CHAPTER 1

Three famous coloring theorems

Ramsey theory, named after Frank P. Ramsey (1903–1930), studies the following phenomenon: If we take a rich mathematical structure, and color (no matter how) each of its elements in one out of finitely many prescribed colors, there will be a rich *monochromatic* substructure, that is, a rich substructure with all elements of the same color. In this chapter, we provide elementary proofs of several beautiful theorems exhibiting this phenomenon.

1. Ramsey's Theorem

For a set A and a natural number d, let $[A]^d := \{ F \subseteq A : |F| = d \}$, the collection of all d-element subsets of A.

A *graph* is a pair $G = (V, E)$, consisting of a set of vertices V and a set E of edges among vertices. Formally, E is a subset of $[V]^2$, and $\{a, b\} \in E$ is interpreted as "there is an edge between a and b". The graph G is *complete* if $E = [V]^2$, that is, there is an edge between every pair of vertices.

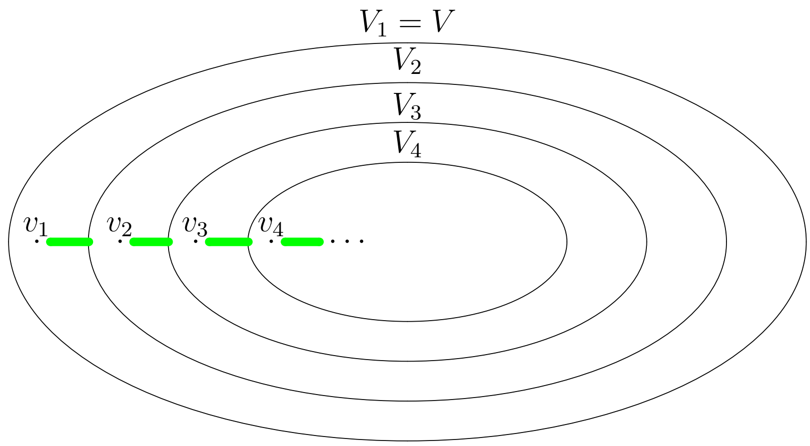
THEOREM 1.1 (Ramsey). *If we color each edge of an infinite, complete graph, with one out of finitely many prescribed colors, then there is an infinite, complete monochromatic subgraph. That is, an infinite set of vertices such that all edges among them have the same color.*

PROOF. The following proof is Ramsey's original. Several alternative proofs were suggested since, but, remarkably, Ramsey's proof remains the most lucid one.

We prove the theorem in the case of two colors, and later instruct the reader how to generalize it to an arbitrary finite number of colors. Assume, thus, that the colors are red and green.

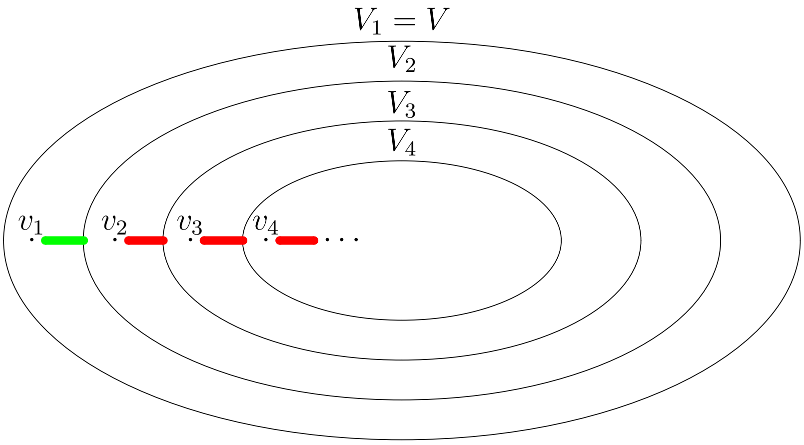
Every two vertices are joined by an edge, a red one or a green one. Let $V_1 := V$. Choose a vertex $v_1 \in V_1$. Either this vertex has infinitely many green edges, or it has infinitely many red edges. We may assume that the case is the former. Let V_2 be the (infinite) set of vertices connected to v_1 by green edges. There are two cases to consider.

The good case:



Assume that there is a vertex $v_2 \in V_2$ with infinitely many green edges connecting it to other vertices in V_2. Let $V_3 \subseteq V_2$ be the set of these vertices. Continue by induction, as long as possible: For each n, assume that there is a vertex $v_n \in V_n$ with infinitely many green edges connecting it to vertices in V_n, and let $V_{n+1} \subseteq V_n$ be the set of these vertices. If this is the case for all n, then all edges of the complete graph with vertices $v_1, v_2, \ldots$ are green. Indeed, for $n < m$, we have that $v_m \in V_{n+1}$, and thus the edge $\{v_n, v_m\}$ is green.

The remaining, even better case:



Assume that, for some n, the above procedure terminates: each $v \in V_n$ has only finitely many green edges. In this case, we restart the procedure, from an arbitrary $v_n \in V_n$, with *red* edges. This time, the procedure cannot terminate, since for $m \geq n$, we have that for each $v_m \in V_m$ all but finitely many edges connecting v_m to other elements of V_m are red! Thus, in the remaining case, we obtain vertices $v_n, v_{n+1}, \ldots$ with all edges among them red. $\square$

For brevity, we make some terminological conventions. Let A be a nonempty set. By *coloring of A* we mean a coloring of the elements of A, each by one color out of a prescribed set of colors. A *finite coloring* of A is a coloring in a finite number of colors. A *k-coloring* of A is a coloring in k colors.

Whenever convenient, we will identify each color with a natural number. For example, a coloring of A with colors red, green and blue is a function $c\colon A \to \{\text{red}, \text{green}, \text{blue}\}$, and we may consider instead a function $c\colon A \to \{1, 2, 3\}$.

In accordance with earlier uses of the word, a subset of A with all elements of the same color will be called *monochromatic*.

EXERCISE 1.2. We have proved Ramsey's Theorem for 2-colorings. Prove it for arbitrary finite colorings. (*Hint:* A color blindness argument.)

Ramsey's Theorem has a more general version. Before stating it note that, in the above formulation of Ramsey's Theorem, we may restrict attention to a countable subgraph of the given graph, and enumerate its vertices by the natural numbers. In short, we may assume that $V = \mathbb{N}$.

THEOREM 1.3 (Ramsey). *Let d be a natural number. For each finite coloring of $[\mathbb{N}]^d$, there is an infinite set $A \subseteq \mathbb{N}$ such that $[A]^d$ is monochromatic.*

PROOF. By induction on d. The case $d = 1$ is immediate. For $d > 1$, given a coloring

$$c\colon [\mathbb{N}]^d \to \{1, \ldots, k\},$$

choose $v_1 \in \mathbb{N}$ and consider the coloring $c_{v_1}\colon [\mathbb{N} \setminus \{v_1\}]^{d-1} \to \{1, \ldots, k\}$, defined by

$$c_{v_1}(\{v_2, \ldots, v_d\}) = c(\{v_1, v_2, \ldots, v_d\}).$$

By the inductive hypothesis, there is an infinite set $V_1 \subseteq \mathbb{N} \setminus \{v_1\}$ such that $[V_1]^{d-1}$ is monochromatic for the coloring c_{v_1}.

Continue as in the proof of Theorem 1.1. □

EXERCISE 1.4. Complete the proof of Theorem 1.3.

EXERCISE 1.5. Let A be an infinite set of points in the plane, such that each line contains at most a finite number of points from A. Using Ramsey's Theorem, prove that there is an infinite set $B \subseteq A$ such that each line contains at most two points from B.

2. Compactness, the Four Color Theorem, and the Finite Ramsey Theorem

Each of the coloring theorems proved in this book also has a version where the colored set is finite. These finite versions follow from the infinite ones, thanks to the following theorem.

THEOREM 2.1 (Compactness). *Let $X = \{x_1, x_2, \dots\}$ be a countable set, and let $\mathcal{A}$ be a family of finite subsets of X. Assume that for each k-coloring of X, there is in $\mathcal{A}$ a monochromatic set. Then there is a natural number n such that, for each k-coloring of $\{x_1, \dots, x_n\}$ there is in $\mathcal{A}$ a monochromatic subset of $\{x_1, \dots, x_n\}$.*

PROOF. Assume, towards a contradiction, that there is for each n a k-coloring c_n of $\{x_1, \dots, x_n\}$ with no monochromatic set in $\mathcal{A}$. Define a k-coloring of $X = \{x_1, x_2, \dots\}$ as follows:

(1) Choose a color i_1 such that the set $I_1 = \{\, n \in \mathbb{N} : c_n(x_1) = i_1 \,\}$ is infinite.
(2) Choose a color i_2 such that the set $I_2 = \{\, n \in I_1 : n \geq 2,\ c_n(x_2) = i_2 \,\}$ is infinite.
(3) By induction, for each $m > 1$ choose a color i_m such that the set

$$\begin{aligned} I_m &= \{\, n \in I_{m-1} : n \geq m,\, c_n(x_m) = i_m \,\} \\ &= \{\, n : n \geq m,\, c_n(x_1) = i_1, c_n(x_2) = i_2, \dots, c_n(x_m) = i_m \,\} \end{aligned}$$

is infinite.

Define a k-coloring of X by

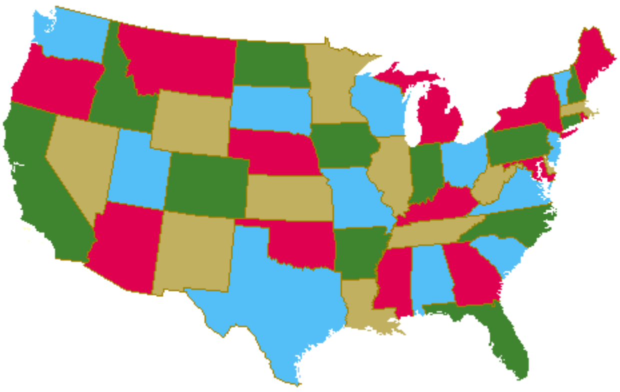
$$c(x_m) = i_m$$

for all m. By the premise of the theorem, there is a set $F \in \mathcal{A}$ that is monochromatic for the coloring c. Since F is finite, there is m such that $F \subseteq \{x_1, \dots, x_m\}$. Fix $n \in I_m$. Then c_n is defined on $x_1, \dots, x_m$, and agrees with c there. It follows that F is monochromatic for c_n; a contradiction. □

To illustrate the compactness theorem, we provide an amusing application. Assume that we have a (real, or imaginary) map of states, and we are interested in coloring each state region in a way that no neighboring states (ones that share a border that is more than a point) have the same color. There are several formal restrictions on the map: That it is embedded in the plane, and that the state regions are "continuous", but the intuitive concept will suffice for our purposes.

What is the minimal number of colors necessary to color a map? The Google Chrome logo forms a minimal example of a map that cannot be colored with fewer than *four* colors:

On the other hand, even the sate map of USA does not necessitate the use of more than four colors:



A nineteenth century conjecture, asserting that four colors suffice to color any map, was only proved in 1976 (by Kenneth Appel and Wolfgang Haken), by reducing the problem to fewer than 2,000 special cases, and checking them all using a computer. There is still no proof that does not necessitate the consideration of hundreds of cases, and we will of course not attempt at providing such a proof here. However, we will show that the infinite version of this theorem can be deduced from it.

Theorem 2.2. *Every infinite state map can be colored with four colors, such that states sharing a border have different colors.*

Proof. We first observe that the number of states in every map is countable. Indeed, think of the map as embedded in $\mathbb{R}^2$. Inside each state, choose a point (q_1, q_2) such that q_1 and q_2 are both rational numbers. The number of states is equal to the number of chosen points, which is not grater than the total number of points in $\mathbb{Q}^2$, which is countable.

Let $X = \{x_1, x_2, \dots\}$ be the set of states in the map. Let $\mathcal{A}$ be the set of neighboring states, that is, $\{x_i, x_j\} \in \mathcal{A}$ if and only if the states x_i and x_j share a border. Assume, towards a contradiction, that for each 4-coloring of X, there are neighboring states of the same color, that is, there is a monochromatic $\{x_i, x_j\} \in \mathcal{A}$. By the Compactness Theorem, there is n such that for each 4-coloring of $\{x_1, \dots, x_n\}$ there is a monochromatic set $\{x_i, x_j\} \in \mathcal{A}$. But this contradicts the *finite* Four Color Theorem. □

We now establish a finite version of Ramsey's Theorem. In the case where $d = 2$, the theorem asserts that every large enough k-colored complete graph has a large complete monochromatic subgraph.

Theorem 2.3 (Finite Ramsey Theorem). *Let k, m and d be natural numbers. There is n such that for each k-coloring of $[\{1, \dots, n\}]^d$ there is a set $A \subseteq \{1, \dots, n\}$ of cardinality m such that the set $[A]^d$ is monochromatic.*

Proof. Write $[\mathbb{N}]^d = \{x_1, x_2, \dots\}$. Let

$$\mathcal{A} = \left\{ [A]^d : A \subseteq \mathbb{N},\ |A| = m \right\}.$$

Every element of $\mathcal{A}$ is a finite subset of $\{x_1, x_2, \dots\}$.

Ramsey's Theorem asserts that for each k-coloring of $\{x_1, x_2, \dots\}$, there is an infinite set $B \subseteq \mathbb{N}$ such that the set $[B]^d$ is monochromatic. In particular, if we fix a subset $A \subseteq B$ of cardinality m, we have that the element $[A]^d$ of $\mathcal{A}$ is monochromatic.

By the Compactness Theorem, there is a natural number N such that, for each k-coloring of $\{x_1, \dots, x_N\}$, there is in $\mathcal{A}$ a monochromatic subset of $\{x_1, \dots, x_N\}$. Let n be the largest element appearing in any x_i, formally $n = \max(x_1 \cup \cdots \cup x_N)$. Then

$$\{x_1, \dots, x_N\} \subseteq [\{1, \dots, n\}]^d,$$

and thus for each k-coloring of $[\{1, \dots, n\}]^d$ there is a monochromatic element $[A]^d$ in $\mathcal{A}$. □

Ramsey's Theorem has numerous applications in mathematics and theoretical computer science. We will present here a beautiful application in number theory.

3. Fermat's Last Theorem and Schur's Theorem

In contrast to the Pythagorean Theorem, *Fermat's Last Theorem* asserts that, for $n > 2$ the equation

$$x^n + y^n = z^n$$

has no solution over the natural numbers. Fermat has stated this assertion without proof, and a proof was discovered only many generations later. The story of this theorem and its immensely complicated proof constitutes the topic of a best-selling book.

Long before Fermat's Last Theorem was proved, Issai Schur considered the problem whether Fermat's Equation has solutions *modulo a prime number.* One might hope that solving this problem for large enough prime numbers (and fixed n) may shed light on Fermat's assertion. Working with large primes also eliminates the following trivial obstacle.

EXERCISE 3.1. Prove that, for each prime number p, there is n such that the equation

$$x^n + y^n = z^n \pmod p$$

has no nontrivial solution $x, y, z \neq 0 \pmod p$. (*Hint:* Consider *Fermat's Little Theorem*: For each $a \in \{0, 1, \dots, p-1\}$, we have that $a^p = a \pmod p$.)

THEOREM 3.2 (Schur). *For each large enough prime number p, the equation $x^n + y^n = z^n \pmod p$ has a solution with $x, y, z \neq 0 \pmod p$.*

The proof of Schur's Theorem uses the following coloring theorem, that is of independent interest.

THEOREM 3.3 (Schur's Coloring Theorem). *For each finite coloring of $\mathbb{N}$, there are natural numbers x,y and z of the same color such that $x+y=z$. In other words, the equation $x+y=z$ has a monochromatic solution.*

PROOF. Let c be a k-coloring of $\mathbb{N}$. Define a k-coloring χ of $[\mathbb{N}]^2$ by

$$\chi(\{i, j\}) = c(j - i)$$

for all $i < j$. By Ramsey's Theorem, there is an infinite set $A \subseteq \mathbb{N}$ such that $[A]^2$ is monochromatic for χ. Let $i, j, m \in A$ be such that $i < j < m$. By the definition of χ we have that $c(j - i) = c(m - j) = c(m - i)$, and we have a monochromatic solution

$$\underbrace{m - j}_{x} + \underbrace{j - i}_{y} = \underbrace{m - i}_{z}. \quad \square$$

EXERCISE 3.4. Show, using the proof of Schur's Coloring Theorem, that the equation $x+y = z$ has infinitely many monochromatic solutions.

COROLLARY 3.5. *For every number of colors k, there is n such that, for each k-coloring of $\{1, \dots, n\}$, the equation $x + y = z$ has a monochromatic solution.*

EXERCISE 3.6. Prove Corollary 3.5, using the Compactness Theorem.

Proof of Theorem 3.2. Consider the finite field $\mathbb{Z}_p = \{0, 1, \dots, p-1\}$, with addition and multiplication modulo p. A fundamental theorem asserts that for each finite field $\mathbb{F}$, there is $g \in \mathbb{F}$ such that every nonzero element of $\mathbb{F}$ is a power of g. Let $0 \neq g \in \mathbb{Z}_p$ be such that $\{g, g^2, \dots, g^{p-1}\} = \mathbb{Z}_p \setminus \{0\}$.

Define a coloring of $\{1, \dots, p-1\}$ as follows: For each $r \in \{1, \dots, p-1\}$, there is a unique $m \in \{1, \dots, p-1\}$ such that $g^m = r$. Set $c(r) := m \bmod n$.

If p is large enough, then by Schur's Coloring Theorem there are $x, y, z \in \{1, \dots, p-1\}$, of the same color, such that $x + y = z$ over $\mathbb{N}$ and, in particular, over $\mathbb{Z}_p$. Continue the argument in $\mathbb{Z}_p$. Write

$$x = g^{nt_1+i}, y = g^{nt_2+i}, z = g^{nt_3+i}.$$

Then

$$g^{nt_1+i} + g^{nt_2+i} = g^{nt_3+i}.$$

Divide by g^i, to obtain

$$(\underbrace{g^{t_1}}_{x})^n + (\underbrace{g^{t_2}}_{y})^n = (\underbrace{g^{t_3}}_{z})^n.$$

We have thus found nonzero elements $x, y, z \in \mathbb{Z}_p$ such that $x^n + y^n = z^n \pmod p$. □

Remark 3.7. The proof of Schur's Theorem shows that for each n, in every large enough finite field $\mathbb{F}$, the equation $x^n + y^n = z^n$ has a nontrivial solution.

The following exercises can be solved using small modifications of the proof of Schur's Theorem.

Exercise 3.8. Let n be a natural number. Prove that for each large enough prime number p, the equation $x^n + y^n + z^n = w^n \pmod p$ has a solution with $x, y, z, w \neq 0 \pmod p$.

Exercise 3.9. Let G be a group with at least 6 elements. Prove that for each 2-coloring of G, there are nonidentity elements $a, b, c \in G$ of the same color, such that $ab = c$.

In the remainder of the book, we will prove coloring theorems that have no simple elementary proofs. We will exploit the interplay between algebra and topology. The next chapter provides the foundations of this method. Readers familiar with these foundations may find it sufficient to skim this chapter quickly, and proceed to the next chapter.

4. Comments for Chapter 1

Ramsey's Theorem is proved in his paper *On a problem of formal logic*, Proceedings of the London Mathematical Society 30 (1928), 264–286. Theorem 3.2 was first proved by Leonard E. Dickson, *On the Last Theorem of Fermat*, Quarterly Journal of Pure and Applied Mathematics, 1908. The proof provided here, via Schur's Coloring Theorem (Theorem 3.3), is due to Issai Schur, *Über die Kongruenz* $x^m + y^m = z^m \pmod p$, Jahresbericht der Deutschen Mathematiker–Vereinigung, 1916.

When a coloring theorem guarantees the existence of an infinite monochromatic set, it may be strictly stronger than its finite version. It follows from Ramsey's Theorem that, for all d, k, and m:

> There is n such that, for each k-coloring of $[\{m, m+1, \dots, n\}]^d$ there is a set $A \subseteq \{m, m+1, \dots, n\}$ such that $|A| > \min A$ and $[A]^d$ is monochromatic.

Exercise 4.1. Prove the last assertion.

The finite Ramsey Theorem is provable in Peano Arithmetic (the basic axiomatic system for number theory). Paris and Harrington, that identified the above consequence of Ramsey's theorem, proved that it is *unprovable* in Peano Arithmetic. This was the first natural statement in the language of Peano Arithmetic that is true but unprovable. The mere existence of such statements follows from Gödel's celebrated Incompleteness Theorem. This topic is covered in Section 6.3 of the Graham–Spencer–Rothschild classic book *Ramsey Theory*.

The finitary theorems may be thought of as shadows of their infinite counterpart. In general, the question how large should the finite colored set be to guarantee a monochromatic set as desired is wide open. For example, let r_m be the minimal n such that the Finite Ramsey Theorem holds for $k = 2$ colors, dimension $d = 2$, and $|A| = m$. It follows from the proof of Ramsey's Theorem that $r_3 = 6$. It is known that $r_4 = 18$. But in general, despite great efforts, only weak bounds are available for r_m. According to Joel Spencer (*Ten Lectures on the Probabilistic Method*, SIAM, 1994),

> Erdős asks us to imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of r_5 or they will destroy our planet. In that case, he claims, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for r_6. In that case, he believes, we should attempt to destroy the aliens.

The Wikipedia entry *Ramsey's theorem* forms an updated source on the status of this direction of research.