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Coverings, Selections and
Games in Topology
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The centennial of Djuro Kurepa

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Djuro R. Kurepa: August 16, 1907 (Majske Poljane) – November 2, 1993 (Beograd)

University of Zagreb (Faculty of Mathematics and Physics) in 1931.


University of Zagreb (1938–1965);

University of Belgrade (1965–1877);

Professor Emeritus (1977–1993)

About 300 scientific works (the first 1933; the last 1993)
The author or coauthor of 147 books with 53 first editions.

"Set Theory", 1951

"Higher Algebra" (Zagreb, 1965; Beograd, 1971) in two volumes, 1519 pages.

Advisor of 49 master theses and 42 doctoral theses.

Attended:

all International Mathematical Congresses from 1952 until 1982 (at the Edinburgh Congress he was an invited speaker on "Some principles of mathematical education"),

all Balkan Mathematical Congresses (1967-1983),

all 8 Yugoslav Mathematical Congresses from 1949 to 1985,

three Congresses on Methodology of Sciences (Jerusalem, 1964; Hannover, 1979; Salzburg, 1983),

two International Congresses on Philosophy (Brussels, 1953; Venezia, 1958).
Head of the Mathematical Institute in Zagreb (1943–1965)


President of the Union of Societies of Mathematicians, Physicists and Astronomers of Yugoslavia (1955-1960)

President of the Balkan Union of Mathematicians (1977-1983)

Vice-president of the ICMI (1952–1962)

Honorary member of the Nikola Tesla Memorial Society for Canada and USA (1982).
Corresponding member of the Yugoslav Academy of Sciences and Arts in Zagreb (1952)

Member of the Serbian Academy of Sciences and Arts in Belgrade (1988)

Member of the Academy of Sciences and Arts of Bosnia and Hertzegovina (1984)

Member of the Academy of Sciences of Macedonia.
AVNOJ Prize (1976) - the greatest scientific and cultural award in former Yugoslavia (only one mathematician yet was awarded this prize: I. Vidav, Ljubljana)

The Order of work with red flag (1965),

Order for merits for people with golden star (1975)

The Bernard Bolzano Medallion, Praha (1981)
SET THEORY

1. Souslin problem

SH: Every ccc compact LOTS is separable

SL: counterexample of SH

GSH: For every LOTS $X$, $c(X) = d(X)$ (Kurepa’s thesis)

Kurepa, 1935: SL exists iff there exists a Suslin tree (i.e. SH $\Leftrightarrow$ there is no a Suslin tree)

[This result is sometimes attributed to W. Miller (1943)]
2. Kurepa tree (1935, also 1942):

an $\omega_1$-tree with more than $\omega_1$ cofinal branches

Kurepa Hypothesis KH: Kurepa trees exist

F. Rowbottom (1965); L. Bukovsky (1966): If $\omega_1$ is inaccessible in $L$ and if no ordinal between $\omega_1$ and $\omega_2$ belongs to $L$, then KH is true

D.H. Stewart (1966): there exists a (standard transitive) model of ZFC in which there is a Kurepa tree

J.H. Silver: there is a model of ZFC such that $2^{\omega_1} > \omega_2$ holds and there exists a Kurepa tree having exactly $\omega_2$ cofinal branches

J.H. Silver (1966): There is a model of ZFC in which Kurepa trees do not exist
R.M. Solovay: Axiom of constructibility $V = L$ implies the existence of Kurepa trees

K. Devlin (1981): (relative) consistency of $\text{MA} \vdash \neg \text{KH}$ with $\text{ZFC}$.
3. Kurepa line (\(=\) KH)

Kurepa family: a family \(\mathcal{K}\) of subsets of \(\omega_1\) such that: (i) \(|\mathcal{K}| > \omega_1\); (ii) \(\{K \in \mathcal{K} : K \cap \alpha \neq \emptyset\}\) is countable for each countable \(\alpha\)

Kurepa continuum: a linearly ordered continuum \(K\) of weight \(\omega_1\) having more than \(\omega_1\) points of uncountable character and such that the closure of every countable subset of \(K\) has a countable base

Kurepa type: an order type \(\phi\) having the following three properties: (i) \(|\phi| > \omega_1\); (ii) \(d(\phi) = \omega_1\); (iii) \(\phi\) does not contain uncountable real type

Kurepa \(\tau\)-partially ordered set

KH (the existence of a Kurepa line) and the existence of a Kurepa family are equivalent
4. **Weak Kurepa tree:** A tree of height $\omega_1$ and cardinality $\omega_1$ with more than $\omega_1$ cofinal branches

The weak Kurepa Hypothesis $wKH$: there exists a weak Kurepa tree

W.J. Mitchel (1972): (relative) consistency of $\neg wKH$

CH implies $wKH$ and that $MA + \neg wKH$ is stronger than $MA + \neg CH$ ($\neg wKH$ is a strengthening of $\neg CH$);

PFA implies $MA + \neg wKH$

PFA $\Rightarrow wKH$ is false

S. Todorčević (1981): $MA + \neg wKH$ is consistent with ZFC
5. $T$ a tree; $ch(T)$ (resp. $ach(T)$) denotes the supremum of cardinalities of all chains (resp., antichains) in $T$; $bT = \sup\{ch(T), ach(T)\}$

Kurepa (1935): For every tree $T$, $|T| \leq 2^{bT}$; in fact, it is easy to check that for every tree $T$, $|T| \leq (chT)^+ ach(T)$

Kurepa’s Ramification Hypothesis (or Rectangle Hypothesis) RH (1936): for every tree $T$ we have $|T| \leq bT$

SH is true iff for every tree $T$, $bT \leq \omega$ implies $|T| \leq \omega$, i.e. SH $\iff$ in every uncountable tree $T$ there exists an uncountable chain or antichain

In 1989, Kurepa introduced a class of mappings called almost strictly increasing and proved an equivalent of RH in terms of these mappings.
6. **Kurepa antichain principle** KAP: AC is equivalent to the conjunction of (1) every set can be linearly ordered, and (2) every partially ordered set has a maximal antichain.

7. **The problem of existence of strictly increasing real-valued functions on trees and partially ordered sets**

1940, 1941: SH is equivalent to the statement: if $T$ is a tree with $bT \leq \omega$, then there exists a strictly increasing function $f : T \hookrightarrow \mathbb{R}$

1942: A partially ordered set $P$ admits a strictly increasing function into the set of rationals iff it can be decomposed into at most countably many antichains (and that such a function from $P$ into $\mathbb{R}$ exists if every well-ordered subset of $P$ is at most countable)

1954: For a partially ordered set $P$ denote by $\sigma P$ the set of all bounded well-ordered subsets
of $P$ ordered by $A \leq B$ iff $A$ is an initial segment of $B$. Then $\sigma P$ is a tree.

$\sigma Q$ (resp. $\sigma R$) does not admit a strictly increasing mapping into $Q$ (resp. $R$).

1989: Kurepa generalized some previously mentioned results to any aleph

8. Kurepa (1953): Conjecture that $c = 2^\omega$ can be any uncountable aleph of cofinality $> \omega$.

Surprisingly, this conjecture was true (R. Cohen and for a generalization to higher cardinality, for regular cardinals, W.E. Easton 1970). For singular cardinals the problem remains open.
9. Partition Calculus:

Very important, but unfortunately not recognized enough, are Kurepa’s contributions to Partition Calculus.

F.P. Ramsey (1930), W. Sierpinski (1933), Kurepa, P. Erdős, Erdős-R. Rado, A. Hajnal and others.

1937 (without proof), 1939 (with a proof): If $X$ is partially ordered, then

$$|X| \leq (2k_s(X))^{k_0(X)};$$

$k_s(X) = \sup\{|Y| : Y \text{ is an antichain in } X\}$,

$k_0(X) = \sup\{|Y| : Y \subseteq X \text{ is WO or conversely WO}\}$.

1939 and 1952/53, again 1959: For every graph $(G, \rho)$ we have

$$|G| \leq (2k_s(G))^{k_c(G)};$$
\[ k_c(G) = \sup\{|X| : X \subset G \text{ is connected}\}, \]

\[ k_s(G) = \sup\{|Y| : Y \subset G \text{ is anticonnected, i.e.} \ a \neq b \text{ in } G \implies a \text{ non}\rho b\}. \]

S. Todorčević (Partition Problems in Topology, 1989) uses:

**Erdős-Hajnal-Kurepa-Rado theorem** for

\[ (\exp_{s-1}\tau)^+ \rightarrow (\tau^+)^s_\tau \]

known usually as Erdős-Rado theorem

**Erdős-Kurepa-Rado-Sierpinski theorem** for

\[ (\tau^+, \tau^+)^2 \rightarrow ((2^\tau)^+, \tau^\tau)^2. \]

I. Juhász (Cardinal Functions in Topology, 1971) gives Kurepa’s result from 1959:

\[ 2^\tau \nrightarrow (\tau^+, \tau^+)^2 \]

(as a generalization of the result of Sierpinski

\[ 2^\omega \nrightarrow (\omega_1, \omega_1)^2. \])
1. Kurepa (1934): Pseudodistancial spaces
(known today usually as $\omega_\mu$-metrizable or linearly uniformizable spaces).

M. Fréchet (1945): linearly uniformizable spaces

J. Colmez (1947): this class coincides with the class of pseudodistancial spaces.

R. Sikorski (1950): $\omega_\mu$-metric spaces

F.W. Stevenson, W.J. Thron (1969): this class is the class of linearly uniformizable spaces.

1936: For every initial ordinal $\omega_\alpha$ the class $(D_\alpha)$ as special pseudodistancial spaces

Kurepa proved: $D_0$ is exactly the class of metric spaces
Kurepa (1963): For every regular uncountable cardinal $\tau$ there exists a $\tau$-metrizable space which is not orderable

Consequence: If $\tau$ is a regular uncountable cardinal, then every dense-in-itself $\tau$-metrizable space is orderable

[The last result was rediscovered by R. Frankiewicz and W. Kulpa (1979)]

For topological groups by P. Nyikos & H.-C. Reichel (1975): $\omega_\mu$-metrizable non metrizable topological group is orderable.
2. Kurepa (1936): \( R\)-spaces (\(= \) "spaces with ramified bases") as \(T_1\)-spaces having a base which is a tree (with respect to \(\supset\)).

These spaces form a class that is wider than the class of linearly uniformizable spaces;

P. Papić (1954): Every pseudodistancial non-metrizable space is an \( R\)-space.

These spaces were redefined a few times by some authors and under different names.

A.F. Monna (1950): Non-archimedean spaces (which are in fact \( R\)-spaces).

Every \(\omega_\mu\)-metrizable space is non-archimedean for \(\mu > 0\) (for \(\mu = 0\) \(X\) is non-archimedean iff \(\dim X = 0\) by a result of de Groot).

Non-archimedean metrics are called ultra-metrics and were known to Hausdoff (1914).
A.V. Arhangel’skii (1962, 1963): spaces with a base of rank 1 which are precisely $R$-spaces.

P. Papić (1954), Kurepa (1956): an $R$-space is metrizable if and only if it has a base which is a tree of countable height.

A.V. Arhangel’skii, V.V. Filippov (1973): Every $R$-space which is also a $p$-space is metrizable.

P. Papić (1963): an $R$-space is orderable if and only if it has a base which is a tree having no compact elements in levels whose order type is a limit ordinal.

This condition is equivalent to the following Kurepa’s condition ($\delta N_2$) from 1956: every member in a level whose order type is a limit ordinal is a union of open and closed pairwise disjoint subsets of $X$. 
3. The notion of **cellularity** of a space was introduced by Kurepa in his thesis in 1935.

Kurepa’s significant results on the cellularity:

1935: \( c(X) \) is accessible for metric spaces \( X \) and for linearly ordered spaces always but in the case when \( c(X) \) is an inaccessible cardinal

1953: \( s(X) \) is accessible whenever \( X \) is a metric space or a linearly ordered space unless \( s(X) \) is an inaccessible cardinal.

1950: If \( S \) is the Suslin line, then \( c(S \times S) > c(S) \) (more precisely, \( c(S \times S) = c(S)^+ \)).

1962: If \( \tau \) is a cardinal and if \( \{X_\lambda : \lambda \in \Lambda\} \) is a family of topological spaces such that \( c(X_\lambda) \leq \tau \) for all \( \lambda \in \Lambda \), then \( c(\prod\{X_\lambda : \lambda \in \Lambda\}) \leq 2^\tau \).
4. Cardinal invariants on LOTS’:

(1) $d(X) \leq c(X)^+$ (1935)

(2) $hd(X) = d(X)$ and $hl(X) = c(X)$ (1939, 1945)

The first of these two results was obtained later by L. Skula in 1965, and the second one by S. Mardešić & P. Papić in 1962 for compact case and by R. Bennett & D. Lutzer in 1969 for any LOTS.

(3) $c(X) = s(X)$.

(4) $d(X) \leq c(X^2)$ (1952)

S. Todorčević: $hd(X) \leq \min\{c(X)^+, c(X^2)\}$.

This result gives the following remarkable consequence: If $X$ is a linearly ordered continuum and if for some $n > 1$, $c(X^n) \leq \omega$, then $X$ is similar to a segment of the real line; so the only exception is the case $n = 1$ (SH).
1971 (left factorial): \( !n = 0! + 1! + \ldots + (n-1)! \)

In this paper and a few papers published in the subsequent years Kurepa stated the following

**Conjecture** (Problem B44 in: Richard K. Guy, Unsolved problems in Number Theory, Springer-Verlag, 1980): for every \( n \in \mathbb{N} \), the greatest common divisor for \( n! \) and \( !n \) is equal to 2.

By the use of computers this hypothesis was checked for all \( n < 10^6 \) and for such numbers it is true

(D. Slavić for \( n < 10^3 \); Wagstaff for \( n < 50000 \); Ž. Mijajlović for \( n < 317000 \); Ž. Mijajlović and G. Gogić for \( n < 10^6 \)).

In 2003: YES
1973: *(K-function)*

Using

\[ \Gamma(n) = \int_0^\infty e^{-x} x^{n-1} \, dx = (n - 1)!, \quad n \in \mathbb{N}, \]

one obtains

\[ !n = \int_0^\infty e^{-x} x^{n-1} \frac{1}{x-1} \, dx. \]

Kurepa defined the same formula for the complex domain:

\[ !z = \int_0^\infty e^{-x} x^z \frac{1}{x-1} \, dx, \quad \text{Re}(z) > 0. \]