PRIMITIVE ALGEBRAS WITH ARBITRARY GELFAND-KIRILLOV DIMENSION

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Abstract. We construct, for every real $\beta \geq 2$, a primitive affine algebra with Gelfand-Kirillov dimension $\beta$. Unlike earlier constructions, there are no assumptions on the base field. In particular, this is the first construction over $\mathbb{R}$ or $\mathbb{C}$.

Given a recursive sequence $\{v_n\}$ of elements in a free monoid, we investigate the quotient of the free associative algebra by the ideal generated by all non-subwords in $\{v_n\}$.

We bound the dimension of the resulting algebra in terms of the growth of $\{v_n\}$. In particular, if $|v_n|$ is less than doubly-exponential then the dimension is 2. This also answers affirmatively a conjecture of Salwa [12].

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1. Preliminaries

Let $A$ be an affine $k$-algebra. The Gelfand-Kirillov dimension [6] of $A$ is defined as

$$GKdim(A) = \limsup_{s \to \infty} \frac{\log \dim(V + V^2 + V^3 + \cdots + V^s)}{\log s}$$

where $V$ is a finite-dimensional subspace that generates $A$ as an algebra. (see [9] for details).

It is easily seen that $GKdim(A) = 0$ iff $A$ is finite dimensional. Otherwise $GKdim(A) \geq 1$, and by Bergman’s gap theorem [3], either $GKdim(A) = 1$ (in which case $A$ is a PI-algebra by [14]), or $GKdim(A) \geq 2$.

If $A$ is PI then $GKdim(A)$ equals the transcendence degree of $A$ over $k$ [2], and is thus an integer.

Affine algebras with GKdim arbitrary real $\beta \geq 2$ were constructed by Borho-Kraft [4] and by Warfield [15] (cf. also [9, 2.9]). These examples fail to be prime. For a semiprime example, see [7].

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In [8] Irving and Warfield constructed primitive algebras with arbitrary GKdim, under the restriction that the base field \( F \) has an infinite-dimensional algebraic extension.

In this note we provide straightforward examples of primitive affine algebras over an arbitrary field, having arbitrary GKdim \( \geq 2 \).

Our construction is a generalization of the Morse algebra. It is a monomial algebra, that is, the quotient of a free associative algebra by an ideal generated by monomials. The growth of such algebras has been studied, for example, in [1], [5] and [11].

We assume the ideal to be generated by all monomials that are not subwords in a given sequence \( \{v_n\} \) of elements in a free monoid, and relate properties of the resulting algebra \( A \) to the sequence \( \{v_n\} \).

In section 2 we prove that under conditions (1), (2) below, \( \text{GKdim}(A) \) is bounded in terms of \( \frac{\log |v_{n+1}|}{\log |v_n|} \). If \( |v_n| \) is less then doubly-exponential (as is the case if \( v_n \) is defined by a constant recursion rule), then \( \text{GKdim}(A) \leq 2 \). Some theory of recurring sequences over finite fields is used in section 3 to choose \( \{v_n\} \) that achieve the bound, thus producing prime algebras of arbitrary GKdim. These examples are shown to be primitive in section 4.

2. MONOMIAL ALGEBRAS

Let \( S \) be a free finitely-generated monoid. If \( L \) is an ideal of \( S \), then \( A = k[S/L] \cong k[S]/k[L] \) is an affine monomial algebra.

Fix a group \( T \) of permutation automorphisms of \( S \). If \( M \subset S \) is a submonoid, let \( M_T \) denote the closure of \( M \) under the operation of \( T \).

Let \( \{v_n\} \) be a sequence in \( S \). Evidently, the set \( L \) of words that are not subwords in any \( v_n \), is an ideal. In this case \( \{\text{subwords of length } s \text{ in } v_n: \text{all } n\} \) is a basis for the \( s \)'th homogeneous part of \( A \).

All algebras discussed below are defined over a fixed (but arbitrary) field \( k \), with ideals \( L \) as above, where we assume

\[
\forall n : |v_n| \leq |v_{n+1}|
\]

(2) \quad \forall n : v_{n+\kappa} \in \{ v_n, v_{n+1}, \ldots, v_{n+\kappa-1} \} \quad \text{for some fixed } \kappa.

Write \( x \leq y \) if \( x \) is a subword in \( y \). As in Salwa [12], there is an obvious criterion for \( A \) to be prime.

**Remark 2.1.** \( A \) is prime iff for any \( i, j \) there exist \( w \in S \) such that \( v_i w v_j < v_k \) for some \( k \).
If \( x \in \mathcal{S}, \ X \subseteq \mathcal{S} \), let \( W_s(x) = \{ w : w \leq x, |w| = s \} \) and \( W_s(X) = \bigcup_{x \in X} W_s(x) \). Obviously, \( |W_s(x)| < |x| \) if \( s > 1 \).

Write \( w_s = |W_s(\{v_n\})| \). Then

\[
GKdim(A) = \limsup_{s \to \infty} \frac{\log(w_1 + w_2 + \cdots + w_s)}{\log s}.
\]

Note that \( GKdim(A) = 0 \) iff \( |v_n| \) is bounded. We assume henceforth that this is not the case, so by Bergman’s gap theorem \( GKdim(A) = 1 \) or \( GKdim(A) \geq 2 \).

The main tool we use to compute \( GKdim(A) \) is the following simple lemma.

**Lemma 2.2.** If \( |x_2|, |x_3|, \ldots, |x_{m-1}| \geq s \), then

\[
W_s(x_1 x_2 \ldots x_m) = W_s(x_1 x_2) \cup W_s(x_2 x_3) \cup \cdots \cup W_s(x_{m-1} x_m).
\]

**Proof.** A subword of length \( s \) of \( x_1 x_2 \ldots x_m \) can never intersect more then two consecutive \( x_i \)'s. \( \square \)

**Theorem 2.3.** Let \( d = \limsup \frac{\log |v_{n+1}|}{\log |v_n|} \). Then \( GKdim(A) \leq 1 + d^\epsilon \).

**Proof.** Fix \( s \) and some \( \epsilon > 0 \).

There is some \( \mu \) such that \( |v_\mu| < s \leq |v_{\mu+1}| \). Iterating assumption (2), we get \( v_{\mu+i} \ll v_{\mu+1}, \ldots, v_{\mu+\kappa} \gg \tau \) for all \( i \geq 1 \). Thus, by Lemma 2.2,

\[
w_s = |W_s(\{v_{\mu+i} : i \geq 1\})|
\leq | \bigcup_{\tau_1, \tau_2 \in \tau, 0 < j_1, j_2 \leq \kappa} W_s(\tau_1 v_{\mu+j_1} \tau_2 v_{\mu+j_2}) |
\leq \sum_{\tau_1, \tau_2 \in \tau, 0 < j_1, j_2 \leq \kappa} |W_s(\tau_1 v_{\mu+j_1} \tau_2 v_{\mu+j_2}) |
\leq \sum_{\tau_1, \tau_2 \in \tau, 0 < j_1, j_2 \leq \kappa} |\tau_1 v_{\mu+j_1} \tau_2 v_{\mu+j_2} |
\leq 2|\tau|^2 \kappa^2 |v_{\mu+\kappa}|
\]

Let \( c = 2|\tau|^2 \kappa^2 \). We have that \( w_1 + w_2 + \cdots + w_s < cs|v_{\mu+\kappa}| \), so

\[
\frac{\log(w_1 + \cdots + w_s)}{\log s} < \frac{\log cs}{\log s} + \frac{\log |v_{\mu+\kappa}|}{\log |v_{\mu}|} \leq 1 + d^\epsilon + \epsilon \text{ for large enough } s.
\]

\( \square \)

If \( |v_n| \) is less then doubly exponential, i.e., \( \log \log |v_{n+1}| - \log \log |v_n| \to 0 \), then \( d = 1 \) and \( GKdim(A) \leq 2 \).

It can be shown that \( GKdim(A) = 1 \) iff for some constant \( C \), almost all the words \( v_n \) are periodic with period \( < C \). We omit the details of the proof.
In many natural examples \( v_n \) are defined recursively. In this case we have

**Corollary 2.4.** Suppose that \( \{v_n\} \) is defined by a constant recursion rule (i.e. the formula for \( v_n \) as a function of \( v_{n-1}, \ldots, v_{n-k} \) does not involve \( n \)), such that assumption (1) is satisfied. Then \( |v_{n+1}| < M|v_n| \) for some constant \( M \), and by Theorem 2.3 we have that \( \text{GKdim}(A) \leq 2 \).

In particular Salwa’s example [12] has Gelfand-Kirillov dimension 2.

We end this section with

**Lemma 2.5.** Assume that for any \( i \), \( v_i < v_k \) for some \( k \). Then \( \text{GKdim}(A) \geq 1 + \limsup \frac{\log w_s}{\log s} \).

**Proof.** The assumption implies that \( w_s \) is nondecreasing. Now

\[
\text{GKdim}(A) = \limsup \frac{\log(w_1 + \cdots + w_{2s})}{\log 2s} \\
\geq \limsup \frac{\log sw_s}{\log 2s} \\
= 1 + \limsup \frac{\log w_s}{\log s}.
\]

\( \Box \)

3. PRIME AFFINE ALGEBRAS WITH ARBITRARY DIMENSION

In this section we present sequences \( \{v_n\} \) that define prime algebras with arbitrary \( \text{GKdim} > 2 \). These examples are shown to be primitive in Section 4.

Some preliminaries from the theory of linear recurring sequences are needed. The reader is referred to [10, Chap. 8] for more details and proofs.

**Proposition 3.1.** Let \( m \geq 1 \) be a natural number. Let \( K \) be the field of order \( 2^m \). Pick a generator \( u \) of the multiplicative group \( K^* \). Let \( g(x) = g_0 + g_1x + \cdots + x^m \) be the minimal polynomial of \( u \) over \( \mathbb{Z}_2 \).

Define a sequence \( \{b_i\} \) over \( \mathbb{Z}_2 \) by \( b_0 = \cdots = b_{m-2} = 0, b_{m-1} = 1 \), and the recursion rule \( b_{i+m} = gb_i + g_1b_{i+1} + \cdots + g_{m-1}b_{i+m-1} \) (\( i \geq 0 \)).

Then \( \{b_i\} \) has period \( 2^m - 1 \), and for every non-zero \( w \in \mathbb{Z}_2^m \), there is a unique \( 0 \leq i < 2^m - 1 \) such that \( w = b_{ib_{i+1}} \cdots b_{i+m-1} \).

Moreover, if \( w \) and \( w' \) are opposite non-zero words (i.e. \( w + w' = (11 \ldots 1) \) of length \( m + 1 \), then exactly one of them appears in \( \{b_i\} \) if \( (11 \ldots 1), (00 \ldots 0) \) do not appear at all).
Definition 3.2. $L_m$ denotes a word of length $2^m + m - 1$ over $\mathbb{Z}_2$, constructed as the first $2^m + m - 2$ elements of a sequence defined as in Proposition 3.1, preceded by a single 0.

For example, we can take $L_1 = 01$, $L_2 = 00110$, $L_3 = 0001011100$ and $L_4 = 0001001101011100$. 

Remark 3.3. Every word of length $m$ appears exactly once as a subword of $L_m$. Two opposite words of length $m+1$ do not appear both in $L_m$ except for the couple $(00\ldots01),(11\ldots10)$. No opposite words of length $m+2$ appear in $L_m$.

Let $S = < x, y >$ be the free monoid on two generators, with the automorphism $v \rightarrow \bar{v}$ defined by $\bar{x} = y, \bar{y} = x$. The substitution of a word $v$ in $L_m$ is defined as the replacement of all 0’s in $L_m$ by $v$ and all 1’s by $\bar{v}$.

For example, $L_1(v) = v\bar{v}$.

Fix a sequence of integers $r_n$. We define $\{v_n\} \subseteq S$ as follows:

$$v_1 = x, \quad v_{n+1} = L_{r_n}(v_n).$$

Define an algebra $A$ using $\{v_n\}$ as in the beginning of section 2. Note that assumptions (1) and (2) are satisfied (with $\kappa = 1$).

Note that if $r_n > 1$ then $v_{n+1}$ starts with $v_nv_n$ and ends with $v_n$. The case $r_n = 1$ gives the well known Morse algebra.

Theorem 3.4. $A$ is prime.

Proof. By Remark 2.1 we must show for any $v_i, v_j$ that $v_iwv_j < v_k$ for some $k$ and a word $w$. Pick $n = \max\{i, j\}$, then $v_i, v_j < v_n$, so pick $w_i, w_j$ such that $v_iw_i$ is a tail of $v_n$ and $w_jv_j$ a head of $v_n$.

If $r_n > 1$ then $v_iw_iw_jv_j < v_nv_n < v_{n+1}$ by definition of $L_{r_n}$. Otherwise suppose $r_n = 1$; if $r_{n+1} > 1$ then $v_iw_i\bar{v}_n w_jv_j < v_{n+1}v_{n+1} < v_{n+2}$, and if $r_{n+2} = 1$ then $v_iw_i\bar{v}_nv_nw_jv_j < v_{n+1}\bar{v}_n = v_{n+2}$.

From now on we assume that $r_n \geq 3$.

Lemma 3.5. If $m \geq (r_n + 2)|v_n|$, then the subwords of length $m$ in $v_{n+1}$ and in $\bar{v}_{n+1}$ are all different.

Proof. Let $k = r_n + 2$. It is enough to prove the assertion in the case $m = k|v_n|$. For $n = 1$ the result follows from 3.3. Let $n > 1$.

Let $a, b$ be two equal subwords of $v_{n+1}$ or $\bar{v}_{n+1}$. Write $v_{n+1}$ as a word on the letters $v_n, \bar{v}_n$ which we call full letters. Then $a, b$ are determined by the full letter $(v_n$ or $\bar{v}_n$) in $v_{n+1}$ or in $\bar{v}_{n+1}$ in which they start, and the relative position in this full letter. The strategy is to show first that $a$ and $b$ start at the same relative position, and then show that they actually start at the same full letter.
Write $a = a_0u_1 \ldots u_{k-1}a_1$ where $|a_0|, |a_1| \leq |v_n|$, and each $u_i$ equals one of the full letters $v_n, \overline{v_n}$; write $b = b_0w_1 \ldots w_{k-1}b_1$ in the same manner. \textit{W.l.o.g.} we assume $|a_0| \geq |b_0|$. Write $a_0 = a_{00}a_{01}$ and $b_1 = b_{10}b_{11}$ where $a_{00} = b_0$ and $b_{11} = a_1$. Also factor $u_i = u_i''u_i'$ and $w_i = w_i''w_i'$ where $|u_i''| = |w_i''| = |a_0| - |b_0|$. Assume $|a_0| - |b_0| \leq \frac{1}{2}|v_n|$ (the other case is treated similarly). Then $u_i = w_i'$ is an equality of words of length $\geq \frac{1}{2}|v_n|$. Since $\frac{1}{2}|v_n| \geq \frac{r_{n-2}+r_{n-1}+r_{n-1}}{2n-2+r_{n-1}}|v_n| = (r_{n-1} + 2)|v_{n-1}|$, the induction hypothesis force $u_i', w_i'$ to begin in the same relative position. But then it follows that $u_i'', w_i''$ are empty words and each of $u_i', w_i'$ is a full letter, $v_n$ or $\overline{v_n}$. By Remark 3.3, the $r_n + 1$ equalities $u_i = u_i' = w_i' = w_i$ force one of two cases: $a, b$ begin in the same position in $v_{n+1}$ or in $\overline{v_{n+1}}$, in which case we are done, or $u_1 \ldots u_{k-1} = w_1 \ldots w_{k-1} = v_n \ldots v_n\overline{v_n}$ and $a \leq v_{n+1}, b \leq v_{n+1}$ (or vice versa). But $v_n \ldots v_n\overline{v_n}$ is the header of $v_{n+1}$, so $a_0, b_0$ must be empty. Then we have that $a_1 = b_1$, the $(r_n + 2)$'th equality of full letters, a contradiction of Remark 3.3.

We can now compute the Gelfand-Kirillov dimension of $A$.

\textbf{Theorem 3.6.} Let $d = \lim \sup \frac{r_1 + \cdots + r_n}{r_1 \cdots r_{n-1}}$. Then $GKdim(A) = d + 1$.

\textit{Proof.} Note that $|v_n| = |L_{r_1}| |L_{r_2}| \cdots |L_{r_{n-1}}|$, so

$$r_1 + \cdots + r_{n-1} < \log_2 |v_n| < n + r_1 + \cdots + r_{n-1}.$$  

By Theorem 2.3 we have

$$GKdim(A) \leq 1 + \lim \sup \frac{\log_2 |v_{n+1}|}{\log_2 |v_n|} = 2 + \lim \sup \frac{\log_2 |L_{r_n}|}{\log_2 |v_n|} \leq$$

$$\leq 2 + \lim \sup \frac{r_n + 1}{r_1 + \cdots + r_{n-1}} = 1 + d.$$  

For the other direction, recall that by Lemma 3.5 all of the subwords of length $s = (r_n + 2)|v_n|$ in $v_{n+1}$ are different. Thus

$$w_s \geq |v_{n+1}| - s = (2^{r_n} - 3)|v_n|$$

(where $w_s$ is the number of subwords of length $s$ in any $v_n$), and, if $d > 1$,

$$\frac{\log_2 w_s}{\log_2 s} > \frac{\log_2 (2^{r_n} - 3) + \log_2 |v_n|}{\log_2 (r_n + 2) + \log_2 |v_n|} > \frac{r_1 + \cdots + r_{n-1}}{\log_2 (r_n) + n + r_1 + \cdots + r_{n-1}}.$$  

The \textit{limsup} of the lower bound is $d$ (whether $d$ is finite or infinite). If $d = 1$, then the expression in the middle already approaches 1. \hfill $\Box$
Finally, let $\beta \in \mathbb{R}$, $\beta \geq 1$.

Take $r_n = \max([\beta^n], 3)$, and define an algebra $A$ as above. Checking the conditions of Theorem 3.6, we arrive at

**Theorem 3.7.** $A$ is an affine prime algebra with $\text{GKdim}(A) = \beta + 1$.

In particular, the bound in 2.3 is tight (at least for $\kappa = 1$).

4. **OUR EXAMPLES ARE PRIMITIVE**

In this section we show that $A$ is primitive. We assume that $r_n \geq 3$, and $r_n > 3$ infinitely often (note that for dimension 2 we must take $r_n = 4$).

**Definition 4.1.** For $u, v \in S$, let $u \leq_l v$ ($u \leq_r v$) denote that $u$ is a head (tail) of $v$. An element $a \in A$ is a left (right) tower if the set of monomials of $a$ is linearly ordered by $\leq_l$ ($\leq_r$).

Being a left tower is invariant under multiplication by a monomial from the left.

**Lemma 4.2.** Let $L < A$ be a left ideal. Then $L$ contains a left tower.

Moreover, for every $a \in A$, $wa$ is a left tower for some monomial $w$.

**Proof.** It is enough to show that if $a_1, a_2$ are monomials, and $wa_1 = 0$ iff $wa_2 = 0$ (all $w \in S$), then $a_1, a_2$ are $\leq_l$-comparable. Assume $|a_1| \geq |a_2|$.

For some $n$ we have $a_2 \leq v_n$ and $r_n > 3$. Write $v_n = \alpha a_2\beta$, $v_{n+1} = uv_n$.

By assumption $u\alpha a_1 \neq 0$, so $u\alpha a_1 \leq v_m$ for some $m$. Writing $v_m$ as a word in the letters $v_{n+1}, v_{n+1}$, the intersection of $u$ with some letter is of length $> \frac{1}{2}|u| > (r_n + 2)|v_n|$. By Lemma 3.5, $u$ (and thus $u\alpha a_1$) appears in $v_m$ as a header of $v_{n+1}$, and we get $u\alpha a_2 \leq_l u\alpha a_1$, as desired. \qed

**Corollary 4.3.** Let $I \trianglelefteq A$ be an ideal. Then $I$ contains a monomial.

**Proof.** By Lemma 4.2 and left-right symmetry, there is some $a \in I$ that is a left and right tower.

Let $u, w$ be two different monomials in $a$, $|u| \leq |w|$. Multiplying by long enough monomials from both sides we may assume that $w = v_n$, and $\frac{1}{2}|v_n| < |u|$.

Now $u \leq_r v_n$ and $u \leq_l v_n$ implies by Lemma 3.5 that $u = v_n$, a contradiction. \qed

Let $J = \langle x, y \rangle \trianglelefteq A$, a maximal ideal in $A$.

Recall that if $J_1, \ldots, J_n \trianglelefteq R$ are maximal ideals in a (unital) ring $R$, and $J_1J_2 \ldots J_n \subseteq I \neq R$, then $I \subseteq J_i$ for some $i$ (for taking a maximal ideal $I_1 \supseteq I$ we have some $J_i \subseteq I_1$, implying $J_i = I_1 \supseteq I$).
Corollary 4.4. $J$ is the unique maximal ideal in $A$. Moreover, for any $I \leq A$, the quotient $A/I$ is a finite dimensional $k$-algebra and $\text{Jac}(A/I) = J/I$.

Proof. By Corollary 4.3, $v_n \in I$ for some $n$. Now every monomial of length $> 2|v_{n+1}|$ contains $v_{n+1}$ or $\overline{v}_{n+1}$ and thus $v_n$ as a subword, so $J^m \subseteq I$ for $m = 2|v_{n+1}|$, and thus $I \subseteq J$.

$A/I$ is finite dimensional spanned by $\{\text{words of length } < m\}$, and $\text{Jac}(A/I) = \text{Jac}(A/J^m)/(I/J^m) = (J/J^m)/(I/J^m) = J/I$. \hfill \Box

Corollary 4.5. The only prime ideals of $A$ are $0, J$.

In particular, the (classical) Krull dimension of $A$ is one.

Proof. $0$ is prime by Remark 2.1. If $I \neq 0$ is prime then $J/I = \text{Jac}(A/I) = 0$ so $I = J$. \hfill \Box

Theorem 4.6. $A$ is primitive.

Proof. Assume, on the contrary, that $A$ is non-primitive. Then the only primitive ideal of $A$ is $J$, so this is the Jacobson radical of $A$. In particular, $x + y \in J$ should be quasi-regular.

But if $a(1-x-y) = 1$, let $w$ be a longest monomial in $a$. Necessarily $wx \neq 0$ or $wy \neq 0$, so $wx$ (say) appears on the left hand side of the equality but not on the right hand side, a contradiction. \hfill \Box

From Corollary 4.4 it follows that $A$ satisfies ACC on two-sided ideals (since a chain going up from $I \leq A$ has length $\leq \dim(A/I)$ as a $k$-algebra). For one-sided ideals the situation is not that pleasant.

Remark 4.7. $A$ is a not left-Noetherian.

Proof. We construct a left ideal that is not finitely-generated.

Ordering the monomials in $A$ by $<_r$ we get a tree $T_1$, in which every node is a root for a tree with at least two, and thus infinitely many, branches (indeed, if $w < v_n$ is a monomial write $v_n = \alpha w \beta$, then $w <_r v_n \alpha w$ and $w <_r \overline{v}_n \alpha w$). Now define a tree $T_n$ and $w_n \in T_n$ ($n \geq 1$) as follows: Let $w \in T_n$ be a monomial with $wx, wy \in T_n$. Take $w_n = wx$, and $T_{n+1}$ the tree with root $wy$.

Then $L = Aw_1 + Aw_2 + \ldots$ is definitely not f.g., since the $w_i$ are $\leq_r$ incomparable. \hfill \Box
References


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