CHAPTER 3

The Stone–Čech compactification

1. The space of ultrafilters

For brevity, let us fix an infinite set $X$ throughout this section.

**Definition 1.1.** $\beta X$ is the set of all ultrafilters on $X$.

We would like to consider the set $\beta X$ as a topological space, and its elements as points in that space. Thus, it would be natural to henceforth denote points $\beta X$ (which happen to be ultrafilters on $X$) by lowercase letters such as $p$, $q$, etc.

**Definition 1.2.** For a set $A \subseteq X$, let $[A] := \{ p \in \beta X : A \in p \}$.

**Exercise 1.3.** Prove that the function $A \mapsto [A]$, defined on $P(X)$, has the following properties:

1. $[\emptyset] = \emptyset$ and $[X] = \beta X$.
2. For all $A, B \subseteq X$:
   a. $[A] \subseteq [B]$ if and only if $A \subseteq B$.
   b. $[A] = [B]$ if and only if $A = B$.
   c. $[A] \cup [B] = [A \cup B]$;
   d. $[A] \cap [B] = [A \cap B]$;
   e. $[A^c] = [A]^c$.

**Exercise 1.4.** Consider the case $X = \mathbb{N}$.

1. Find sets $A_1, A_2, \ldots \subseteq \mathbb{N}$ such that $\bigcup_{n=1}^{\infty} A_n \neq \bigcup_{n=1}^{\infty} [A_n]$.
2. Find sets $A_1, A_2, \ldots \subseteq \mathbb{N}$ such that $\bigcap_{n=1}^{\infty} A_n \neq \bigcap_{n=1}^{\infty} [A_n]$.

*Hint:* (1) implies (2). Consider one-element sets.

By Exercise 1.3, the family $\mathcal{B} = \{ [A] : A \subseteq X \}$ satisfies the conditions of Lemma 4.8 for being a basis for a topology on $\beta X$.

**Definition 1.5.** The topology of $\beta X$ is the one with basic open sets $[A]$ (for $A \subseteq X$).

Since $[A^c] = [A]^c$ for all $A \subseteq X$, the sets $[A]$ are clopen, that is, simultaneously closed and open.

**Theorem 1.6.** The topological space $\beta X$ is compact.

**Proof.** Assume otherwise, and consider an open cover of $\beta X$ with no finite subcover. We may assume that this cover is of the form $\{ [A_\alpha] : \alpha \in I \}$, for some index set $I$. For all $\alpha_1, \ldots, \alpha_n \in I$, since

$$[A_{\alpha_1} \cup \cdots \cup A_{\alpha_n}] = [A_{\alpha_1}] \cup \cdots \cup [A_{\alpha_n}] \neq \beta X = [X],$$

we have that $A_{\alpha_1} \cup \cdots \cup A_{\alpha_n} \neq X$, or, equivalently, that $A_{\alpha_1}^c \cap \cdots \cap A_{\alpha_n}^c \neq \emptyset$. It follows that the family $\{ A_{\alpha}^c : \alpha \in I \}$ extends to an ultrafilter $p \in \beta X$. Let $\alpha \in I$ be such that $p \in [A_\alpha]$. Then $A_\alpha, A_\alpha^c \in p$; a contradiction. □

**Definition 1.7.** For $x \in X$, let $p_x \in \beta X$ be the principal ultrafilter determined by $x$. 21
The function from $X$ to $\beta X$ defined by $x \mapsto p_x$ is bijective. We identify each principal ultrafilter $p_x$ with the point $x$. Under this identification, we have that $X \subseteq \beta X$, and the set $X$ becomes a topological subspace of $\beta X$. We will see that the topology induced on $X$ is the simplest possible.

Recall that a topological space $X$ is discrete if all subsets of $X$ are open. A point $x \in X$ is isolated if the set $\{x\}$ is open. In other words, if there is a neighborhood of $x$ containing no other point. A topological space is discrete if and only if all of its points are isolated.

A subset of a topological space is dense if its closure is the entire space. One may interpret the following lemma as asserting that every dictatorship is isolated, but in every neighborhood of any government one may find a dictatorship.

**Lemma 1.8.** Every point of $X$ is isolated in $\beta X$, but the set $X$ is dense in $\beta X$.

Formally: For each $x \in X$, the set $\{p_x\}$ is open in $\beta X$, and the closure of the set $\{p_x : x \in X\}$ is $\beta X$.

**Proof.** Let $x \in X$. By definition, $p \in \{\{x\}\}$ if and only if $\{x\} \in p$, and the latter property is equivalent to $p = p_x$. Thus, $\{p_x\} = \{\{x\}\}$, a basic open set.

Let $q \in \beta X$. For each basic neighborhood $[\alpha]$ of $q$, we have that $A \in q$ and, therefore, $A \neq \emptyset$. Fix $x \in A$. Then $A \in p_x$, that is, $p_x \in [\alpha]$. □

**Exercise 1.9.** Prove the following generalization of Lemma 1.8: For each $A \subseteq X$, we have that $\overline{A} = [A]$. (Formally: $\{p_x : x \in A\} = [A]$).

We say that a topological space $X$ is a subspace of another space $Y$ if $X \subseteq Y$ and the induced topology on $X$ coincides with the original one.

**Definition 1.10.** A compactification of a topological space $X$ is a compact space $K$ such that $X$ is a dense subspace of $K$.

If we think of a set $X$ with no prescribed topology as a discrete topological space, then Lemma 1.8 implies that the space $\beta X$ is a compactification of $X$. The space $\beta X$ is called the Stone–Čech compactification of $X$.

### 2. Excursion: The origin of the open sets in $\beta X$

We will see here that the topology we have defined on $\beta X$ is the natural one. Let $I$ be a set, and consider the set $\{0, 1\}$ as a discrete topological space. By Tychonoff’s Product Theorem (Theorem 5.7), the space $\{0, 1\}^I = \prod_{\alpha \in I}\{0, 1\}$ is compact. The basic open sets in this space are

$$\{\alpha_1, \ldots, \alpha_n ; \{i_1, \ldots, i_n\}\} = \{ (x_{\alpha})_{\alpha \in I} \in \{0, 1\}^I : x_{\alpha_1} = i_1, \ldots, x_{\alpha_n} = i_n \},$$

where $n$ is a natural number, $\alpha_1, \ldots, \alpha_n \in I$, and $i_1, \ldots, i_n \in \{0, 1\}$. By reordering the elements, we may assume that there is $l \leq n$ such that $i_1 = \cdots = i_l = 0$ and $i_{l+1} = \cdots = i_n = 1$. This way, the basic open sets may be denoted as

$$\{\alpha_1, \ldots, \alpha_l ; \beta_1, \ldots, \beta_m\} := \{ (x_{\alpha})_{\alpha \in I} \in \{0, 1\}^I : x_{\alpha_1} = \cdots = x_{\alpha_l} = 0, x_{\beta_1} = \cdots = x_{\beta_m} = 1 \}.$$

We may identify the set $\{0, 1\}^I$ with the family $P(I)$ of all subsets of $I$, by identifying each set $J \subseteq I$ with its characteristic sequence $(x_{\alpha})_{\alpha \in I} \in \{0, 1\}^I$, define by

$$x_{\alpha} = \begin{cases} 1 & \alpha \in J \\ 0 & \alpha \notin J \end{cases}$$
for all $\alpha \in I$. This identification transports the Tychonoff product topology on $\{0, 1\}^I$ into $P(I)$. The basic open sets in $P(I)$ are those of the form

$$[\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m] = \{ J \in P(I) : \alpha_1, \ldots, \alpha_l \notin J, \beta_1, \ldots, \beta_m \in J \}.$$

This is the product topology on $P(I)$.

The set $I$ could be any set. In particular, we can take $I = P(X)$ for a prescribed set $X$. Then the set $\beta X$ of all ultrafilters of $X$ is a subset of $P(I)$. The following proposition asserts, in effect, that the topology on $\beta X$ is the one you get from the product topology.

**Proposition 2.1.** Let $X$ be a set and $I = P(X)$. Equip the set $P(I)$ with the product topology. Then the topology of $\beta X$ coincides with the subspace topology induced by the space $P(I)$.

**Proof.** Consider the induced topology on $\beta X$. The basic open sets are the intersections of basic open sets in $P(I)$ with $\beta X$. By the above discussion, a basic open set in $P(I)$ is determined by elements $A_1, \ldots, A_l, B_1, \ldots, B_m \in I = P(X)$, and the basic open set is

$$[A_1, \ldots, A_l; B_1, \ldots, B_m] = \{ p \in \beta X : A_1, \ldots, A_l \notin p, B_1, \ldots, B_m \in p \}.$$

Let $p \in \beta X$. As $p$ is an ultrafilter, we have that the following assertions are equivalent:

1. $A_1, \ldots, A_l \notin p, B_1, \ldots, B_m \in p$;
2. $A_1^{\ast} \cap \cdots \cap A_l^{\ast}, B_1, \ldots, B_m \in p$;
3. $A_1^{\ast} \cap \cdots \cap A_l^{\ast} \cap B_1 \cap \cdots \cap B_m \in p$.

Thus, taking $A := A_1^{\ast} \cap \cdots \cap A_l^{\ast} \cap B_1 \cap \cdots \cap B_m$, we see that the basic open set is

$$[A] := \{ p \in \beta X : A \in p \}.$$

These are exactly the basic open sets in the original topology of $\beta X$. \hfill \Box

With this understanding, compactness of $\beta X$ follows naturally.

**Theorem 2.2.** The space $\beta X$ is compact.

**Proof.** Let $I = P(X)$, and consider the space $P(I)$ with the product topology. Since $P(I)$ is a compact space, it suffices to observe that $\beta X$ is a closed subset of $P(I)$.

Let $q \in P(I) \setminus \beta X$. We will show that $q$ has a neighborhood disjoint of $\beta X$. Indeed, every possible reason why $q$ is not an ultrafilter defines such an open set. This is so since all reasons are in terms of membership or non-membership of certain sets to $q$. For example:

1. If $X \notin q$ then $q \in [X ; ]$, which is disjoint of $\beta X$.
2. If $\emptyset \in q$ then $q \in [ ; \emptyset]$, which is disjoint of $\beta X$.
3. If there is a set $B \subseteq X$ such that $B \supseteq A \in q$ and $B \notin q$, then $q \in [B ; A]$, which is disjoint of $\beta X$. \hfill \Box

**Exercise 2.3.** Complete the consideration of all cases in the proof of Theorem 2.2.

### 3. The Extension Theorem

**Lemma 3.1 (Regularity).** Let $K$ be a compact space and $x \in K$. For each neighborhood $U$ of $x$, there is a neighborhood $V$ of $x$ such that $V \subseteq U$.

**Proof.** As the space $K$ is compact, its closed subset $U^c$ is compact, too. For each $y \in U^c$, choose disjoint open neighborhoods $V_y$ and $U_y$ of $x$ and $y$, respectively. Then $U^c \subseteq \bigcup_{y \in U^c} U_y$.

As $U^c$ is compact, there are $y_1, \ldots, y_n \in U^c$ such that $U^c \subseteq U_{y_1} \cup \cdots \cup U_{y_n}$. Let

$$C = U_{y_1} \cap \cdots \cap U_{y_n},$$

$$V = V_{y_1} \cap \cdots \cap V_{y_n}.$$
Then $V$ is a neighborhood of $x$ and $V \subseteq C$. As the set $C$ is closed, we have that $\overline{V} \subseteq C$. \hfill \Box

**Exercise 3.2.** Let $F$ be a filter in a topological space that converges to a point $x$. Prove that for each $A \in F$ we have that $x \in A$.

**Theorem 3.3 (Extension).** Let $K$ be a compact space. Every function $f: X \to K$ extends, in a unique manner, to a continuous function $\bar{f}: \beta X \to K$.

\[\begin{array}{ccc}
\beta X & \xrightarrow{\bar{f}} & K \\
\text{id} & \Downarrow & \downarrow \quad \bar{f} \\
X & \xrightarrow{f} & K
\end{array}\]

**Proof.** Uniqueness follows from the density of $X$ in $\beta X$. We prove existence.

For each $p \in \beta X$, $f(p)$ is an ultrafilter in the compact space $K$, and thus converges to a unique point in $K$. Define

$$\bar{f}(p) := \lim f(p).$$

The function $\bar{f}$ extends $f$: For each $x \in X$, we have that

$$f(p_x) = \{ f(A) : A \in p_x \},$$

and the set $\{f(x)\}$ is in the latter set. Thus, $\bar{f}(x) = \lim f(p_x) = f(x)$.

The function $\bar{f}$ is continuous: Let $p \in \beta X$. Let $V$ be a neighborhood of $\bar{f}(p)$ in $K$. As $\lim f(p) = \bar{f}(p) \in V$, there is $f(A) \in f(p)$ (with $A \in p$) such that $f(A) \subseteq V$. The set $[A]$ is a neighborhood of $p$, and for $q \in [A]$ we have that $A \in q$, and thus $f(A) \in f(q)$. As $V$ is closed, we have that $\bar{f}(q) = \lim f(q) = f(\bar{f}(A)) \subseteq \overline{V}$. This is almost what we need.

By the Regularity Lemma, there is a neighborhood $W$ of $\bar{f}(p)$ such that $\overline{W} \subseteq V$. Carrying out the preceding argument with $W$ instead of $V$, we have that $\bar{f}(q) \in \overline{W} \subseteq V$ for all $q \in [A]$. \hfill \Box

**Exercise 3.4.** Prove, using the Hausdorff property, the uniqueness of the extension $\bar{f}$ in the Extension Theorem.

A tip for the next exercise: By the characterization of ultrafilters as maximal filters, in order to establish equality of an ultrafilter $p$ to a set $q$, it suffices to prove that $q$ is a filter and $p \subseteq q$.

**Exercise 3.5.** Let $f: X \to X$. As $X \subseteq \beta X$, we have in particular that $f: X \to \beta X$. Let $\bar{f}: \beta X \to \beta X$ be the unique continuous extension of $f$. Prove that, in this case,

$$\bar{f}(p) = f(p)^\uparrow := \{ B \subseteq X : \exists A \in p, f(A) \subseteq B \}$$

for all $p \in \beta X$.

**Exercise 3.6.** Let $f: X \to \beta X$, and let $\bar{f}: \beta X \to \beta X$ be its unique continuous extension. Prove that, in this case,

$$\bar{f}(p) = \{ B \subseteq X : \exists A \in p \forall x \in A, B \in f(x) \}$$

for all $p \in \beta X$.

**4. Multiplication in $\beta S$**

**Definition 4.1.** A *semigroup* is a nonempty set $S$ equipped with an associative binary operator $\ast$. Explicitly:

1. To each pair of elements $a, b \in S$, a unique element $a \ast b \in S$ is assigned.
2. For all $a, b, c \in S$, we have that $(a \ast b) \ast c = a \ast (b \ast c)$. 
The multiplication symbols is usually omitted, writing \( ab \) instead of \( a \ast b \).

Let \( S \) be a semigroup. We extend the multiplication operator from \( S \times S \) to \( \beta S \times \beta S \) by applying the Extension Theorem theorem twice: First, we extend it from \( S \times S \) to \( S \times \beta S \) by fixing the left coordinate and using the extension theorem on the right coordinate, and then we extend it from \( S \times \beta S \) to \( \beta S \times \beta S \) by fixing the right coordinate and using the extension theorem on the left coordinate. The exact details are provided in the following definition.

**Definition 4.2.** Let \( S \) be a semigroup.

1. Fix an element \( a \in S \). Let \( L_a : S \to \beta S \subseteq \beta S \) be the function of left multiplication by \( a \), that is,
   \[
   L_a(x) := ax
   \]
   for all \( x \in X \). By the Extension Theorem, the function \( L_a \) extends uniquely to a continuous function \( \bar{L}_a : \beta S \to \beta S \).

2. Fix an element \( q \in \beta S \). Let \( R_q : S \to \beta S \) be the function of right multiplication by \( q \), that is,
   \[
   R_q(x) := xq
   \]
   for all \( x \in X \). By the Extension Theorem, the function \( R_q \) extends uniquely to a continuous function \( \bar{R}_q : \beta S \to \beta S \).

Define \( aq := \bar{L}_a(q) \) for all \( q \in \beta S \). Keep in mind that this function is continuous in its argument \( q \).

Exercise 4.3. Prove that every composition of continuous functions is continuous.

**Lemma 4.4.** The set \( \beta S \), with the binary operator \((p,q) \mapsto pq\), is a semigroup.

**Proof.** Let \( p,q,r \in \beta S \). We need to prove that \((pq)r = p(qr)\).

1. For all \( x,y,z \in S \), we have that
   \[
   \bar{L}_{xy}(z) = L_{xy}(z) = (xy)z = x(yz) = L_x \circ L_y(z) = \bar{L}_x \circ \bar{L}_y(z),
   \]
that is, the continuous functions $\bar{L}_{xy}$ and $\bar{L}_x \circ \bar{L}_y$ coincide on $S$. As $S$ is dense in $\beta S$, these two functions coincide on all of $\beta S$. In particular, we have that 

$$(xy)r = \bar{L}_{xy}(r) = \bar{L}_x \circ \bar{L}_y(r) = x(yr).$$

2. By the last equation, we have that $\bar{R}_r \circ \bar{L}_x(y) = \bar{L}_x \circ \bar{R}_r(y)$ for all $y \in S$, that is, the continuous functions $\bar{R}_r \circ \bar{L}_x$ and $\bar{L}_x \circ \bar{R}_r$ coincide on $S$. As $S$ is dense in $\beta S$, these two functions coincide on all of $\beta S$. In particular, we have that 

$$(xq)r = \bar{R}_r \circ \bar{L}_x(q) = \bar{L}_x \circ \bar{R}_r(q) = x(qr).$$

3. By the last equation, the continuous functions $\bar{R}_r \circ \bar{R}_q$ and $\bar{R}_{qr}$ coincide on $S$, and thus on $\beta S$. In particular, we have that $(pq)r = p(qr)$. □

A right topological semigroup is a semigroup with a topology such that, for each constant $c \in S$, multiplication by $c$ on the right, $x \mapsto xc$, is continuous. Following the tradition of naming algebraic structures as groups, rings, bands, etc., we introduce the following name for an algebraic structure.

**Definition 4.5.** A company is a compact right topological semigroup.

**Corollary 4.6.** For each semigroup $S$, the semigroup $\beta S$ is a company. □

The following combinatorial characterization of multiplication in $\beta S$ will help shortening many later arguments. This characterization is due to Glazer. Intuitively, it asserts that $A \in pq$ if and only if there are $p$-many elements $b$ for which there are $q$-many elements $c$ with $bc \in A$.

**Theorem 4.7 (Product Characterization).** Let $S$ be a semigroup and $p, q \in \beta S$. Then:

1. $pq = \{ A \subseteq S : \exists B \in p \land B \subseteq C \in q, bC \subseteq A \}$. In other words, a set $A$ is in $pq$ if and only if there is $B \in p$ such that, for each $b \in B$, $bC \subseteq A$ for some $C \in q$.
2. For each $A \in pq$ there are $b \in S$ and $C \in q$ such that $bC \subseteq A$.

**Proof.** (2) follows from (1). We prove (1). Note that $[A] \cap S = A$ (in particular, $A \subseteq [A]$) and $\overline{A} = [A]$ for all sets $A \subseteq S$.

$(\Rightarrow) pq \in [A]$. By continuity of right multiplication by $q$ ($\bar{R}_q$), there is a neighborhood $[B]$ of $p (B \in p)$ such that $Bq \subseteq [B]q \subseteq [A]$.

Let $b \in B$. Then $bq \in Bq \subseteq [A]$. By continuity of left multiplication by an element of $S (\bar{L}_b)$, there is a neighborhood $[C]$ of $q (C \in q)$ such that $bC \subseteq b[C] \subseteq [A]$. Thus, $bC \subseteq [A] \cap S = A$.

$(\Leftarrow)$ For each $b \in B$, let $C \in q$ be such that $bC \subseteq A$. By continuity of left multiplication by $b$, we have that $b\overline{C} \subseteq \overline{bC} \subseteq \overline{A}$. Since $q \subseteq [C] = \overline{C}$, we have that $bq \in \overline{A}$. In summary, $Bq \subseteq \overline{A}$. By continuity of right multiplication by $q$, we have that $\overline{Bq} \subseteq \overline{Bq} \subseteq \overline{A}$. Since $p \subseteq [B] = \overline{B}$, $pq \subseteq \overline{A} = [A]$, that is, $A \in pq$. □

A diagram containing some of the information in Theorem 4.7 may help remembering it:

$$
\begin{array}{c}
A \\
\quad B \\
A \supseteq b \\
\end{array}
\begin{array}{c}
p \\
\quad q \\
\end{array}
\quad C
$$

According to our identification of the elements of $S$ with the principal ultrafilters in $\beta S$, we have that $S$ is a subsemigroup of $\beta S$.

**Exercise 4.8.** Let $S$ be a semigroup. Prove, using the Product Characterization Theorem, that for all $a, b \in S$ we have that $p_ap_b = p_{ab}$. 
5. When is $\beta S \setminus S$ a company?

As the set $\{s\}$ is open in $\beta S$ for all $s \in S$, the set $S$ is open in $\beta S$. Thus, the set $\beta S \setminus S$ is topologically closed in $\beta S$. If, in addition, the set $\beta S \setminus S$ is closed under multiplication, then it is a company, and there is an idempotent in $\beta S \setminus S$. We describe this sufficient condition in terms of the semigroup $S$.

**Definition 5.1.** A semigroup $S$ is **moving** if $S$ is infinite, and for all finite $F \subseteq S$ and infinite $A \subseteq S$, there are $a_1, \ldots, a_k \in A$ such that, for all but finitely many $s \in S$,

$$\{a_1s, \ldots, a_ks\} \notin F.$$

A **group** is a semigroup with an element $e$ such that $se = es = s$ for all $s$, and for each $s$ there is $t$ such that $st = ts = e$. A semigroup $S$ is **left cancellative** if, for all $a, b, c \in S$, if $ca = cb$ then $a = b$. **Right cancellative** semigroups are defined similarly. A function $f : X \to Y$ is **finite-to-one** if for all $y \in Y$, the set of preimages $f^{-1}(y) = \{ x \in X : f(x) = y \}$ of $y$ is finite.

**Exercise 5.2.** Let $S$ be a semigroup. If $S$ is a group, then it is right cancellative and left cancellative. If $S$ is right cancellative or left cancellative, then it is moving. If left multiplication in $S$ is finite-to-one (i.e., the functions $L_a : x \mapsto ax$ are finite-to-one), then $S$ is moving.

**Theorem 5.3.** Let $S$ be a semigroup. The following assertions are equivalent:

1. $\beta S \setminus S$ is a subsemigroup of $\beta S$.
2. $S$ is moving.

**Proof.** (2) $\Rightarrow$ (1): Let $p, q \in \beta S \setminus S$ and assume that $pq = s \in S$. Then $\{s\} \in pq$. By the Product Characterization Theorem, there is $A \in p$ such that for each $a \in A$ there is $B_a \in q$ such that $aB_a \subseteq \{s\}$. Let $a_1, \ldots, a_k \in A$. Then the set $B := B_{a_1} \cap \cdots \cap B_{a_k}$ is in $q$, and thus infinite, and

$$\{a_1, \ldots, a_k\}B \subseteq \{s\}.$$ 

Thus, $S$ is not moving.

(1) $\Rightarrow$ (2): Let $F \subseteq S$ be finite and $A \subseteq S$ be infinite, such that for all $a_1, \ldots, a_k \in A$, the set $\{ s \in S : a_1s, \ldots, a_ks \in F \}$ is infinite. For each $a \in A$, let

$$B_a = \{ s \in S : as \in F \}.$$ 

Then, for all $a_1, \ldots, a_k \in A$, the intersection $B_{a_1} \cap \cdots \cap B_{a_k}$ is infinite. Let $q \in \beta S \setminus S$ be an ultrafilter containing all sets $B_a$ for $a \in A$. Since the set $A$ is infinite, there is an ultrafilter $p \in \beta S \setminus S$ with $A \in p$. We will show that $F \in pq$, and therefore $pq \in S$. Indeed, $A \in p$ and for all $a \in A$, the set $B_a$ is in $q$, and $aB_a \subseteq F$. By the Product Characterization Theorem, we have that $F \in pq$. □

6. Idempotents

**Lemma 6.1 (Finite Intersection Property).** Let $K$ be a compact space, and let $\{C_\alpha : \alpha \in I\}$ be a family of closed sets in $K$ such that every intersection of finitely many members of this family is nonempty. Then the entire intersection, $\bigcap_{\alpha \in I} C_\alpha$, is nonempty.

**Proof.** If $\bigcap_{\alpha \in I} C_\alpha = \emptyset$, then the open cover $\bigcup_{\alpha \in I} C_\alpha = K$ has a finite subcover $C_{\alpha_1}^* \cup \cdots \cup C_{\alpha_n}^* = K$. Then $C_{\alpha_1} \cap \cdots \cap C_{\alpha_n} = \emptyset$; a contradiction. □

**Lemma 6.2.** Let $K$ be a compact space. A set $C \subseteq K$ is closed if and only if $C$ is compact.

**Proof.** We already know that closed subsets of compact spaces are compact. The proof of the converse implication is similar to the proof of the Regularity Lemma (Lemma 3.1). □
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Exercise 6.3. Complete the proof of Lemma 6.2.

Thus, when working in compact spaces, we will freely switch among the terms “closed” and “compact”.

Exercise 6.4. Prove that for each point \( x \) in a topological space \( X \), the one-point set \( \{x\} \) is closed.

An idempotent in a semigroup \( S \) is an element \( e \in S \) satisfying \( e^2 := ee = e \).

Theorem 6.5. Every company \( S \) has an idempotent element.

Proof. We will use the following variation of Zorn’s Lemma (Lemma 1.12): Let \( \mathcal{A} \) be a nonempty family of sets, with the property that for each chain \( \{ A_\alpha : \alpha \in I \} \) in \( \mathcal{A} \), we have that \( \bigcap_{\alpha \in I} A_\alpha \in \mathcal{A} \). Then there is a minimal element \( A \in \mathcal{A} \). (I.e., such that there is no \( B \in \mathcal{A} \) with \( B \subset A \).

A subcompany of \( S \) is a subset that is a company with respect to the (induced) multiplication and topology of \( S \). We will apply Zorn’s Lemma to find a minimal subcompany of \( S \), and show that this subcompany must be of the form \( \{ e \} \). (In particular, \( ee = e \).

The family of all subcompanies of \( S \) satisfies the conditions of Zorn’s Lemma: \( S \) is there, and the intersection of a chain of subcompanies is a company, by the finite intersection property of compact sets. By Zorn’s Lemma, there is a minimal subcompany \( T \subseteq S \).

Fix an element \( e \in T \). As right multiplication by \( e \) is continuous and the set \( T \) is compact, the set \( Te = \{ te : t \in T \} \) is also compact. As \( e \in T \), we have that \( Te \subseteq T \) and \( Te \) is closed under multiplication. Thus, the set \( Te \) is a company. By the minimality of \( T \), we have that \( Te = T \), and therefore \( e \in Te \). It follows that the stabilizer of \( e \) in \( T \), defined as

\[
\text{stab}_T(e) := \{ t \in T : te = e \} = R_e^{-1}(\{e\}) \cap T,
\]

is nonempty. The stabilizer is a subsemigroup of \( T \): For \( t_1, t_2 \) in the stabilizer, we have that \( (t_1 t_2)e = t_1(t_2e) = t_1 e = e \). As the set \( \{e\} \) is closed and the function \( R_e \) is continuous, the set \( R_e^{-1}(\{e\}) \) is closed, and therefore so is its intersection with \( T \). Thus, the set \( \text{stab}_T(e) \) is compact, and is therefore a subcompany of \( T \). By the minimality of \( T \), we have that \( \text{stab}_T(e) = T \) and, in particular, that \( e \in \text{stab}_T(e) \), that is, \( ee = e \). This establishes the theorem, but it also follows that \( \{e\} \) is a company contained in \( T \), and therefore \( T = \{e\} \).

\[\square\]

Corollary 6.6. Let \( S \) be a semigroup. In each closed subsemigroup of \( \beta S \) there is an idempotent element.

\[\square\]

We will usually consider \( \mathbb{N} \) as a semigroup with respect to its additive structure. Thus, an idempotent element of \( \beta \mathbb{N} \) is an element \( e \in \mathbb{N} \) with \( e + e = e \).

Exercise 6.7. Prove that for every idempotent \( e \in \beta \mathbb{N} \) and each \( n \), we have that \( nN \in e \).

Hint: Let \( \varphi : \mathbb{N} \to \mathbb{Z}_n \) be the canonical homomorphism \( \varphi(k) = k \mod n \). The finite semigroup \( \mathbb{Z}_n \), with the discrete topology, is compact. Thus, \( \varphi \) extends to a continuous homomorphism \( (!) \bar{\varphi} : \beta \mathbb{N} \to \mathbb{Z}_n \). Then \( \bar{\varphi}(e) \) is an idempotent in \( \mathbb{Z}_n \), and is therefore equal to 0. Apply the continuity of \( \bar{\varphi} \).

Theorem 6.8 (Idempotent Characterization). Let \( S \) be a semigroup and let \( e \in \beta S \) be an idempotent. For each \( A \in e \), there is \( a \in A \) such that:

There is a set \( A' \subseteq A \) in \( e \) such that \( aA' \subseteq A \).

Moreover:

1. The set of elements \( a \) with the quoted property is in \( e \).
2. If \( e \) is nonprincipal, then we may request that \( a \notin A' \).
7. Comments for Chapter 3

(3) Item (1) above characterizes the property “e is an idempotent in βS”.

Proof. Let \( A \subseteq e = e^2 \). By the Product Characterization Theorem, there is \( B \subseteq e \) such that, for each \( a \in B \), there is \( C \subseteq e \) with \( aC \subseteq A \). Fix \( a \in B \cap A \), and take the corresponding set \( C \). Then the set \( A' = C \cap A \) is in \( e \), and \( aA' \subseteq aC \subseteq A \). This establishes (1) and, in particular, the quoted assertion.

(2) Since \( e \) is nonprincipal, the set \( \{ a \} \) is not in \( e \) and \( \{ a \}^e \subseteq e \). Take \( A' = C \cap A \cap \{ a \}^e = C \cap A \setminus \{ a \} \).

(3) By the Product Characterization Theorem and (1), we have that \( e \subseteq e^2 \). As \( e \) and \( e^2 \) are ultrafilters, equality holds. \( \square \)

It follows directly from the definition that every subsemigroup of a moving semigroup is moving.

Theorem 6.9. Let \( S \) be a semigroup. If \( S \) has a moving subsemigroup, then there is an idempotent \( e \in \beta S \setminus S \).

Proof. Let \( T \) be a moving subsemigroup of \( S \). By Theorem 5.3, there is an idempotent \( e \in \beta T \setminus T \). Let

\[
\{ A \subseteq S : \exists B \in e, A \supseteq B \}
\]

the closure of \( e \) under taking supersets. Then \( e^\uparrow \subseteq \beta S \). By the Idempotent Characterization Theorem, \( e^\uparrow \) is an idempotent in \( \beta S \). As all elements of \( e^\uparrow \) are infinite (given that \( e \in \beta T \setminus T \)), we have that \( e^\uparrow \subseteq \beta S \setminus S \). \( \square \)

Exercise 6.10. Prove that if there are no idempotents in a semigroup \( S \), then \( S \) has a subsemigroup isomorphic to \( \mathbb{N} \). In particular, in this case \( S \) has a moving subsemigroup.

7. Comments for Chapter 3


We have described the basic open sets in \( \beta X \). This determines the open sets of \( \beta X \) as unions of basic open sets. We provide here an explicit description of the open sets in \( \beta X \).

Let \( U = \bigcup_{\alpha \in I} \{ A_\alpha \} \) be a union of basic open sets. Let \( p \in \beta X \) be such that \( p \notin U \). Then \( A_\alpha \notin p \), and thus \( B_\alpha := A_\alpha \setminus p \), for all \( \alpha \in I \). The sets \( B_\alpha \) generate a filter \( F \) contained in \( p \). The converse implications also hold, and we have that \( p \notin U \) if and only if \( F \subseteq p \). Thus, the open sets in \( \beta X \) are the sets of the form

\[
[\mathcal{F}] = \{ p \in \beta X : \mathcal{F} \not\subseteq p \}
\]

for \( \mathcal{F} \) a filter on \( X \). Equivalently, closed sets in \( \beta X \) are those of the form \( \{ p \in \beta X : \mathcal{F} \subseteq p \} \).

Theorem 5.3 is due to Hindman, The ideal structure of the space of \( \kappa \)-uniform ultrafilters on a discrete semigroup, Rocky Mountain Journal of Mathematics, 1986 (Theorem 2.5 with \( \kappa = \omega \)).

De Gruyter Expositions in Mathematics, Theorem 4.28. We did not include in Exercise 5.2 semigroups with finite-to-one right multiplication. The reason is that there are such semigroups that are not moving (Benjamin Steinberg, Answer to Mathoverflow question 164050, 2014). Fortunately, for our purposes this is not crucial: A semigroup \( S \) is periodic if every elements of \( S \) generates a finite semigroup. A semigroup \( S \) is right (left) zero if \( ab = b \) (\( ab = a \)) for all \( a, b \in S \). Lev N. Shevris (On the theory of periodic semigroups, Izvestia Vysših Učebnyh
Zavedení Matematika, 1974) proved that every infinite semigroup has a subsemigroup of one of the following types:

1. \((\mathbb{N}, +)\).
2. An infinite periodic group.
3. An infinite right zero or left zero semigroup.
4. \((\mathbb{N}, \vee)\), where \(m \lor n := \max\{m, n\}\).
5. \((\mathbb{N}, \wedge)\), where \(m \land n := \min\{m, n\}\).
6. An infinite semigroup \(S\) with \(S^2\) finite.
7. The fan semilattice \((\mathbb{N}, \land)\), with \(m \land n = 1\) for distinct \(m, n\) (and \(n \land n = n\) for all \(n\)).

Assume that right multiplication in \(S\) is not finite-to-one. Then \(S\) does not have a subsemigroup of the type (6) or (7). Thus, it must have a subsemigroup of one of the remaining types (1)–(5), which are all moving (!). Thus, Theorem 6.9 applies to semigroups with finite-to-one right multiplication as well.