Gamma^j_{1[1}\Gamma^2_{2]j}\right)$$