CHAPTER 5

Monochromatic arithmetic progressions

1. Ideals in semigroups

Definition 1.1. Let $S$ be a semigroup. A left ideal (of $S$) is a nonempty set $L \subseteq S$ such that $SL := \{ sa : s \in S, a \in L \} \subseteq L$, that is, $sL := \{ sa : a \in L \} \subseteq L$ for all $s \in S$. A left ideal $L$ is minimal if no left ideal of $S$ is properly contained in $L$. Elements of minimal left ideals are called minimal elements.

It follows that every left ideal of a semigroup $S$ is a subsemigroup of $S$, and that, for each $a \in S$, the set $Sa$ is a left ideal of $S$.

Lemma 1.2. Let $S$ be a semigroup.

(1) If $a$ is a minimal element, then the element $sa$ is minimal for each $s \in S$.

(2) For each minimal left ideal $L$ and every $a \in L$, we have that $Sa = L$.

(3) For each minimal element $a \in S$, the set $Sa$ is a minimal left ideal.

(4) If $S$ is a company, then every minimal left ideal of $S$ is compact.

Proof. (1) Let $L$ be a minimal left ideal containing $a$. Then $sa \in L$.

(2) $Sa \subseteq L$ is a left ideal. By minimality of $L$, $Sa = L$.

(3) By (2).

(4) By (3), the minimal left ideal is of the form $Sa$. As right multiplication by $a$ is a continuous function and $S$ is compact, the left ideal $Sa$ is compact.

Lemma 1.3 (Fixing). Let $S$ be a semigroup. Let $a \in S$. The following assertions are equivalent:

(1) $a$ is minimal.

(2) For each $b \in S$, there is $c \in S$ such that $cba = a$.

Proof. (1) $\Rightarrow$ (2): Let $L$ be a minimal left ideal with $a \in L$. Then $ba \in L$, and therefore $L = Sba$. As $a \in L$, there is $c \in S$ such that $a = cba$.

(2) $\Rightarrow$ (1): Let $L$ be a left ideal with $L \subseteq Sa$. Take $ba \in L$. Then there is $c$ such that $a = cba \in L$. Thus, $Sa \subseteq L$, and therefore $L = Sa$. This shows that $Sa$ is a minimal left ideal and $a \in L = Sa$.

An element $a$ of a semigroup $S$ is a minimal idempotent element if it is both a minimal element and an idempotent.

Lemma 1.4. Let $S$ be a company. Every left ideal of $S$ contains a minimal idempotent element.

Proof. We first show that every left ideal of $S$ contains a minimal left ideal. Let $L$ be a left ideal. Fix an element $a \in L$. Then $Sa \subseteq L$, and $Sa$ is a compact left ideal. Thus, the family of all compact left ideals contained in $L$ is nonempty. By the finite intersection property of compact sets, this family satisfies the conditions of Zorn’s Lemma, and therefore has a minimal element $M$. Let $I \subseteq M$ be a left ideal. Then, for any $b \in I$, $Sb$ is a compact left ideal contained in $I$. By minimality $I$, we have that $Sb = I = M$.
Being a minimal left ideal in a company, the set \( M \) is compact and thus a company. Thus, there is an idempotent in \( M \).

Minimal elements are minimal in every company where they belong.

**Lemma 1.5.** Let \( S \) be a company and \( a \in S \) a minimal element. For each subcompany \( T \) of \( S \) such that \( a \in T \), the element \( a \) is also minimal in \( T \).

**Proof.** The set \( Ta \) is a left ideal of \( T \). Take a minimal idempotent \( e \in Ta \). Let \( t \in T \) be with \( e = ta \). By the Fixing Lemma, there is \( s \in S \) such that \( se = a \). Then \( a = se = see = ae \).

As \( e \) is minimal in \( T \) and \( a \in T \), the element \( a = ae \) is also minimal in \( T \).

**Definition 1.6.** An *ideal* (of \( S \)) is a nonempty set \( I \subseteq S \) with \( IS, SI \subseteq I \).

**Lemma 1.7.** Let \( I \) be an ideal of a semigroup \( S \). For each minimal element \( a \), we have that \( a \in I \).

**Proof.** Let \( b \in I \). By the Fixing Lemma, there is \( c \in S \) such that \( a = cba \in I \).

Let \( S \) be a semigroup and \( m \) be a natural number. For visual clarity, we present elements of \( S^m \) as columns. The set \( S^m \) is a semigroup with the coordinate-wise product:

\[
\begin{pmatrix}
  (a_1) \\
  \vdots \\
  (a_m)
\end{pmatrix}
\cdot
\begin{pmatrix}
  (b_1) \\
  \vdots \\
  (b_m)
\end{pmatrix} :=
\begin{pmatrix}
  (a_1b_1) \\
  \vdots \\
  (a_mb_m)
\end{pmatrix}.
\]

Elements of \( S^m \) will be denoted by boldface letters: \( \mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots \).

**Lemma 1.8.** Let \( S \) be a semigroup and \( a_1, \ldots, a_m \in S \) be minimal. Then the element

\[
\mathbf{a} :=
\begin{pmatrix}
  a_1 \\
  \vdots \\
  a_m
\end{pmatrix}
\]

is minimal in \( S^m \).

**Proof.** By the Fixing Lemma, it suffices to prove that for each \( \mathbf{b} \in S^m \) there is \( \mathbf{c} \in S^m \) such that \( \mathbf{c} \mathbf{b} \mathbf{a} = \mathbf{a} \). Let

\[
\mathbf{b} \mathbf{a} =
\begin{pmatrix}
  (b_1) \\
  \vdots \\
  (b_m)
\end{pmatrix}
\cdot
\begin{pmatrix}
  (a_1) \\
  \vdots \\
  (a_m)
\end{pmatrix} :=
\begin{pmatrix}
  (b_1a_1) \\
  \vdots \\
  (b_ma_m)
\end{pmatrix}.
\]

By the Fixing Lemma, there is for each \( i = 1, \ldots, m \) an element \( c_i \in S \) such that \( c_ib_ia_i = a_i \).

Let

\[
\mathbf{c} :=
\begin{pmatrix}
  c_1 \\
  \vdots \\
  c_m
\end{pmatrix}.
\]

Then

\[
\mathbf{c} \mathbf{b} \mathbf{a} =
\begin{pmatrix}
  (c_1) \\
  \vdots \\
  (c_m)
\end{pmatrix}
\cdot
\begin{pmatrix}
  (b_1a_1) \\
  \vdots \\
  (b_ma_m)
\end{pmatrix} =
\begin{pmatrix}
  (c_1b_1a_1) \\
  \vdots \\
  (c_mb_ma_m)
\end{pmatrix} =
\begin{pmatrix}
  (a_1) \\
  \vdots \\
  (a_m)
\end{pmatrix} = \mathbf{a}.
\]

Recall that, for a company \( S \), the product topology on \( S^m \) is the one with the basic open sets

\[
U_1 \times \cdots \times U_m,
\]

where \( U_1, \ldots, U_m \subseteq S \).
Exercise 1.9. Let $S$ be a company. The set $S^m$, with the product topology and coordinate-wise multiplication, is a company.

2. van der Waerden’s Theorem

The following theorem, due to Bartel L. van der Waerden, proves a conjecture of Schur.

**Theorem 2.1 (van der Waerden).** For each finite coloring of $\mathbb{N}$, there are arbitrarily long monochromatic arithmetic progressions.

Towards the proof of van der Waerden’s Theorem, assume that $\mathbb{N}$ is finitely colored. Fix a natural number $m$. We wish to find a monochromatic arithmetic progression of length $m$. It suffices to find an ultrafilter $p \in \beta \mathbb{N}$ such that every element of $p$ contains an arithmetic progression of length $m$. Explicitly, fix $A \in p$. We need that there are $a, d \in \mathbb{N}$ such that

$$
\begin{pmatrix}
a \\
a + d \\
\vdots \\
a + (m-1)d
\end{pmatrix} \in [A]^m.
$$

The set $[A]^m$ is a neighborhood of the element

$$
\begin{pmatrix}
p \\
\vdots \\
p
\end{pmatrix}
$$

in the product topology of $(\beta \mathbb{N})^m$.

Consider the following two subsets of $\mathbb{N}^m$. These sets are, in particular, subsets of $(\beta \mathbb{N})^m$:

$$
\begin{align*}
\text{AP} & := \left\{ \begin{pmatrix} a \\ a + d \\ \vdots \\ a + (m-1)d \end{pmatrix} : a, d \in \mathbb{N} \right\} ; \\
\text{AP}_0 & := \text{AP} \cup \left\{ \begin{pmatrix} a \\ \vdots \\ a \end{pmatrix} : a \in \mathbb{N} \right\} .
\end{align*}
$$

**Exercise 2.2.** For each $s \in \mathbb{N}^m$, The function on $(\beta \mathbb{N})^m$, defined by $q \mapsto s + q$, is continuous.

**Lemma 2.3.** $\overline{\text{AP}_0}$ is a subcompany of $(\beta \mathbb{N})^m$.

**Proof.** As every sum of two arithmetic progressions is an arithmetic progression, the set $\text{AP}_0$ is a subsemigroup of $\mathbb{N}^m$. The lemma follows by continuity: Let $p, q \in \overline{\text{AP}_0}$. We need to show that $p + q \in \overline{\text{AP}_0}$, that is, every neighborhood of $p + q$ intersects $\text{AP}_0$.

Let $U$ be a neighborhood of $p + q$. By right continuity, there is a neighborhood $V$ of $p$ such that $V + q \subseteq U$. As $p \in \overline{\text{AP}_0}$, there is $s \in V \cap \text{AP}_0$. In particular, $s + q \in U$. By continuity of left multiplication by an element of $\mathbb{N}^m$, there is a neighborhood $W$ of $q$ such that $s + W \subseteq U$. As $q \in \overline{\text{AP}_0}$, there is $t \in W \cap \text{AP}_0$ and, in particular $s + t \in U$. As $s$ and $t$ are in $\text{AP}_0$, so is their sum. Thus, $s + t \in U \cap \text{AP}_0$ and the latter intersection is nonempty. \qed
LEMMA 2.4. Let \( p \in \beta \mathbb{N} \), and
\[
\mathbf{p} = \begin{pmatrix} p \\ \vdots \\ p \end{pmatrix}.
\]
Then:
(1) Every neighborhood of \( \mathbf{p} \) contains one of the form \([A]^m\), with \( A \in p \).
(2) \( p \in \overline{\mathbf{AP}}_0 \).

PROOF. (1) Let \([A_1] \times \cdots \times [A_m]\) be a basic open neighborhood of \( \mathbf{p} \) contained in the given neighborhood. As \( A_1, \ldots, A_m \in p \), we have that \( A := A_1 \cap \cdots \cap A_m \in p \). Then \( p \in [A]^m \subseteq [A_1] \times \cdots \times [A_m] \).
(2) It suffices to consider neighborhoods of the form (1). As \( A \in \mathbf{p} \), there is \( a \in A \). Then
\[
\begin{pmatrix} a \\ \vdots \\ a \end{pmatrix} \in [A]^m \cap \overline{\mathbf{AP}}_0.
\]
\( \square \)

LEMMA 2.5. The set \( \overline{\mathbf{AP}} \) is an ideal of \( \overline{\mathbf{AP}}_0 \).

PROOF. The set \( \mathbf{AP} \) is an ideal of \( \mathbf{AP}_0 \). The assertion follows, by continuity considerations as in the proof of Lemma 2.3.
\( \square \)

EXERCISE 2.6. Prove Lemma 2.5.

LEMMA 2.7. Let \( p \in \beta \mathbb{N} \) be a minimal element. Then
\[
\mathbf{p} := \begin{pmatrix} p \\ \vdots \\ p \end{pmatrix} \in \overline{\mathbf{AP}}.
\]

PROOF. We collect the information gained thus far: By Lemma 1.8, The element \( \mathbf{p} \) is minimal \( (\beta \mathbb{N})^m \). Since \( \mathbf{p} \in \overline{\mathbf{AP}}_0 \), it is also minimal in this subcompany. As \( \overline{\mathbf{AP}} \) is an ideal of \( \overline{\mathbf{AP}}_0 \), the minimal element \( \mathbf{p} \) is in \( \overline{\mathbf{AP}} \) (Lemma 1.7).
\( \square \)

We can now conclude the proof of van der Waerden’s Theorem. Given a finite coloring of \( \mathbb{N} \), take a minimal element \( p \in \beta \mathbb{N} \) and a monochromatic set \( A \in p \). By the last lemma, we have that \( \mathbf{p} \in \overline{\mathbf{AP}} \), and thus its neighborhood \([A]^m\) intersects the set \( \mathbf{AP} \). Therefore, there is
\[
\begin{pmatrix} a \\ a + d \\ \vdots \\ a + (m - 1)d \end{pmatrix} \in [A]^m
\]
with \( a, d \in \mathbb{N} \). Then the elements \( a, a + d, \ldots, a + (m - 1)d \) are in \( A \), and are thus of the same color.

EXERCISE 2.8. Find a 2-coloring of \( \mathbb{N} \) with no infinite monochromatic arithmetic progression.

EXERCISE 2.9. Formulate and prove a finite version of van der Waerden’s Theorem.

The proof of van der Waerden’s Theorem shows that for each \( p \in \beta \mathbb{N} \) minimal in \( (\beta \mathbb{N}, +) \), every element \( A \in p \) contains arbitrarily long arithmetic progressions. Similarly, if \( p \) is minimal in \( (\beta \mathbb{N}, \cdot) \), then every element \( A \in p \) contains arbitrarily long geometric progressions \( a, aq, \ldots, aq^{m-1} \). The following theorem is stronger.
THEOREM 2.10. For each finite coloring of \( \mathbb{N} \), there is a color with arbitrarily long arithmetic and geometric progressions of that color.

PROOF. The proof is similar to the proof of Theorem 1.7. Say that a set \( A \subseteq \mathbb{N} \) is an AP set if there are arbitrarily long arithmetic progressions in \( A \). Let

\[
L = \{ p \in \beta \mathbb{N} : \text{each } A \in p \text{ is an AP set} \}.
\]

We have seen that every minimal element \( p \) in \( (\beta \mathbb{N},+) \) is in \( L \). It is easy to see that \( L \) is a left ideal of the company \( (\beta \mathbb{N},\cdot) \). Let \( p \in L \) be a minimal element of \( (\beta \mathbb{N},\cdot) \).

Take a monochromatic set \( A \in p \). By minimality of \( p \), there are arbitrarily long geometric progressions in \( A \). As \( p \in L \), there are arbitrarily long arithmetic progressions in \( A \). \( \square \)

EXERCISE 2.11. Complete the proof of Theorem 2.10, by showing that \( L \) is a left ideal of the company \( (\beta \mathbb{N},\cdot) \).

EXERCISE 2.12. Prove that, in Theorem 2.10, we may request that there are, in addition, FS and FP sets of the same color.

Hint: In the definition of \( L \), request that every \( A \) is AP and FS. Prove that \( L \) is nonempty. Prove that it is a left ideal of \( (\beta \mathbb{N},\cdot) \). Take a minimal idempotent in \( L \).

3. Excursion: the game EquiDist

We introduce a two-player game based on a concrete realization of van der Waerden’s Theorem. We describe here its simplest variation. Additional variations are easy to come up with, search “EquiDist game” online for some examples.

By van der Waerden’s Theorem and the Compactness Theorem, we know that there is a natural number \( N \) such that for each 2-coloring of the numbers \( 1,2,\ldots,N \) there is a monochromatic arithmetic progression of length 3.

PROPOSITION 3.1. For each coloring of the numbers 1,2,3,4,5,6,7,8 and 9 in red and green, there is a monochromatic 3-element arithmetic progression.

PROOF. Assume that we are given a coloring with no monochromatic 3-element arithmetic progression. We may assume that the color of 5 is red.

Assume that 3 or 7 is red. Then, by symmetry, we may assume that 3 is red. Then 1, 4 and 7 green:

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9
\end{array}
\]

We obtain a green arithmetic progression:

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9
\end{array}
\]

a contradiction.

Thus, we the coloring of 3, 5 and 7 must be of the following form:

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9
\end{array}
\]
The numbers 1, 5, and 9 cannot all be red. Thus, 1 or 9 is green. By symmetry, we may assume that 1 is green. Then:

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\end{array}
\]

\[\text{a contradiction.} \]

\[\square\]

**Exercise 3.2.** Find a 2-coloring of the numbers 1, 2, 3, 4, 5, 6, 7, 8 with no monochromatic 3-element arithmetic progression.

We now describe the game. The game board looks as follows.

The players, that will be called *Alice* and *Bob*, have red and green pieces. Each player, in turn, places a red or a green piece on an empty cell. The first player to place a piece such that there are three equidistant pieces of the same color looses.

Following is an example of a play:

Alice:

Bob:

Alice:

Bob:

Alice:

Bob:

Alice:

Bob:

Alice:

Bob:

Alice:

Bob:

Alice:

Bob:

Alice:

Bob:

Bob looses, since placing a piece of any color in any cell would result in three equidistant pieces of the same color.

As in Tic-Tac-Toe, one may play the game without the board and pieces, by drawing the board on a page and writing X and O in the cells instead of placing there pieces.

**Exercise 3.3.** Consider the variation of EquiDist where Alice is only allowed to use red pieces, and Bob is only allowed to place green ones. Prove that Bob has a winning strategy.
van der Waerden’s Theorem (Theorem 2.1) was first conjectured by Issai Schur and later, independently, by Pierre J. H. Baudet.

The proof idea of van der Waerden’s Theorem, spanned through the first and second sections, is due to Hillel Furstenberg and Yitzhak Katznelson (*Idempotents in compact semigroups and Ramsey Theory*, Israel Journal of Mathematics, 1989). Furstenberg and Katznelson used the language of topological dynamical systems and enveloping semigroups. Their proof was converted to the one included here by Vitaly Bergelson and Neil Hindman (*Nonmetrizable topological dynamics and Ramsey Theory*, Transactions of the American Mathematical Society, 1990). Furstenberg’s original proof was in the language of dynamical systems. According to the Hindman–Strauss monograph, this proof shows

> how much one can get for how little . . . It is enough to make someone raised on the work ethic feel guilty.

Theorem 2.10 is due to Vitaly Bergelson and Neil Hindman, *Nonmetrizable topological dynamics and Ramsey Theory*, Transactions of the American Mathematical Society, 1990. It is easy to see that the set $L$ in the proof of Theorem 2.10 is in fact an ideal. Not only of $(\beta\mathbb{N}, \cdot)$, but also of $(\beta\mathbb{N}, +)$. The existence of this simultaneous ideal was first observed in the Hindman–Strauss monograph (Theorem 14.5).

I have suggested this game to my family in 2011. My son, Avraham Tsaban, and my nephew, Ariel Vishne, were, respectively, 12 and 18 years old then. They soon came up with two results: Avraham came up with Exercise 3.3, and Ariel proved, by exhaustive computer search, that Bob has a winning strategy in (9-cell) EquiDist. Ariel found out that whom has a winning strategy depends on the number of cells. A proof of Vishne’s observation not using computers is unknown to us.