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Convergence properties of hyperspaces
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$2^X$ the set of closed subsets of $X \in T_2$

$A \subset X$, $\mathcal{A}$ a family of subsets of $X$:

$$
A^c = X \setminus A \quad \text{and} \quad \mathcal{A}^c = \{A^c : A \in \mathcal{A}\}, \\
A^- = \{F \in 2^X : F \cap A \neq \emptyset\}, \\
A^+ = \{F \in 2^X : F \subset A\}.
$$

$\Delta \subset 2^X$ closed for finite unions and containing all singletons

*upper $\Delta$-topology* $\Delta^+$ has a base

$$
\{(D^c)^+ : D \in \Delta\} \cup \{2^X\}.
$$

1. $\Delta = \text{CL}(X)$: *upper Vietoris topology* $V^+$

2. $\Delta = \text{K}(X)$: *upper Fell topology* $F^+$

3. $\Delta = \text{F}(X)$: $Z^+$-topology
The lower Vietoris topology $V^- \subseteq X$ is generated by all the sets $U^-, U \subset X$ open.

The $\Delta$-topology: $\tau_{\Delta} = \Delta^+ \lor V^-$. $\tau_{\Delta}$-basic sets:

$$(D^c)^+ \cap (\bigcap_{i \leq m} V_i^-), \; D \in \Delta, V_1, \cdots, V_m \text{ open in } X.$$ 

\(\Delta\)-topologies:

the Vietoris topology $V = V^+ \lor V^-$

the Fell topology $F = F^+ \lor V^-$

the topology $Z = Z^+ \lor V^-$
Let $\mathcal{A}$ and $\mathcal{B}$ be sets whose elements are families of subsets of an infinite set $X$. Then:

$S_1(\mathcal{A}, \mathcal{B})$ denotes the selection principle:

For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(b_n : n \in \mathbb{N})$ such that for each $n$, $b_n \in A_n$ and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

$S_{fin}(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis:

For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ of finite sets such that for each $n$, $B_n \subset A_n$ and $\bigcup_{n \in \mathbb{N}} B_n$ is an element of $\mathcal{B}$. 
For a space $X$, $x \in X$, $\Delta \subset 2^X$:

- $\mathcal{O}$: the collection of open covers of $X$;
- $\Omega$: the collection of $\omega$-covers of $X$;
- $\mathcal{K}$: the collection of $k$-covers of $X$;
- $\Gamma$: the collection of $\gamma$-covers;
- $\Gamma_k$: the collection of $\gamma_k$-covers;
- $\Gamma_\Delta$: the collection of $\gamma_\Delta$-covers;
- $\Omega_x$: the set \{\(A \subset X \setminus \{x\} : x \in \overline{A}\)\};
- $\Sigma_x$: the set of all nontrivial sequences in $X$ that converge to $x$. 
Let us recall that if $\Delta \subset 2^X$, then an open cover $\mathcal{U}$ of $X$ is called a $\Delta$-cover if each $D \in \Delta$ is contained in an element of $\mathcal{U}$ and $X$ does not belong to $\mathcal{U}$ (i.e. the cover is not trivial).

$\mathcal{F}(X)$-covers (resp. $\mathcal{K}(X)$-covers) are called $\omega$-covers (resp. $k$-covers).

An open cover $\mathcal{U}$ of $X$ is said to be a $\gamma_\Delta$-cover if it is infinite and for each $D \in \Delta$ the set $\{U \in \mathcal{U} : D \notin U\}$ is finite.

$\gamma_{\mathcal{F}(X)}$-covers (resp. $\gamma_{\mathcal{K}(X)}$-covers) are called $\gamma$-covers (resp. $\gamma_k$-covers).

Observe that each infinite subset of a $\gamma_\Delta$-cover is still a $\gamma_\Delta$-cover. So, we may suppose that such covers are countable.
\( \alpha_i \)-properties in hyperspaces

A space \( X \) has property \( \alpha_i \), \( i = 1, 2, 3, 4 \), if for each \( x \in X \) and each sequence \( (\sigma_n : n \in \mathbb{N}) \) of elements of \( \Sigma_x \) there is a \( \sigma \in \Sigma_x \) such that:

\( \alpha_1 \): for each \( n \in \mathbb{N} \) the set \( \sigma_n \setminus \sigma \) is finite;

\( \alpha_2 \): for each \( n \in \mathbb{N} \) the set \( \sigma_n \cap \sigma \) is infinite;

\( \alpha_3 \): for infinitely many \( n \in \mathbb{N} \) the set \( \sigma_n \cap \sigma \) is infinite;

\( \alpha_4 \): for infinitely many \( n \in \mathbb{N} \) the set \( \sigma_n \cap \sigma \) is nonempty.

Evidently,

\[ \alpha_1 \Rightarrow \alpha_2 \Rightarrow \alpha_3 \Rightarrow \alpha_4. \]
**Theorem 1** For a space $X$ and a collection $\Delta \subset 2^X$ the following statements are equivalent:

(1) $(2^X, \Delta^+) \text{ is an } \alpha_2\text{-space;}$

(2) $(2^X, \Delta^+) \text{ is an } \alpha_3\text{-space;}$

(3) $(2^X, \Delta^+) \text{ is an } \alpha_4\text{-space;}$

(4) For each $E \in 2^X$, $(2^X, \Delta^+) \text{ satisfies } S_1(\Sigma_E, \Sigma_E);$ 

(5) Each open set $Y \subset X$ satisfies $S_1(\Gamma_\Delta, \Gamma_\Delta).$
Remark 2  *The statement* (1) *implies* (2):

(1) *For each* $E \in 2^X$, $(2^X, \tau_{\Delta})$ *satisfies* $S_1(\Sigma_E, \Sigma_E)$;

(2) *Each open set* $Y \subset X$ *satisfies* $S_1(\Gamma_{\Delta}, \Gamma_{\Delta})$ *(equivalently, $S_{fin}(\Gamma_{\Delta}, \Gamma_{\Delta})$)*.

Two consequences of Theorem 1.

**Corollary 3**  *For a space* $X$ *TFAE*:

(1) $(2^X, F^+)$ *is an* $\alpha_4$-*space*;

(2) *Each open set* $Y \subset X$ *is an* $S_1(\Gamma_k, \Gamma_k)$-*set*.

**Corollary 4**  *For a space* $X$ *TFAE*:

(1) $(2^X, Z^+)$ *is an* $\alpha_4$-*space*;

(2) *Each open set* $Y \subset X$ *is an* $S_1(\Gamma, \Gamma)$-*set*. 
A space is said to be a $\gamma$-set (resp. $\gamma_k$-set) if it satisfies the selection principle $S_1(\Omega, \Gamma)$ (resp. $S_1(\mathcal{K}, \Gamma_k)$).

$(2^X, Z^+)$ is FU ($(2^X, F^+)$ is SFU) if and only if each open set $Y \subset X$ is a $\gamma$-set (a $\gamma_k$-set).

$\Gamma \subset \Omega$ and $\Gamma_k \subset \mathcal{K}$, it follows that $S_1(\Omega, \Gamma) \subset S_1(\Gamma, \Gamma)$ and $S_1(\mathcal{K}, \Gamma_k) \subset S_1(\Gamma_k, \Gamma_k)$.

So:

**Proposition 5** For a space $X$ the following statements hold:

(1) *If* $(2^X, Z^+)$ *is a Fréchet-Urysohn space, then it is an $\alpha_2$-space;*

(2) *If* $(2^X, F^+)$ *is strongly Fréchet-Urysohn, then it is an $\alpha_2$-space.*
Theorem 6  For a space $X$ and a collection $\Delta \subset 2^X$ the following are equivalent:

(1) $(2^X, \Delta^+)$ is an $\alpha_1$-space;

(2) For each open set $Y \subset X$ and each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of $\gamma_\Delta$-covers of $Y$ there is a $\gamma_\Delta$-cover $\mathcal{U}$ of $Y$ intersecting each $\mathcal{U}_n$ in all but finitely many elements.
FU-type properties

$X$ is FU if $\forall x \in X$ and $\forall A \in \Omega_x$ $\exists$ a sequence $(x_n : n \in \mathbb{N})$ in $A$ belonging to $\Sigma_x$. $X$ is sequential if for each non-closed set $A \subset X$ there are a point $x \in X \setminus A$ and a sequence $(x_n : n \in \mathbb{N})$ in $A$ that belongs to $\Sigma_x$. $X$ has countable tightness if for each $x \in X$ and each $A \in \Omega_x$ there is a countable element $B \in \Omega_x$ such that $B \subset A$.

A space $X$ has the Reznichenko property if for each $x \in X$ and each $A \in \Omega_x$ there is a sequence $(B_n : n \in \mathbb{N})$ of pairwise disjoint, finite subsets of $A$ such that each neighborhood $U$ of $x$ intersects all but finitely many sets $B_n$.

A space $X$ has the Pytkeev property if for each $x \in X$ and each $A \in \Omega_x$ there is a sequence $(B_n : n \in \mathbb{N})$ of infinite, countable subsets of $A$ such that each neighborhood $U$ of $x$ contains some $B_n$. 
A space $X$ is said to be:

**FF:** *filter-Fréchet* if for each $x \in X$ and each $A \in \Omega_x$ there is a sequence $(\mathcal{F}_n : n \in \mathbb{N})$ of filter-bases on $A$ such that:

**(FF1)** For each $n \in \mathbb{N}$, there is an $F_n \in \mathcal{F}_n$ such that $x \notin \overline{F_n}$;

**(FF2)** For each neighborhood $U$ of $x$ there is $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies $F_n \subset U$ for some $F_n \in \mathcal{F}_n$.

**SFF:** *strongly filter-Fréchet* if for each $x \in X$ and each $A \in \Omega_x$ there is a sequence $(\mathcal{F}_n : n \in \mathbb{N})$ of filter-bases on $A$ satisfying (FF1) and (FF2) above and the condition

**(FF3)** For each $n \in \mathbb{N}$ there is a countable $F \in \mathcal{F}_n$. 
SSF: *strongly set-Fréchet* if for each \( x \in X \) and each \( A \in \Omega_x \) there is a sequence \( (B_n : n \in \mathbb{N}) \) of pairwise disjoint subsets of \( A \) such that the following conditions hold:

(SF1) \( x \notin \overline{B_n} \) for each \( n \in \mathbb{N} \);

(SF2) each neighborhood \( U \) of \( x \) intersects all but finitely many sets \( B_n \);

(SF3) each \( B_n \) is countable.

SF: *set-Fréchet* if only conditions (SF1) and (SF2) in SSF are satisfied.
**Theorem 7** For a space $X$ and a family $\Delta \subset 2^X$ the following statements are equivalent:

1. $(2^X, \Delta^+)$ is a filter-Fréchet space;

2. For each open set $Y \subset X$ and each $\Delta$-cover $\mathcal{U}$ of $Y$ there is a sequence $(\mathcal{B}_n : n \in \mathbb{N})$ of filter-bases on $\mathcal{U}$ such that:

   (i) For each $n$, there is $C_n \in \mathcal{B}_n$ which is not a $\Delta$-cover of $Y$;

   (ii) For each $D \in \Delta$ with $D \subset Y$ there is $n_0 \in \mathbb{N}$ such that whenever $n \geq n_0$, then there exists $\mathcal{H}_n \in \mathcal{B}_n$ satisfying $D \subset H$ for every $H \in \mathcal{H}_n$. 
**Theorem 8** For a space $X$ the following statements are equivalent:

(1) $(2^X, \Delta^+)$ is a strongly filter-Fréchet space;

(2) For each open set $Y \subset X$ and each $\Delta$-cover $\mathcal{U}$ of $Y$ there is a sequence $(\mathcal{B}_n : n \in \mathbb{N})$ of filter-bases on $\mathcal{U}$ such that:

(i) For each $n$, there is $\mathcal{C}_n \in \mathcal{B}_n$ which is not a $\Delta$-cover of $Y$;

(ii) For each $D \in \Delta$ with $D \subset Y$ there is $n_0 \in \mathbb{N}$ such that whenever $n \geq n_0$, then there exists $\mathcal{H}_n \in \mathcal{B}_n$ satisfying $D \subset H$ for every $H \in \mathcal{H}_n$;

(iii) For each $n \in \mathbb{N}$ there is some countable element in $\mathcal{B}_n$. 
The implication $(1) \Rightarrow (2)$ in Theorem 7 and in Theorem 8 is still valid if the $\Delta^+$-topology is replaced with the $\tau_\Delta$-topology.

**Example 9** [CH] *There exists a space $X$ satisfying the condition (2) in the previous theorem, but $(2^X, \tau_\Delta)$ is not a strongly filter-Fréchet space.*

Let $X$ be the Hausdorff, compact, hereditarily Lindelöf, non hereditarily separable space constructed by Kunene, and let $\Delta = \mathbb{K}(X)$, hence $\tau_\Delta = F$. $(\text{CL}(X), F)$ has uncountable tightness so that it is not SFF.

Let us show that the condition (2) in Theorem 8 holds. Let $Y$ be any open subset of $X$ and let $\mathcal{U}$ be a $k$-cover of $Y$ in $X$. As $Y$ is locally compact and Lindelöf, it is hemicompact. Let $(K_n : n \in \mathbb{N})$ be an increasing countable family
of compact subsets of $Y$ such that each compact subset of $Y$ is contained in some $K_n$. For each $n$ pick a set $U_n \in \mathcal{U}$ such that $K_n \subset U_n$. Since $\mathcal{U}$ is a $k$-cover of $Y$ in $X$, the family \{ $Y \cap U : U \in \mathcal{U}$ \} is a $k$-cover of $Y$ in $Y$, so that $Y$ is not a member of \{ $Y \cap U : U \in \mathcal{U}$ \}. Thus for each $n \in \{1, \cdots, n_0\}$, where $n_0$ is some element in $\mathbb{N}$, pick a point $x_n \in Y \setminus U_n$ and for each $n \in \mathbb{N}$ define

$$\mathcal{B}_n = \{ \{U_i : n \leq i \leq n^*\} : n^* \geq n\}.$$ 

It is clear that \{ $U_i : n \leq i \leq n_1^*$ \} $\subset$ \{ $U_i : n \leq i \leq n_2^*$ \} for $n \leq n_1^* \leq n_2^*$, so that the collection $\mathcal{B}_n$ is linearly ordered by inclusion and in particular it is a filter base. We show that the sequence $(\mathcal{B}_n : n \in \mathbb{N})$ satisfies:

(i) For each $n$, there is $C_n \in \mathcal{B}_n$ which is not a $k$-cover of $Y$;

(ii) For each compact set $K \subset Y$ there is $n_0 \in \mathbb{N}$ such that whenever $n \geq n_0$, then there
exists $\mathcal{H}_n \in \mathcal{B}_n$ satisfying $K \subset H$ for every $H \in \mathcal{H}_n$;

(iii) For each $n \in \mathbb{N}$ there is some countable element in $\mathcal{B}_n$.

Of course, (iii) is satisfied because every element of $\mathcal{B}_n$ is finite. The condition (i) is also true. Indeed, for a given $i \in \mathbb{N}$, we have by the choice of points $x_n$ that no element of $\{U_i : n \leq i \leq n^*\}$ includes the set $\{x_i : n \leq i \leq n^*\}$ which is a compact subset of $Y$. Finally, let us prove that (ii) holds. Let $K$ be a compact subset of $Y$. There exists $n_0 \in \mathbb{N}$ such that $K \subset K_{n_0}$. For each $n \geq n_0$ take any element $\mathcal{H}_n := \{U_i : n \leq i \leq \overline{n}\}$ in $\mathcal{B}_n = \{\{U_i : n \leq i \leq n^*\} : n^* \geq n\}$, where $\overline{n}$ is an element of $\mathbb{N}$ with $\overline{n} \geq n$. Then we have $K \subset H$ for each $H \in \mathcal{H}_n$. Indeed, for each $i \in \mathbb{N}$ with $n \leq i \leq \overline{n}$ we have $K \subset K_{n_0} \subset K_n \subset K_i \subset U_i$. □
Theorem 10 For a space $X$ the following statements are equivalent:

(1) $(2^X, \Delta^+)$ has the strong set-Fréchet property;

(2) For each open set $Y \subset X$ and each $\Delta$-cover $U$ of $Y$ there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of countable, pairwise disjoint subsets of $U$ such that:

(i) no $\mathcal{V}_n$ is a $\Delta$-cover of $Y$;

(ii) each $D \in \Delta$ which is a subset of $Y$ is contained in an element of $\mathcal{V}_n$ for all but finitely many $n$. 
Theorem 11 For a space $X$ the following statements are equivalent:

(1) $(2^X, \Delta^+) \text{ is a set-Fréchet space;}$

(2) For each open set $Y \subset X$ and each $\Delta$-cover $\mathcal{U}$ of $Y$ there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of pairwise disjoint subsets of $\mathcal{U}$ such that:

(i) no $\mathcal{V}_n$ is a $\Delta$-cover of $Y$;

(ii) each $D \subset Y$ such that $D \in \Delta$ is contained in an element of $\mathcal{V}_n$ for all but finitely many $n$. 
References


