Jointly Continuous Utility Functions defined on submetrizable $k_\omega$-spaces.

Rita Ceppitelli
PERUGIA - Italy

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A preference relation \( \preceq \) on a set (of alternatives) \( X \) is a preorder, that is a reflexive and transitive binary relation. The preference relation \( \preceq \) is complete or total if every pair of elements of \( X \) is comparable.

In Economics, preference relations are often described by means of utility functions.
A function $u : X \rightarrow \mathbb{R}$ is a utility function representing a preference relation $\preceq$ if:

(i) $\forall x, y \in X \text{ t. c. } x \preceq y \Rightarrow u(x) \leq u(y)$;

(ii) $\forall x, y \in X \text{ t. c. } x \prec y \Rightarrow u(x) < u(y)$.
utility functions

$X$ a commodity set

$\preceq$ a customer preference relation

$x \preceq y, \ x, y \in X$

means that the commodity $x$ is weakly preferred to $y$

to represent $\preceq$ by a utility function $u : X \rightarrow \mathbb{R}$ means to numerically measure the ranking of a customer preference by associating to each possible consumption bundle a real number that measures its utility: **the greater the utility, the more preferred is the bundle, and conversely.**
It has been interesting to introduce some structures (topological, linear, algebrical. . . .) on \((X, \preceq)\) and to require that the utility function has properties connected with the introduced structure.

We are interested in continuous utility functions.

**Definition**

A preference relation on a topological space \(X\) is continuous if for every \(x \in X\) the sets \((-\infty, x]\) and \([x, +\infty)\) are closed in \(X\).
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Another frequently taken assumption is \( \preceq \) to be closed (cf. Nachbin (1965) and Levin (1983)).

**Definition**

A preference relation \( \preceq \) on a topological space \( X \) is said to be closed if its graph \( \{(x, y) \in X \times X : x \preceq y\} \) is a closed subset of the topological product \( X \times X \).

Continuous and closedness properties are equivalent in the total case.
In general a closed preorder is always continuous (Nachbin (1965)).
Peleg (*) was the first who presented sufficient conditions for the existence of a continuous utility function for a partial order on a topological space. Peleg solved a problem which was posed by Aumann in the context of expected utility.

Aumann observed that a rational decision-maker may express *indecisiveness* (or equivalently *incomparability*) between two alternatives, so that he is not forced to express *indifference*.

Existence of jointly continuous utility functions

Let $X$ be a topological space and $\Gamma$ a set of closed preorders on $X$.

The Problem of the Existence of Jointly Continuous Utility Functions is
to find topological conditions on $\Gamma$ and $X$ in order to exist a
continuous function

$$u : \Gamma \times X \to \mathbb{R}$$

such that $u(\preceq, \cdot)$ is a utility function for every $\preceq \in \Gamma$.

Clearly if $\Gamma = \{\leq\}$ we have the classic continuous representation
problem of a continuous preference relation.
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Levin’s Theorem

**Theorem (Levin, 1983)**

Let $\Gamma$ be metrizable and let $X$ be locally compact and second countable. Moreover, assume that the set

$$G = \{ (\preceq, x, y) : x \preceq y \}$$

is closed in $\Gamma \times X \times X$.

Then there exists a continuous function $u : \Gamma \times X \to [0, 1]$ such that, for each $\preceq \in \Gamma$, $u(\preceq, \cdot)$ is a continuous utility function.

A natural topology on the set $\Gamma$ of preorders of $X$ should satisfy the following condition:

$$x_n \to x, \ y_n \to y, \ \preceq_n \to \preceq, \ x_n \preceq_n y_n \implies x \preceq y.$$  

If the spaces $\Gamma$ and $X$ are metrizable, the former condition is equivalent to require the set $G = \{(\preceq, x, y) : x \leq y\}$ to be closed in $\Gamma \times X \times X$. 

Rita Ceppitelli  PERUGIA - Italy

Jointly Continuous Utility Functions defined on submetrizable $k\omega$-spaces.
Let $\mathcal{P}$ be a space of closed preorders defined on closed subsets $D \subset X$ (preorders with moving domain $D(\preceq)$) and
$\Phi = \{(\preceq, x) : \preceq \in \mathcal{P}, x \in D(\preceq)\}$

**Theorem (Levin, 1983)**

*If $\mathcal{P}$ is metrizable and $X$ is locally compact second countable and $M = \{(\preceq, x, y) : \preceq \in \mathcal{P}, x, y \in D(\preceq), x \preceq y\}$ is closed in $\mathcal{P} \times X \times X$, there exists a continuous function $u : \Phi \rightarrow \mathbb{R}$ such that $u(\preceq, \cdot)$ is a utility function for every $\preceq \in \mathcal{P}$.***
Theorem (Back, 1986)

Let $X$ be a locally compact and second countable space. There exists a continuous map $\nu : \mathcal{P} \to \mathcal{U}_\tau$ such that $\nu(\preceq)$ is a utility function for every $\preceq \in \mathcal{P}$. Any such map $\nu$ is actually a homeomorphism of $\mathcal{P}_{\text{Ins}}$ onto $\nu(\mathcal{P}_{\text{Ins}})$, where $\mathcal{P}_{\text{Ins}}$ is the family of total locally non-sati ed preorders.

$\mathcal{P}$ is the space of total closed preorders defined on closed subsets of $X$, endowed with the Fell topology.

$\mathcal{U}_\tau$ is the space of all continuous utility functions defined on closed subsets of $X$ with the $\tau_c$ topology, a generalized compact-open topology.

The Fell topology on $CL((X, \tau))$, has as a subbase

$$U^-=\{B \in CL((X, \tau)) : B \cap U \neq \emptyset\}, \ U \in \tau \text{ and}$$

$$(K^c)^+ = \{B \in CL((X, \tau)) : B \cap K = \emptyset\}, \ K \text{ compact in } (X, \tau).$$

The $\tau_c$-topology on $\mathcal{U}_\tau$ has as a subbase

$$[G] = \{(D, u) \in \mathcal{U}_\tau : D \cap G \neq \emptyset\}$$

$$[K : I] = \{(D, u) \in \mathcal{U}_\tau : u(D \cap K) \subset I\}$$

where $G$ is an open subset of $X$, $K \subset X$ is compact and $I \subset \mathbb{R}$ is open (possibly empty).
A preference relation $\preceq$ on a topological space $X$ is said to be locally non satisfied if for every $x \in X$ and for every neighbourhood $U$ of $x$ there is $y \in U$ such that $x \prec y$. 
Theorem (CCH, 2010-11)

Let \((X, \tau)\) be a regular space submetrizable by a boundedly compact metric \(\rho\). There exists a continuous map

\[ \nu : (\mathcal{P}, \tau(\mathcal{L})) \to (\mathcal{U}_\tau, \tau_c) \]

such that \(\nu(\preceq)\) is a utility function for \(\preceq\), for every \(\preceq \in \mathcal{P}\).

Jointly Continuous Utility Functions defined on submetrizable $k\omega$-spaces.
The space $\mathcal{S}'$ of tempered distributions (Example 3.3)(*) is an example of a submetrizable $k_\omega$-space, not submetrizable by a boundedly compact metric.

A Hausdorff topological space is a submetrizable $k_\omega$-space if it is the inductive limit of a nondecreasing sequences of metrizable compact subspaces.

$$X = \bigcup_n K_n$$

We will say that $(K_n)_n$ determines the topology of $X$. 
Theorem

Every submetrizable $k_\omega$-space $X$ is a quotient space of a locally compact second countable space.

\[
\pi : (\hat{X}, \eta) \rightarrow (X, \tau)
\]

if $X = \bigcup_n K_n$

then $\hat{X} = \bigoplus_n \{n\} \times K_n$

$\pi = \nabla_n i|_{K_n}$
We put

\[ \mathcal{P} = \{ \preceq : \preceq \text{ is a preorder on } D(\preceq) \subset X \text{ and } \preceq \in CL((X, \tau) \times (X, \tau)) \} . \]

For every \( \preceq \in \mathcal{P} \) let \( \tilde{\preceq} \) be the preorder so defined:

- \( D(\tilde{\preceq}) = \pi^{-1}(D(\preceq)) \)
- for every \( a, b \in D(\tilde{\preceq}) \), \( a \tilde{\preceq} b \) if and only if \( \pi(a) \leq \pi(b) \).

\[ \tilde{\mathcal{P}} = \{ \tilde{\preceq} = p^{-1}(\preceq) : \preceq \in \mathcal{P} \} \subset CL(\hat{X} \times \hat{X}) \]

where

\[ p = \pi \times \pi : \hat{X} \times \hat{X} \rightarrow X \times X. \]
Let $(X, \tau)$ be a submetrizable $k_\omega$-space. There exists a continuous map

$$\nu : (\mathcal{P}, F(\eta \times \eta)) \to (\mathcal{U}_\tau, \tau_c)$$

such that $\nu(\preceq)$ is a utility function for $\preceq$, for every $\preceq \in \mathcal{P}$. 
\[ (\tilde{\mathcal{P}}, F(\eta \times \eta)) \xrightarrow{\uparrow \Gamma} (\mathcal{V}, \eta_c) \]
\[ (\mathcal{P}, F(\eta \times \eta)) \xrightarrow{\nu = L \circ \tilde{\nu} \circ \Gamma} (\mathcal{U}_\tau, \tau_c) \]

\( \Gamma \) is a homeomorphism
\( L \) is continuous
\( L(\tilde{u}) = u : D(\preceq) \to \mathbb{R} \) with \( u(x) = \tilde{u}(\pi^{-1}(x)) \)
is a utility function for \( \preceq \).
Theorem

There exists a continuous map

\[ \nu_0 : (\mathcal{P}, F(\tau \times \tau)) \to (\mathcal{U}_\tau, \tau_c) \]

such that \( \nu_0(\leq) \) is a utility function for \( \leq \), for every \( \leq \in \mathcal{P} \).
\[ (\mathcal{P}, F(\eta \times \eta)) \xrightarrow{\nu} (U_\tau, \tau_c) \]
\[ \downarrow i \]
\[ (\mathcal{P}, F(\tau \times \tau)) \xrightarrow{\nu_0} \]
\[ \nu_0(\preceq) = \nu(\preceq) \text{ for every } \preceq \in \mathcal{P}. \]