Compact-like properties, normality and $C^*$-embeddedness of the hyperspace of compact sets

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\( \mathcal{CL}(X) \) denote the hyperspace of non-empty closed sets of \( X \) with the Vietoris topology. \( \mathcal{K}(X) \) is the subspace of compact sets.

The Vietoris topology has the sets of the form

\[
V^+ = \{ A \in \mathcal{CL}(X) : A \subseteq V \} \quad \text{and} \quad V^- = \{ A \in \mathcal{CL}(X) : A \cap V \neq \emptyset \}
\]

like a subbase, when \( V \) is an open set of \( X \).

Given open sets of \( X \), \( U_1, \ldots, U_n \), define

\[
< U_1, \ldots U_n > = \{ T \in \mathcal{CL}(X) : T \in \bigcup_{1 \leq k \leq n} U_k^+, \ T \in U_k^- \}. 
\]
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Compact-like properties, normality and $C^*$-embeddedness of $\mathcal{K}(X)$
Theorem

\((M.) \mathcal{C} \mathcal{L}(X) \) is:

1. \(T_2 \) iff \(X\) is \(T_3\),
2. \(T_3 \) iff \(\mathcal{C} \mathcal{L}(X)\) is Tychonoff iff \(X\) is \(T_4\),
3. \(T_4 \) iff \(\mathcal{C} \mathcal{L}(X)\) is compact iff \(X\) is compact.

Theorem

\((M.) \mathcal{K}(X) \) is:

1. \(T_2 \) iff \(X\) is \(T_2\),
2. \(T_3 \) iff \(X\) is \(T_3\),
3. Tychonoff iff \(X\) is Tychonoff.
Theorem

(M.) $C\mathcal{L}(X)$ is:

1. $T_2$ iff $X$ is $T_3$,
2. $T_3$ iff $C\mathcal{L}(X)$ is Tychonoff iff $X$ is $T_4$,
3. $T_4$ iff $C\mathcal{L}(X)$ is compact iff $X$ is compact.

Theorem

(M.) $\mathcal{K}(X)$ is:

1. $T_2$ iff $X$ is $T_2$,
2. $T_3$ iff $X$ is $T_3$,
3. Tychonoff iff $X$ is Tychonoff.
About the normality of $\mathcal{K}(X)$

Theorem

(M.) $\mathcal{K}(X)$ is metrizable iff $X$ is it.

Note that $\mathcal{CL}(X)$ is metrizable iff $X$ is compact metrizable.

Theorem

(Moresco and Artico) If $L$ is the Sorgenfrey line then $\mathcal{K}(L)$ is not normal.
About the normality of $\mathcal{K}(X)$

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About the normality of $\mathcal{K}(X)$

**Theorem**

(M.) $\mathcal{K}(X)$ is metrizable iff $X$ is it.

Note that $\mathcal{C}\mathcal{L}(X)$ is metrizable iff $X$ is compact metrizable.

**Theorem**

(Moresco and Artico) If $L$ is the Sorgenfrey line then $\mathcal{K}(L)$ is not normal.
Theorem

Let $\gamma$ an ordinal number.

1. if $\text{cof}(\gamma) = \omega$ then $K([0, \gamma))$ is normal.

2. (K.) if $\text{cof}(\gamma) > \omega$ then $K([0, \gamma))$ is normal iff $\gamma$ is regular.

3. (K. Hirata) if $\text{cof}(\gamma) > \omega$ then $K([0, \gamma))$ is orthocompact iff $\gamma$ is regular.

Questions:

1. For which other class of spaces the hyperspace $K$ is normal?.

2. Are there conditions $C$ such that: $K(X)$ is normal iff $X$ has $C$?
Theorem

Let $\gamma$ an ordinal number.

1. If $\text{cof}(\gamma) = \omega$ then $\mathcal{K}([0, \gamma))$ is normal.

2. (K.) If $\text{cof}(\gamma) > \omega$ then $\mathcal{K}([0, \gamma))$ is normal iff $\gamma$ is regular.

3. (K. Hirata) If $\text{cof}(\gamma) > \omega$ then $\mathcal{K}([0, \gamma))$ is orthocompact iff $\gamma$ is regular.

Questions:

1. For which other class of spaces the hyperspace $\mathcal{K}$ is normal?.

2. Are there conditions $\mathcal{C}$ such that: $\mathcal{K}(X)$ is normal iff $X$ has $\mathcal{C}$?.
Compact-like properties, normality and $C^*$-embeddedness of $\mathcal{K}(X)$
Theorem

(G.) $\mathcal{CL}(X)$ is:

1. $\omega$-bounded (ultrapseudocompact) iff $X$ is it,
2. $p$-compact ($p$-pseudocompact) iff $X$ is it,
3. $\alpha$-bounded iff $X$ is it.

Questions: Are there conditions $\mathcal{C}$ such that: $\mathcal{CL}(X)$ is countable compact (pseudocompact) iff $X$ has $\mathcal{C}$?
Theorem

\((G.) C\mathcal{L}(X)\) is:

1. \(\omega\)-bounded (ultrapseudocompact) iff \(X\) is it,
2. \(p\)-compact (\(p\)-pseudocompact) iff \(X\) is it,
3. \(\alpha\)-bounded iff \(X\) is it.

Questions: Are there conditions \(\mathcal{C}\) such that: \(C\mathcal{L}(X)\) is countable compact (pseudocompact) iff \(X\) has \(\mathcal{C}\)?
In $\mathcal{K}(X)$.

**Theorem**

(A.O.T.) TFSE:

1. $X$ is $\alpha$-hyperbounded,
2. $\mathcal{K}(X)$ is initially $\alpha$-compact.
3. $\mathcal{K}(X)$ is $\alpha$-bounded, and
4. $\mathcal{K}(X)$ is $\alpha$-hyperbounded.

Milovančević made this prove for $\alpha = \omega$
Theorem

(A.O.T.) Let $X$ be a space. Then the next statements are equivalent:

1. $X$ is pseudo-$\omega$-bounded,
2. $\mathcal{K}(X)$ is pseudo-$\omega$-bounded,
3. $\mathcal{K}(X)$ pseudo-$\mathcal{D}$-bounded for some $\mathcal{D} \subseteq \mathbb{N}^*$,
4. $\mathcal{K}(X)$ is strongly-$p$-pseudocompact for some $p \subseteq \mathbb{N}^*$,
5. $\mathcal{K}(X)$ is $p$-pseudocompact for some $p \subseteq \mathbb{N}^*$ and
6. $\mathcal{K}(X)$ is pseudocompact.
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Pseudocompactness has a different approach. Let \( I : \mathcal{CL}(X) \rightarrow \mathcal{CL}(\beta X) : I(A) = Cl_{\beta X}A. \)

When \( \beta(\mathcal{CL}(X)) = \mathcal{CL}(\beta X) \)? or when is \( \mathcal{CL}(X) \) is natural (I) \( C^* \)-embedded in \( \mathcal{CL}(\beta X) \)?

**Theorem**

*Let \( X \) be normal.*

1. (K.G.) If \( \beta(\mathcal{CL}(X)) = \mathcal{CL}(\beta X) \) then \( \mathcal{CL}(X) \) (and so \( \mathcal{CL}(X) \times \mathcal{CL}(X) \)) is pseudocompact.

2. (G.) If \( \mathcal{CL}(X) \times \mathcal{CL}(X) \) is pseudocompact then \( \beta(\mathcal{CL}(X)) = \mathcal{CL}(\beta X) \).

Natsheh proved the converse but we think it is wrong.
Pseudocompactness has a different approach. Let \( I : CL(X) \longrightarrow CL(\beta X) : I(A) = Cl_{\beta X}A \).

When \( \beta(CL(X)) = CL(\beta X) \)? or when is \( CL(X) \) is natural (I) \( C^* \)-embedded in \( CL(\beta X) \)?

**Theorem**

*Let \( X \) be normal.*

1. *(K.G.) If \( \beta(CL(X)) = CL(\beta X) \) then \( CL(X) \) (and so \( CL(X) \times CL(X) \)) is pseudocompact.
2. *(G.) If \( CL(X) \times CL(X) \) is pseudocompact then \( \beta(CL(X)) = CL(\beta X) \).*

Natsheh proved the converse but we think it is wrong.
Pseudocompactness has a different approach. Let
\[ I : \mathcal{CL}(X) \longrightarrow \mathcal{CL}(\beta X) : I(A) = Cl_{\beta X} A. \]
When \( \beta(\mathcal{CL}(X)) = \mathcal{CL}(\beta X) \)? or when is \( \mathcal{CL}(X) \) is natural
(1) \( C^* \)-embedded in \( \mathcal{CL}(\beta X) \)?

**Theorem**

*Let \( X \) be normal.*

1. *(K.G.)* If \( \beta(\mathcal{CL}(X)) = \mathcal{CL}(\beta X) \) then \( \mathcal{CL}(X) \) (and so \( \mathcal{CL}(X) \times \mathcal{CL}(X) \)) is pseudocompact.

2. *(G.)* If \( \mathcal{CL}(X) \times \mathcal{CL}(X) \) is pseudocompact then \( \beta(\mathcal{CL}(X)) = \mathcal{CL}(\beta X) \).

Natsheh proved the converse but we think it is wrong.
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Question: When is $\mathcal{K}(X)$ $C^*$-embedded in $\mathcal{CL}(X)$? Is there some relation between this problem and the problem: When is $\beta(\mathcal{CL}(X)) = \mathcal{CL}(\beta X)$?

Theorem

(H.) If $\mathcal{K}(X)$ is normal and $C^*$-embedded in $\mathcal{CL}(X)$ then $\mathcal{K}(X)$ is $\omega$-bounded (and so $\mathcal{K}(X)$ is $C$-embedded in $\mathcal{CL}(X)$).

So if $\mathcal{K}(X)$ is normal and $C^*$-embedded in $\mathcal{CL}(X)$ then $\beta(\mathcal{CL}(X)) = \mathcal{CL}(\beta X)$ and the converse is not true. We don’t know what happens if $\mathcal{K}(X)$ is not normal.
Question: When is $\mathcal{K}(X)$ $C^*$-embedded in $\mathcal{CL}(X)$? Is there some relation between this problem and the problem: When is $\beta(\mathcal{CL}(X)) = \mathcal{CL}(\beta X)$?

**Theorem**

(H.) If $\mathcal{K}(X)$ is normal and $C^*$-embedded in $\mathcal{CL}(X)$ then $\mathcal{K}(X)$ is $\omega$-bounded (and so $\mathcal{K}(X)$ is $C$-embedded in $\mathcal{CL}(X)$).

So if $\mathcal{K}(X)$ is normal and $C^*$-embedded in $\mathcal{CL}(X)$ then $\beta(\mathcal{CL}(X)) = \mathcal{CL}(\beta X)$ and the converse is not true. We don’t know what happens if $\mathcal{K}(X)$ is not normal.
Corollary

Let $X$ be a metrizable space. Then $\mathcal{K}(X)$ is $C^*$-embedded in $\mathcal{CL}(X)$ iff $X$ is a compact space.

Theorem

(A. O. T.) Suppose $\mathcal{K}(X)$ is normal and $C^*$-embedded in $\mathcal{CL}(X)$. TFAE:

1. $X$ is $\tau$-bounded,
2. $X$ is $\tau$-hyperbounded,
3. $\mathcal{K}(X)$ is $\tau$-pseudocompact,
4. $\mathcal{K}(X)$ is initially $\tau$-compact,
5. $\mathcal{K}(X)$ is $\tau$-bounded, and
6. $\mathcal{K}(X)$ is $\tau$-hyperbounded.
Corollary

Let $X$ be a metrizable space. Then $\mathcal{K}(X)$ is $C^\ast$-embedded in $\mathcal{CL}(X)$ iff $X$ is a compact space.

Theorem

(A. O. T.) Suppose $\mathcal{K}(X)$ is normal and $C^\ast$-embedded in $\mathcal{CL}(X)$. TFAE:

1. $X$ is $\tau$-bounded,
2. $X$ is $\tau$-hyperbounded,
3. $\mathcal{K}(X)$ is $\tau$-pseudocompact,
4. $\mathcal{K}(X)$ is initially $\tau$-compact,
5. $\mathcal{K}(X)$ is $\tau$-bounded, and
6. $\mathcal{K}(X)$ is $\tau$-hyperbounded.
Theorem

(A. O. T.) Suppose $\mathcal{K}(X)$ is $C^*$-embedded in $\mathcal{CL}(X)$.

TFAE:

1. $X$ is compact,
2. $X$ is $\sigma$-compact,
3. $\mathcal{K}(X)$ is compact,
4. $\mathcal{K}(X)$ is $\sigma$-compact,
5. $\mathcal{K}(X)$ is Lindelöf,
6. $\mathcal{K}(X)$ is paracompact,
7. $\mathcal{K}(X)$ is normal and metacompact,
8. $\mathcal{CL}(X)$ is compact, and
9. $\mathcal{CL}(X)$ is $\sigma$-compact.
Our main result:

**Theorem**

(K. O. R.) Let $\gamma$ be an ordinal number. TFAE:

1. $\mathcal{K}([0, \gamma))$ is $C$-embedded in $\mathcal{CL}([0, \gamma))$.
2. $\mathcal{K}([0, \gamma))$ is $C^*$-embedded in $\mathcal{CL}([0, \gamma))$.
3. $\text{cof}(\gamma) \neq \omega$
Our main result:

**Theorem**

(K. O. R.) Let $\gamma$ be an ordinal number. TFAE:

1. $\mathcal{K}([0, \gamma))$ is $C$-embedded in $\mathcal{CL}([0, \gamma))$.
2. $\mathcal{K}([0, \gamma))$ is $C^\ast$-embedded in $\mathcal{CL}([0, \gamma))$.
3. $\text{cof}(\gamma) \neq \omega$
Theorem

(K. O.) Let $\gamma$ be an infinite ordinal number. TFAE:

1. $\text{cof}(\gamma) \neq \omega$,
2. $[0, \gamma)$ is pseudocompact,
3. $\beta(\mathcal{CL}([0, \gamma))) = \mathcal{CL}(\beta([0, \gamma)))$,
4. $\beta(\mathcal{K}([0, \gamma))) = \mathcal{K}(\beta([0, \gamma)))$,
5. $\beta(\mathcal{CL}([0, \gamma))) = \mathcal{CL}([0, \gamma])$, and
6. $\beta(\mathcal{K}([0, \gamma))) = \mathcal{K}([0, \gamma])$. 

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Theorem

(K. O.) Let $\gamma$ be an infinite ordinal number. TFAE:

1. $\text{cof}(\gamma) \neq \omega$,
2. $[0, \gamma)$ is pseudocompact,
3. $\beta(\mathcal{CL}([0, \gamma))) = \mathcal{CL}(\beta([0, \gamma)))$,
4. $\beta(\mathcal{K}([0, \gamma))) = \mathcal{K}(\beta([0, \gamma)))$,
5. $\beta(\mathcal{CL}([0, \gamma))) = \mathcal{CL}([0, \gamma])$, and
6. $\beta(\mathcal{K}([0, \gamma))) = \mathcal{K}([0, \gamma])$. 

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INTRODUCCIÓN

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5. $C^*$-embeddedness
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Theorem

1. (M.) $\mathcal{K}(X)$ is 0-dimensional iff $X$ is it.
2. (K. T.) $\mathcal{C}\mathcal{L}(\omega)$ is strong 0-dimensional.
3. (K. T.) $\mathcal{K}([0, \gamma))$ is strong 0-dimensional for every $\gamma$.

Theorem

1. (O. O.) If $\text{cof}(\gamma) \neq \omega$ then $\mathcal{C}\mathcal{L}(\omega)$ is strong 0-dimensional.
2. (O.) If $\text{cof}(\gamma) \neq \omega$ then $\mathcal{K}([0, \gamma))$ is strongly 0-dimensional.
**Theorem**

1. (M.) $\mathcal{K}(X)$ is 0-dimensional iff $X$ is it.
2. (K. T.) $\mathcal{CL}(\omega)$ is strong 0-dimensional.
3. (K. T.) $\mathcal{K}([0, \gamma))$ is strong 0-dimensional for every $\gamma$.

**Theorem**

1. (O. O.) If $\text{cof}(\gamma) \neq \omega$ then $\mathcal{CL}(\omega)$ is strong 0-dimensional.
2. (O.) If $\text{cof}(\gamma) \neq \omega$ then $\mathcal{K}([0, \gamma))$ is strongly 0-dimensional.
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Compact-like properties, normality and $C^*$-embeddedness of $\mathcal{K}(X)$
J. Angoa, Y. F. Ortiz-Castillo, A. Tamariz-Mascarua
*Compact like properties on hyperspaces*, to appear.


Gracias gracias gracias!!!

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