Contemporary Mathematics
Volume **533**, 2011

MENGER'S AND HUREWICZ'S PROBLEMS: SOLUTIONS FROM "THE BOOK" AND REFINEMENTS

BOAZ TSABAN

Abstract. We provide simplified solutions of Menger's and Hurewicz's problems and conjectures, concerning generalizations of σ-compactness. The reader who is new to this field will find a self-contained treatment in Sections 1, 2, and 5.

Sections 3 and 4 contain new results, based on the mentioned simplified solutions. The main new result is that there are concrete uncountable sets of reals X (indeed, $|X| = \mathfrak{b}$), which have the following property:

> Given point-cofinite covers $\mathcal{U}_1, \mathcal{U}_2, \dots$ of X, there are for each n sets $U_n, V_n \in \mathcal{U}_n$, such that each member of X is contained in all but finitely many of the sets $U_1 \cup V_1, U_2 \cup V_2, \dots$

This property is strictly stronger than Hurewicz's covering property. Miller and the present author showed that one cannot prove the same result if we are only allowed to pick one set from each $\mathcal{U}_n$.

Dedicated to Professor Gideon Schechtman

Contents

1. Menger's Conjecture

In 1924, Menger [14] introduced the following basis property for a metric space X:

> For each basis $\mathcal{B}$ for the topology of X, there are $B_1, B_2, \dots \in \mathcal{B}$ such that $\lim_{n\to\infty} \operatorname{diam}(B_n) = 0$, and $X = \bigcup_n B_n$.

Soon thereafter, Hurewicz [10] observed that Menger's basis property can be reformulated as follows:

1991 *Mathematics Subject Classification.* Primary: 37F20; Secondary 26A03, 03E75 03E17 .
Key words and phrases. Menger property, Hurewicz property, Rothberger property, Selection principles, special sets of real numbers.

> For all given open covers $\mathcal{U}_1, \mathcal{U}_2, \dots$ of X, there are finite $\mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \dots$ such that $\bigcup_n \mathcal{F}_n$ is a cover of X.

We introduce some convenient notation, suggested by Scheepers in [20]. We say that $\mathcal{U}$ is a *cover* of X if $X = \bigcup \mathcal{U}$,[1] but $X \notin \mathcal{U}$. Let X be a topological space, and $\mathscr{A}, \mathscr{B}$ be families of covers of X. We consider the following statements.

- $\mathsf{S}_1(\mathscr{A}, \mathscr{B})$**:** For all $\mathcal{U}_1, \mathcal{U}_2, \dots \in \mathscr{A}$, there are $U_1 \in \mathcal{U}_1, U_2 \in \mathcal{U}_2, \dots$ such that $\{U_n : n \in \mathbb{N}\} \in \mathscr{B}$.
- $\mathsf{S}_{\mathrm{fin}}(\mathscr{A}, \mathscr{B})$**:** For all $\mathcal{U}_1, \mathcal{U}_2, \dots \in \mathscr{A}$, there are finite $\mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \dots$ such that $\bigcup_n \mathcal{F}_n \in \mathscr{B}$.
- $\mathsf{U}_{\mathrm{fin}}(\mathscr{A}, \mathscr{B})$**:** For all $\mathcal{U}_1, \mathcal{U}_2, \dots \in \mathscr{A}$, none containing a finite subcover, there are finite $\mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \dots$ such that $\{\bigcup \mathcal{F}_n : n \in \mathbb{N}\} \in \mathscr{B}$.

Let $\mathrm{O}(X)$ be the family of all open covers of X. We say that X satisfies $\mathsf{S}_1(\mathrm{O}, \mathrm{O})$ if the statement $\mathsf{S}_1(\mathrm{O}(X), \mathrm{O}(X))$ holds. This way, $\mathsf{S}_1(\mathrm{O}, \mathrm{O})$ is a property of topological spaces. A similar convention applies to all properties of this type.

Hurewicz's observation tells that for metric spaces, Menger's basis property is equivalent to $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$. This is a natural generalization of compactness. Note that indeed, every *σ-compact* space (a countable union of compact spaces) satisfies $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$. Menger made the following conjecture.

Conjecture 1.1 (Menger [14])**.** *A metric space X satisfies $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$ if, and only if, X is σ-compact.*

Hurewicz proved that when restricted to analytic spaces, Menger's Conjecture is true.

Recall that a set $M \subseteq \mathbb{R}$ is *meager* (or *of Baire first category*) if M is a union of countably many nowhere dense sets. A set $L \subseteq \mathbb{R}$ is a *Luzin set* if L is uncountable, and for each meager set M, $L \cap M$ is countable.

Luzin sets can be constructed assuming the Continuum Hypothesis: Every meager set is contained in a Borel (indeed, F_σ) meager set. Let M_α, $\alpha < \aleph_1$ be all Borel meager sets. For each $\alpha < \aleph_1$, take $x_\alpha \in \mathbb{R} \setminus \bigcup_{\beta < \alpha} M_\beta$. Then $L = \{x_\alpha : \alpha < \aleph_1\}$ is a Luzin set.

A subset of $\mathbb{R}$ is *perfect* if it is nonempty, closed, and has no isolated points. In [11], Hurewicz quotes an argument of Sierpiński, proving the following.

Theorem 1.2 (Sierpiński)**.** *Every Luzin set satisfies $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$, and is not σ-compact.*

Proof. Let $\mathcal{U}_1, \mathcal{U}_2, \dots$ be open covers of a Luzin set $L \subseteq \mathbb{R}$. Let $D = \{d_n : n \in \mathbb{N}\}$ be a dense subset of L. For each n, pick $U_n \in \mathcal{U}_n$ such that $d_n \in U_n$. Let $U = \bigcup_n U_n$. Then $L \setminus U$ is nowhere dense, and thus countable. Enumerate $L \setminus U = \{x_n : n \in \mathbb{N}\}$. For each n, pick $V_n \in \mathcal{U}_n$ such that $x_n \in V_n$. Then $L \setminus U \subseteq \bigcup_n V_n$, and thus $\{U_n, V_n : n \in \mathbb{N}\}$ is a cover of L, with at most two elements from each $\mathcal{U}_n$.[2]

Lemma 1.3 (Cantor-Bendixon)**.** *Every uncountable σ-compact set $X \subseteq \mathbb{R}$ contains a perfect set.*

[1] We follow the set theoretic standard that, for a family of sets $\mathcal{F}$, $\bigcup \mathcal{F}$ means the union of all elements of $\mathcal{F}$.

[2] The interested reader may wish to show in a similar manner that actually, every Luzin set satisfies $\mathsf{S}_1(\mathrm{O}, \mathrm{O})$. We will not use this fact.

Proof. By moving to a subset, we may assume that X is an uncountable compact, and thus closed, set. By the Cantor-Bendixon Theorem, X contains a perfect set. $\square$

As perfect sets contain perfect nowhere dense subsets, a Luzin set cannot be σ-compact. $\square$

Thus, Menger's Conjecture is settled if one assumes the Continuum Hypothesis. In 1988, Fremlin and Miller [7] settled Menger's Conjecture in ZFC. They used the concept of a scale, which we now define. This concept is normally defined using $\mathbb{N}^{\mathbb{N}}$, but for our purposes it is easier to work with $P(\mathbb{N})$ (this will become clear later).

Let $P(\mathbb{N})$ be the family of all subsets of $\mathbb{N}$, and $[\mathbb{N}]^{<\infty}, [\mathbb{N}]^{\infty} \subseteq P(\mathbb{N})$ denote the family of all finite subsets of $\mathbb{N}$ and the family of all infinite subsets of $\mathbb{N}$, respectively. For $a \in [\mathbb{N}]^{\infty}$ and $n \in \mathbb{N}$, $a(n)$ denotes the n-th element in the increasing enumeration of a.

For $a, b \in [\mathbb{N}]^{\infty}$, let $a \leq^* b$ mean: $a(n) \leq b(n)$ for all but finitely many n. A subset Y of $[\mathbb{N}]^{\infty}$ is *dominating* if for each $a \in [\mathbb{N}]^{\infty}$ there is $b \in Y$ such that $a \leq^* b$. Let $\mathfrak{d}$ denote the minimal cardinality of a dominating subset of $[\mathbb{N}]^{\infty}$. A *scale* is a dominating set $S \subseteq [\mathbb{N}]^{\infty}$, which has a $\leq^*$-increasing enumeration $S = \{s_\alpha : \alpha < \mathfrak{d}\}$, that is, such that $s_\alpha \leq^* s_\beta$ for all $\alpha < \beta < \mathfrak{d}$.

Scales require special hypotheses to be constructed. Indeed, say that a subset Y of $[\mathbb{N}]^{\infty}$ is *unbounded* if it is unbounded with respect to $\leq^*$, that is, for each $a \in [\mathbb{N}]^{\infty}$ there is $b \in Y$ such that $b \not\leq^* a$. Let $\mathfrak{b}$ denote the minimal cardinality of an unbounded subset of $[\mathbb{N}]^{\infty}$. $\mathfrak{b} \leq \mathfrak{d}$, and strict inequality is consistent. (Indeed, $\mathfrak{b} < \mathfrak{d}$ holds in the Cohen real model.)

Lemma 1.4 (folklore). *There is a scale if, and only if,* $\mathfrak{b} = \mathfrak{d}$.

Proof. ($\Leftarrow$) Let $\{d_\alpha : \alpha < \mathfrak{b}\} \subseteq [\mathbb{N}]^{\infty}$ be dominating. For each $\alpha < \mathfrak{b}$, choose s_α to be a $\leq^*$-bound of $\{d_\beta, s_\beta : \beta < \alpha\}$.

($\Rightarrow$) Let $S = \{s_\alpha : \alpha < \mathfrak{d}\}$ be a scale, and assume that $\mathfrak{b} < \mathfrak{d}$. Let $\{b_\alpha : \alpha < \mathfrak{b}\} \subseteq [\mathbb{N}]^{\infty}$ be unbounded. For each α, take $\beta_\alpha < \mathfrak{d}$ such that $b_\alpha \leq^* s_{\beta_\alpha}$.

Let $c \in [\mathbb{N}]^{\infty}$ witness that $\{s_{\beta_\alpha} : \alpha < \mathfrak{b}\}$ is not dominating, and let $\gamma < \mathfrak{d}$ be such that $c \leq^* s_\gamma$. For each $\alpha < \mathfrak{b}$, $s_\gamma \not\leq^* s_{\beta_\alpha}$, and thus $b_\alpha \leq^* s_{\beta_\alpha} \leq^* s_\gamma$. Thus, $\{b_\alpha : \alpha < \mathfrak{b}\}$ is bounded. A contradiction. $\square$

The canonical way to construct sets of reals from scales (more generally, from subsets of $P(\mathbb{N})$) is as follows. $P(\mathbb{N})$ is identified with Cantor's space $\{0,1\}^{\mathbb{N}}$, via characteristic functions. This defines the canonical topology on $P(\mathbb{N})$. Cantor's space is homeomorphic to the canonical middle-third Cantor set $C \subseteq [0,1]$, and the homeomorphism is (necessarily, uniformly) continuous in both directions. Thus, subsets of $P(\mathbb{N})$ exhibiting properties preserved by taking (uniformly) continuous images may be converted into subsets of $[0,1]$ with the same properties. We may thus work in $P(\mathbb{N})$.

The *critical cardinality* of a (nontrivial) property P of set of reals, denoted $\mathrm{non}(P)$, is the minimal cardinality of a set of reals X such that X does not have the property P. The following is essentially due to Hurewicz [11].

Lemma 1.5 (folklore). $\mathrm{non}(\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})) = \mathfrak{d}$.

Proof. ($\geq$) Let X be a set of reals with $|X| < \mathfrak{d}$. Let $\mathcal{U}_1, \mathcal{U}_2, \ldots$ be open covers of X. Since X is Lindelöf, we may assume that these covers are countable, and enumerate them $\mathcal{U}_n = \{U^n_m : m \in \mathbb{N}\}$.

Define for each $x \in X$ a set $a_x \in [\mathbb{N}]^\infty$ by

$$a_x(n) = \min\{m > a_x(n-1) : x \in U_1^n \cup U_2^n \cup \cdots \cup U_m^n\}.$$

As $|\{a_x : x \in X\}| < \mathfrak{d}$, there is (in particular) $c \in [\mathbb{N}]^\infty$ such that for each $x \in X$, $a_x(n) \le c(n)$ for some n. Take $\mathcal{F}_n = \{U_1^n, \dots, U_{c(n)}^n\}$ for all n. Then $\bigcup_n \mathcal{F}_n$ is a cover of X.

($\le$) Let D be a dominating subset of $[\mathbb{N}]^\infty$. Consider the open covers $\mathcal{U}_n = \{U_m^n : m \in \mathbb{N}\}$, $n \in \mathbb{N}$, where

$$U_m^n = \{a \in [\mathbb{N}]^\infty : a(n) = m\}.$$

For all finite $\mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \dots$, there is $x \in D$ such that for all but finitely many n, $x(n) > \max\{m : U_m^n \in \mathcal{F}_n\}$ (and thus $x \notin \bigcup \mathcal{F}_n$).

But if X satisfies $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$, then for all open covers $\mathcal{U}_1, \mathcal{U}_2, \dots$ of X, there are finite $\mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \dots$, such that for each $x \in X$, x belongs to $\bigcup \mathcal{F}_n$ for infinitely many n: To see this, split the given sequence $\mathcal{U}_1, \mathcal{U}_2, \dots$ into infinitely many disjoint subsequences, and apply $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$ to each of these subsequences separately.

Thus, dominating subsets of $[\mathbb{N}]^\infty$ do not satisfy $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$. $\square$

Let κ be an infinite cardinal. A set of reals X is *κ-concentrated* on a set Q if, for each open set U containing Q, $|X \setminus U| < \kappa$.

Lemma 1.6 (folklore [23])**.** *Assume that a set of reals X is $\mathfrak{c}$-concentrated on a countable set Q. Then X does not contain a perfect set.*

Proof. Assume that X contains a perfect set P. Then $P \setminus Q$ is Borel and uncountable. A classical result of Alexandroff tells that every uncountable Borel set contains a perfect set. Let $C \subseteq P \setminus Q$ be a perfect set.[3] Then $U = \mathbb{R} \setminus C$ is open and contains Q, and $C = P \setminus U \subseteq X \setminus U$ has cardinality $\mathfrak{c}$. Thus, X is not $\mathfrak{c}$-concentrated on Q. $\square$

Theorem 1.7 (Fremlin-Miller [7])**.** *Menger's Conjecture is false.*

Proof. As perfect sets of reals have cardinality continuum, we have by Lemma 1.3 that if $\mathfrak{b} < \mathfrak{d}$, then any set of reals of cardinality $\mathfrak{b}$ is a counter-example.

Thus, assume that $\mathfrak{b} = \mathfrak{d}$ (this is the interesting case), and let $S = \{s_\alpha : \alpha < \mathfrak{d}\} \subseteq [\mathbb{N}]^\infty$ be a scale (Lemma 1.4).

$S \cup [\mathbb{N}]^{<\infty}$ satisfies $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$: This is similar to the argument about Luzin sets satisfying $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$. Given open covers $\mathcal{U}_1, \mathcal{U}_2, \dots$ of $S \cup [\mathbb{N}]^{<\infty}$, take $U_1 \in \mathcal{U}_1, U_2 \in \mathcal{U}_2, \dots$, such that $[\mathbb{N}]^{<\infty} \subseteq \bigcup_n U_n$. We can do that because $[\mathbb{N}]^{<\infty}$ is countable. Let $U = \bigcup_n U_n$. $P(\mathbb{N}) \setminus U$ is closed and thus compact. For each n, the evaluation map $e_n : [\mathbb{N}]^\infty \to \mathbb{N}$ defined by $e_n(a) = a(n)$ is continuous. Thus, $e_n[P(\mathbb{N}) \setminus U]$ is compact and thus finite, for all n. Therefore, there is a $\le^*$-bound b for $P(\mathbb{N}) \setminus U$. Take $\alpha < \mathfrak{d}$ such that $b <^* s_\alpha$. Then

$$S \setminus U = S \cap (P(\mathbb{N}) \setminus U) \subseteq \{s_\beta : \beta < \mathfrak{d}, s_\beta \le^* b\} \subseteq \{s_\beta : \beta < \alpha\}$$

has cardinality $< \mathfrak{d}$, and thus satisfies $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$. Let $\mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \dots$ be such that $S \setminus U \subseteq \bigcup_n \mathcal{F}_n$. Then $S \cup [\mathbb{N}]^{<\infty} \subseteq \bigcup_n \mathcal{F}_n \cup \{U_n\}$.

$S \cup [\mathbb{N}]^{<\infty}$ is not σ-compact: We have just seen that it is $\mathfrak{d}$-concentrated on the countable set $[\mathbb{N}]^{<\infty}$. Use Lemmata 1.3 and 1.6. $\square$

[3] As Q is countable, one can alternatively prove directly that $P \setminus Q$ contains a perfect set.

A reader not familiar with dichotomic proofs may be perplexed by the proof of the Fremlin-Miller Theorem 1.7. It gives a ZFC result by considering an undecidable statement. Indeed, it shows that there is a certain set of reals, but does not tell us what this set is (unless we know in advance whether $\mathfrak{b} < \mathfrak{d}$ or $\mathfrak{b} = \mathfrak{d}$). Another way to view this is as follows.

Sets of reals X satisfying P because $|X| < \mathrm{non}(P)$ are in a sense *trivial* examples for this property. From this point of view, the real question is, given a property P, whether there are sets of reals of cardinality at least $\mathrm{non}(P)$, which satisfy P. The proof of Theorem 1.7 answers this in the positive only when $\mathfrak{b} = \mathfrak{d}$. However, with a small modification we get a complete answer.

Definition 1.8. A $\mathfrak{d}$-*scale* is a dominating set $S = \{s_\alpha : \alpha < \mathfrak{d}\} \subseteq [\mathbb{N}]^\infty$, such that for all $\alpha < \beta < \mathfrak{d}$, $s_\beta \not\leq^* s_\alpha$.

Lemma 1.9. *There are* $\mathfrak{d}$*-scales.*

Proof. Let $\{d_\alpha : \alpha < \mathfrak{d}\} \subseteq [\mathbb{N}]^\infty$ be dominating. For each $\alpha < \mathfrak{d}$, choose s_α to be a witness that $\{s_\beta : \beta < \alpha\}$ is not dominating, such that in addition, $d_\alpha \leq^* s_\alpha$. □

An argument similar to that in the proof of Theorem 1.7 gives the following.

Lemma 1.10. *Every* $\mathfrak{d}$*-scale is* $\mathfrak{d}$*-concentrated on* $[\mathbb{N}]^{<\infty}$. □

We therefore have the following.

Theorem 1.11 (Bartoszyński-Tsaban [3])**.** *For each* $\mathfrak{d}$*-scale* S, $S \cup [\mathbb{N}]^{<\infty}$ *satisfies* $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$, *and is not* σ*-compact. In other words,* $S \cup [\mathbb{N}]^{<\infty}$ *is a counter-example to Menger's Conjecture.* □

Theorem 1.11 is generalized in Tsaban-Zdomskyy [23].

We conclude the section with some easy improvements of statements made above.

Define the following subfamily of $\mathrm{O}(X)$: $\mathcal{U} \in \Gamma(X)$ if $\mathcal{U}$ is infinite, and each element of X is contained in all but finitely many members of $\mathcal{U}$. If $\mathcal{U} \in \Gamma(X)$, then every infinite subset of $\mathcal{U}$ belongs to $\Gamma(X)$. Thus, we may assume for our purposes that elements of $\Gamma(X)$ are countable.

Corollary 1.12 (Just, et al. [12])**.** $\mathsf{S}_1(\Gamma,\mathrm{O})$ *implies* $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$.

Proof. Let X be a set of reals satisfying $\mathsf{S}_1(\Gamma,\mathrm{O})$, and let $\mathcal{U}_1,\mathcal{U}_2,\dots \in \mathrm{O}(X)$. The claim is trivial if some $\mathcal{U}_n$ contains a finite subcover. Thus, assume that this is not the case.

As sets of reals are Lindelöf, we may assume that each $\mathcal{U}_n$ is countable, say $\mathcal{U}_n = \{U^n_m : m \in \mathbb{N}\}$. Let

$$\mathcal{V}_n = \left\{ \bigcup_{k\leq m} U^n_k : m \in \mathbb{N} \right\}.$$

Then $\mathcal{V}_n \in \Gamma(X)$. Applying $\mathsf{S}_1(\Gamma,\mathrm{O})$ there are m_n, $n \in \mathbb{N}$, such that $\{\bigcup_{k\leq m_n} U^n_k : n \in \mathbb{N}\}$ is a cover of X. For each n, the finite sets $\mathcal{F}_n = \{U^n_k : k \leq m_n\} \subseteq \mathcal{U}_n$ are as required in the definition of $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$. □

A modification of the proof of Lemma 1.5 yields the following.

Lemma 1.13 (Just, et al. [12])**.** $\mathrm{non}(\mathsf{S}_1(\Gamma,\mathrm{O})) = \mathfrak{d}$. □

Proof. By Corollary 1.12 and Lemma 1.5,

$$\mathrm{non}(\mathsf{S}_1(\Gamma,\mathrm{O})) \leq \mathrm{non}(\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})) = \mathfrak{d}.$$

To prove the remaining inequality, assume that $|X| < \mathfrak{d}$, and $\mathcal{U}_1, \mathcal{U}_2, \dots \in \Gamma(X)$. We may assume that for each n, $\mathcal{U}_n$ is countable, and enumerate it $\mathcal{U}_n = \{U^n_m : m \in \mathbb{N}\}$. For each $x \in X$, let

$$a_x(n) = \min\{k > a_x(n-1) : (\forall m \geq k)\ x \in U^n_m\}$$

for all n. (In the case $n = 1$, omit the restriction $k > a_x(n-1)$.) $|\{a_x : x \in X\}| < \mathfrak{d}$. Let $d \in [\mathbb{N}]^\infty$ exemplify that $\{a_x : x \in X\}$ is not dominating, and take $\mathcal{F}_n = \{U^n_1, \dots, U^n_{d(n)}\}$. Then each $x \in X$ belongs to $\bigcup \mathcal{F}_n$ for infinitely many n. $\square$

Corollary 1.14. *Each set which is $\mathfrak{d}$-concentrated on a countable subset, satisfies $\mathsf{S}_1(\Gamma,\mathrm{O})$.* $\square$

Corollary 1.15 (Bartoszyński-Tsaban [3])**.** *For each $\mathfrak{d}$-scale S, $S \cup [\mathbb{N}]^{<\infty}$ satisfies $\mathsf{S}_1(\Gamma,\mathrm{O})$.* $\square$

$\mathsf{S}_1(\Gamma,\mathrm{O})$ is strictly stronger that $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$. While every σ-compact set satisfies the latter, we have the following.

Lemma 1.16 (Just, et al. [12])**.** *If X satisfies $\mathsf{S}_1(\Gamma,\mathrm{O})$, then X has no perfect subsets.*

Proof. We give Sakai's proof [18, Lemma 2.1]. Assume that X has a perfect subset and satisfies $\mathsf{S}_1(\Gamma,\mathrm{O})$. Then X has a subset C homeomorphic to Cantor's space $\{0,1\}^{\mathbb{N}}$. C is compact, and thus closed in X, and therefore satisfies $\mathsf{S}_1(\Gamma,\mathrm{O})$ as well.[4] Thus, it suffices to show that $\{0,1\}^{\mathbb{N}}$ does not satisfy $\mathsf{S}_1(\Gamma,\mathrm{O})$. We show instead that its homeomorphic copy $(\{0,1\}^{\mathbb{N}})^{\mathbb{N}}$ does not satisfy $\mathsf{S}_1(\Gamma,\mathrm{O})$.

Let $C_1, C_2, \dots$ be pairwise disjoint nonempty clopen subsets of $\{0,1\}^{\mathbb{N}}$. Let $U_1, U_2, \dots$ be the complements of $C_1, C_2, \dots$, respectively. For each n, let $\pi_n : (\{0,1\}^{\mathbb{N}})^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ be the projection on the n-th coordinate. Then $\mathcal{U}_n = \{\pi_n^{-1}[U_m] : m \in \mathbb{N}\} \in \Gamma(X)$ for all n. But for all $\pi_1^{-1}[U_{m_1}] \in \mathcal{U}_1, \pi_2^{-1}[U_{m_2}] \in \mathcal{U}_2, \dots$, we have that $\Pi_n C_n$ is disjoint of $\bigcup_n \pi_n^{-1}[U_{m_n}]$. $\square$

2. Hurewicz's Conjecture

Hurewicz suspected that Menger's Conjecture was false. For this reason, he introduced in [10] a formally stronger property, which in our notation is $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$. It is easy to see that every σ-compact set satisfies, in fact, $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$, and analogously to Menger, Hurewicz made the following.

Conjecture 2.1 (Hurewicz [10])**.** *A metric space X satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$ if, and only if, X is σ-compact.*

The following easy fact is instructive.

Lemma 2.2. *X satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$ if, and only if, for all $\mathcal{U}_1, \mathcal{U}_2, \dots$, none having a finite subcover of X, there is a decomposition $X = \bigcup_k X_k$, such that for each k, there are finite subsets $\mathcal{F}^k_1 \subseteq \mathcal{U}_1, \mathcal{F}^k_2 \subseteq \mathcal{U}_2, \dots$, such that for each $x \in X_k$, $x \in \bigcup \mathcal{F}^k_n$ for all but finitely many n.*

[4] It is easy to see that all properties involving open covers, considered in this paper, are hereditary for closed subsets [12].

Proof. For each n, take $\mathcal{F}_n = \bigcup_{k\le n} \mathcal{F}_n^k$. Then $\{\bigcup \mathcal{F}_n : n \in \mathbb{N}\} \in \Gamma(X)$. $\square$

$S \subseteq \mathbb{R}$ is a *Sierpiński set* if S is uncountable, and for each Lebesgue measure zero set N, $S \cap N$ is countable. Since every perfect set contains a perfect set of Lebesgue measure zero, a Sierpiński set cannot contain a perfect subset, and therefore is not σ-compact (Lemma 1.3). A construction similar to that of a Luzin set described above, shows that the Continuum Hypothesis implies the existence of Sierpiński sets. We do not know when the following observation was made first.

Theorem 2.3 (folklore)**.** *Every Sierpiński set satisfies* $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$.

Proof. The following proof is a slightly simplified version of the one given in [12].

Let S be a Sierpiński set. $S = \bigcup_n S \cap [-n, n]$, and thus by Lemma 2.2, we may assume that the outer measure p of S is finite. Since S is Sierpiński, $p > 0$.[5] Let $B \supseteq S$ be a Borel set of measure p.

Let $\mathcal{U}_1, \mathcal{U}_2, \dots$ be open covers of S. We may assume that each $\mathcal{U}_n$ is countable, and enumerate $\mathcal{U}_n = \{U^n_m : m \in \mathbb{N}\}$. We may assume that all U^n_m are Borel subsets of B. For each n, $\bigcup_m U^n_m \supseteq S$, and thus has measure p for each n. Thus, for each N there is $f_N \in \mathbb{N}^\mathbb{N}$ such that $\bigcup_{k=1}^{f_N(n)} U^n_k$ has measure $\ge (1 - 1/2^{n+N})p$, and consequently, $A_N = \bigcap_n \bigcup_{k=1}^{f_N(n)} U^n_k$ has measure $\ge (1 - 1/2^N)p$.

Then $A = \bigcup_N A_N$ has measure p, and thus $S \setminus A$ is countable. The countable decomposition $S = (S \setminus A) \cup \bigcup_N A_N$ is as required in Lemma 2.2, by the countability of $S \setminus A$ and the definition of A_N. $\square$

A stronger statement can be proved in a similar manner.

Theorem 2.4 (Just, et al. [12])**.** *Every Sierpiński set satisfies* $\mathsf{S}_1(\Gamma, \Gamma)$ *(even when we consider Borel covers instead of open ones).*

Proof. Replace, in the proof of Theorem 2.3, U^n_m by $\bigcap_{k\ge m} U^n_k$. Let $f \in \mathbb{N}^\mathbb{N}$ be such that for each $x \in S \setminus A$, $x \in \bigcap_{k\ge f(n)} U^n_k$ for all but finitely many n. Let g be a $\le^*$-bound of $\{f_N : N \in \mathbb{N}\} \cup \{f\}$. Then the choice $U^1_{g(1)} \in \mathcal{U}_1, U^2_{g(2)} \in \mathcal{U}_2, \dots$ is as required. $\square$

Thus, the Continuum Hypothesis implies the failure of Hurewicz's Conjecture. A complete refutation, however, was only discovered in 1996, by Just, Miller, Scheepers, and Szeptycki, in their seminal paper [12].

Theorem 2.5 (Just, et al. [12])**.** *Hurewicz's Conjecture is false.*

We will not provide the full solution from [12] here (since we provide a simpler one), but just discuss its main ingredients. The argument in [12] is dichotomic. Recall that $\mathfrak{b}$ is the minimal cardinality of a set $B \subseteq [\mathbb{N}]^\infty$ which is unbounded with respect to $\le^*$. A proof similar to that of Lemma 1.5 gives the following two results, which are also essentially due to Hurewicz [11].

Lemma 2.6 (folklore)**.** *An unbounded subset of* $[\mathbb{N}]^\infty$ *cannot satisfy* $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$. $\square$

Lemma 2.7 (folklore)**.** $\mathrm{non}(\mathsf{S}_1(\Gamma, \Gamma)) = \mathrm{non}(\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)) = \mathfrak{b}$. $\square$

Thus, if $\mathfrak{b} > \aleph_1$ then any set of cardinality $\aleph_1$ is a counter-example to Hurewicz's Conjecture.

[5] Otherwise, S would have measure zero, and thus be countable.

Definition 2.8. A $\mathfrak{b}$-*scale* is an unbounded set $\{b_\alpha : \alpha < \mathfrak{b}\} \subseteq [\mathbb{N}]^\infty$, such that the enumeration is increasing with respect to $\leq^*$ (i.e., $b_\alpha \leq^* b_\beta$ whenever $\alpha < \beta < \mathfrak{b}$).

Like $\mathfrak{d}$-scales, $\mathfrak{b}$-scales can be constructed without special hypotheses.

Lemma 2.9 (folklore)**.** *There are* $\mathfrak{b}$*-scales.*

Proof. Let $\{x_\alpha : \alpha < \mathfrak{b}\} \subseteq [\mathbb{N}]^\infty$ be unbounded. For each $\alpha < \mathfrak{b}$, choose b_α to be a $\leq^*$-bound of $\{b_\beta : \beta < \alpha\}$, such that $x_\alpha \leq^* b_\alpha$. □

The argument in [12] proceeds as follows. We have just seen that the case $\mathfrak{b} > \aleph_1$ is trivial. Thus, assume that $\mathfrak{b} = \aleph_1$. Then there is a $\mathfrak{b}$-scale $B = \{b_\alpha : \alpha < \mathfrak{b}\} \subseteq [\mathbb{N}]^\infty$ such that in addition, for all $\alpha < \beta < \mathfrak{b}$, $b_\beta \setminus b_\alpha$ is finite.[6] It is proved in [12] that for such B, $B \cup [\mathbb{N}]^{<\infty}$ satisfies $\mathsf{U}_{\text{fin}}(\mathrm{O}, \Gamma)$. An argument similar to the one given in Theorem 1.7 for scales shows the following.

Lemma 2.10. *Every* $\mathfrak{b}$*-scale* B *is* $\mathfrak{b}$*-concentrated on* $[\mathbb{N}]^{<\infty}$*. In particular,* $B \cup [\mathbb{N}]^{<\infty}$ *is not* σ*-compact.* □

Unfortunately, the existence of $\mathfrak{b}$-scales as in the proof of [12] is undecidable. This is so because Scheepers proved that for this type of $\mathfrak{b}$-scales, $B \cup [\mathbb{N}]^{<\infty}$ in fact satisfies $\mathsf{S}_1(\Gamma, \Gamma)$ [21] (see also [16]), and we have the following.

Theorem 2.11 (Miller-Tsaban [16])**.** *It is consistent that for each set of reals satisfying* $\mathsf{S}_1(\Gamma, \Gamma)$*,* $|X| < \mathfrak{b}$*. Indeed, this is the case in Laver's model.*

Bartoszyński and Shelah have discovered an ingenious direct solution to Hurewicz's Conjecture, which can be reformulated as follows.

Theorem 2.12 (Bartoszyński-Shelah [2])**.** *For each* $\mathfrak{b}$*-scale* B*,* $B \cup [\mathbb{N}]^{<\infty}$ *satisfies* $\mathsf{U}_{\text{fin}}(\mathrm{O}, \Gamma)$.

We provide a simplified proof of this theorem, using a method of Galvin and Miller from [8]. For natural numbers n, m, let $[n, m) = \{n, n+1, \dots, m-1\}$.

Lemma 2.13 (folklore)**.** *Let* $Y \subseteq [\mathbb{N}]^\infty$*. The following are equivalent:*

(1) Y *is bounded;*
(2) *There is* $s \in [\mathbb{N}]^\infty$ *such that for each* $a \in Y$*,* $a \cap [s(n), s(n+1)) \neq \emptyset$ *for all but finitely many* n*.*

Proof. $(1 \Rightarrow 2)$ Let $b \in [\mathbb{N}]^\infty$ be a $\leq^*$-bound for Y. Define inductively $s \in [\mathbb{N}]^\infty$ by

$$\begin{aligned} s(1) &= b(1) \\ s(n+1) &= b(s(n)) + 1 \end{aligned}$$

For each $a \in Y$ and all but finitely many n, $s(n) \leq a(s(n)) \leq b(s(n)) < s(n+1)$, that is, $a(s(n)) \in [s(n), s(n+1))$.

$(2 \Rightarrow 1)$ Let s be as in (2). s has countably many cofinite subsets. Let $b \in [\mathbb{N}]^\infty$ be a $\leq^*$-bound of all cofinite subsets of s. Let $a \in Y$ and choose n_0 such that for each $n \geq n_0$, $a \cap [s(n), s(n+1)) \neq \emptyset$. Choose m_0 such that $a(m_0) \in [s(n_0), s(n_0+1))$. By induction on n, we have that $(a(n) \leq) a(m_0 + n) \leq s(n_0 + 1 + n)$ for all n. For large enough n, we have that $s(n_0 + 1 + n) \leq b(n)$, thus $a \leq^* b$. □

[6] We will not use this fact here, but here is a proof: Fix an unbounded family $\{x_\alpha : \alpha < \mathfrak{b}\} \subseteq [\mathbb{N}]^\infty$. At step α, we have a countable set $B_\alpha = \{b_\beta : \beta < \alpha\}$ such that for all $\gamma < \beta < \mathfrak{b}$, $b_\beta \setminus b_\gamma$ is finite. In particular, each finite subset of B_α has an infinite intersection. Enumerate $B_\alpha = \{s_n : n \in \mathbb{N}\}$, and for each n pick $m_n \in s_1 \cap \cdots \cap s_n$ such that $m_n > m_{n-1}$. Let c be a $\leq^*$-bound of B_α, and let b_α be a subset of $\{m_n : n \in \mathbb{N}\}$, such that $\max\{c, x_\alpha\} \leq^* b_\alpha$.

Lemma 2.14 (Galvin-Miller [8]). *Assume that* $[\mathbb{N}]^{<\infty} \subseteq X \subseteq P(\mathbb{N})$. *For each* $\mathcal{U} \in \Gamma(X)$,[7] *there are* $a \in [\mathbb{N}]^{\infty}$ *and distinct* $U_1, U_2, \dots \in \mathcal{U}$, *such that for each* $x \subseteq \mathbb{N}$, $x \in U_n$ *whenever* $x \cap [a(n), a(n+1)) = \emptyset$.

Proof. Let $a(1) = 1$. For each $n \geq 1$: As $\mathcal{U} \in \Gamma(X)$, each finite subset of X is contained in infinitely many elements of $\mathcal{U}$. Take $U_n \in \mathcal{U} \setminus \{U_1, \dots, U_{n-1}\}$, such that $P([1, a(n))) \subseteq U_n$. As U_n is open, for each $s \subseteq [1, a(n))$ there is k_s such that for each $x \in P(\mathbb{N})$ with $x \cap [1, k_s) = s$, $x \in U_n$. Let $a(n+1) = \max\{k_s : s \subseteq [1, a(n))\}$. □

Given the methods presented thus far, the following proof boils down to the fact that, if we throw fewer than n balls into n bins, at least one bin remains empty.

Proof of Theorem 2.12. Let $B = \{b_\alpha : \alpha < \mathfrak{b}\}$ be a $\mathfrak{b}$-scale. Let $\mathcal{U}_1, \mathcal{U}_2, \dots \in \Gamma(B \cup [\mathbb{N}]^{<\infty})$.

For each n, take a_n and distinct $U^n_1, U^n_2, \dots$ for $\mathcal{U}_n$ as in Lemma 2.14. We may assume that $a_n(1) = 1$. Let α be such that $I = \{n : a_n(n+1) < b_\alpha(n)\}$ is infinite. As $|\{x_\beta : \beta < \alpha\}| < \mathfrak{b}$, $\{x_\beta : \beta < \alpha\}$ satisfies $\mathsf{S}_1(\Gamma, \Gamma)$ (Lemma 2.7). Thus, there are m_n, $n \in I$, such that $\{U^n_{m_n} : n \in I\} \in \Gamma(\{x_\beta : \beta < \alpha\})$. Take $\mathcal{F}_n = \emptyset$ for $n \notin I$, and $\mathcal{F}_n = \{U^n_1, \dots, U^n_n\} \cup \{U^n_{m_n}\}$ for $n \in I$.

As $\{\bigcup \mathcal{F}_n : n \in \mathbb{N}\} = \{\bigcup \mathcal{F}_n : n \in I\} \cup \{\emptyset\}$, it suffices to show that for each $x \in X$, $x \in \bigcup \mathcal{F}_n$ for all but finitely many $n \in I$. If $x \in [\mathbb{N}]^{<\infty}$, then for each large enough $n \in I$, $x \cap [a_n(n), a_n(n+1)) = \emptyset$ (because $a_n(n) \geq n$), and thus $x \in U^n_n \in \mathcal{F}_n$. For $\beta < \alpha$, $b_\beta \in U^n_{m_n} \subseteq \bigcup \mathcal{F}_n$ for all large enough n.

For $\beta \geq \alpha$ (that's the interesting case!) and all but finitely many $n \in I$, $b_\beta(n) \geq b_\alpha(n) > a_n(n+1)$. Thus, $|b_\beta \cap [1, a_n(n+1))| < n$. As $[1, a_n(n+1)) = \bigcup_{i=1}^{n} [a_n(i), a_n(i+1))$ is a union of n intervals, there must be $i \leq n$ such $b_\beta \cap [a_n(i), a_n(i+1)) = \emptyset$, and thus $b_\beta \in U^n_i \subseteq \bigcup \mathcal{F}_n$. □

A multidimensional version of the last proof gives the following.

Theorem 2.15 (Bartoszyński-Tsaban [3]). *For each* $\mathfrak{b}$*-scale* B, *all finite powers of the set* $B \cup [\mathbb{N}]^{<\infty}$ *satisfy* $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$. □

Indeed, Zdomskyy and the present author proved in [23] that any finite product $(B_1 \cup [\mathbb{N}]^{<\infty}) \times \dots \times (B_1 \cup [\mathbb{N}]^{<\infty})$, with $B_1, \dots, B_k$ $\mathfrak{b}$-scales, satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$.

In a work in progress, the method introduced here is used to prove the following, substantially stronger, result.

Theorem 2.16 (Miller-Tsaban-Zdomskyy). *For each* $\mathfrak{b}$*-scale* B *and each set of reals* H *satisfying* $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$, $(B \cup [\mathbb{N}]^{<\infty}) \times H$ *satisfies* $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$.

3. Strongly Hurewicz sets of reals, in ZFC

Consider, for each $f \in \mathbb{N}^{\mathbb{N}}$, the following selection hypothesis.

$\mathsf{U}_f(\mathscr{A}, \mathscr{B})$**:** For all $\mathcal{U}_1, \mathcal{U}_2, \dots \in \mathscr{A}$, none containing a finite subcover, there are finite $\mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \dots$ such that such that $|\mathcal{F}_n| \leq f(n)$ for all n, and $\{\bigcup \mathcal{F}_n : n \in \mathbb{N}\} \in \mathscr{B}$.

Remark 3.1. One may require in the definition of $\mathsf{U}_f(\mathscr{A}, \mathscr{B})$ that each $\mathcal{F}_n$ is nonempty. This will not change the property when $\mathscr{A}, \mathscr{B} \in \{\mathrm{O}, \Gamma\}$, since we may assume that the given covers get finer and finer. This can be generalized to most types of covers considered in the field.

[7] Less than that is required of the given covers. See the proof.

$\mathsf{U}_f(\mathscr{A},\mathscr{B})$ depends only on $\limsup_n f(n)$.

Lemma 3.2. *Assume that for each $\mathcal{V}\in\mathscr{B}$, $\{\emptyset\}\cup\mathcal{V}\in\mathscr{B}$. For all $f,g\in\mathbb{N}^\mathbb{N}$ with $\limsup_n f(n)=\limsup_n g(n)$, $\mathsf{U}_f(\mathscr{A},\mathscr{B})=\mathsf{U}_g(\mathscr{A},\mathscr{B})$.*

Proof. The argument is as in the proofs of [9, 3.2–3.5] and [24, Lemma 3], concerning similar concepts in other contexts.

Let $\mathcal{U}_1,\mathcal{U}_2,\dots\in\mathscr{A}(X)$. Let $m_1<m_2<\dots$ be such that $f(n)\le g(m_n)$ for all n. Apply $\mathsf{U}_f(\mathscr{A},\mathscr{B})$ to the sequence $\mathcal{U}_{m_1},\mathcal{U}_{m_2},\dots$, to obtain $\mathcal{F}_{m_1}\subseteq\mathcal{U}_{m_1},\mathcal{F}_{m_2}\subseteq\mathcal{U}_{m_2},\dots$, such that $|\mathcal{F}_{m_n}|\le f(n)$ for all n, and $\{\bigcup\mathcal{F}_{m_n}:n\in\mathbb{N}\}\in\mathscr{B}(X)$. For $k\notin\{m_n:n\in\mathbb{N}\}$ we can take $\mathcal{F}_k=\emptyset$. Then $\{\bigcup\mathcal{F}_n:n\in\mathbb{N}\}=\{\emptyset\}\cup\{\bigcup\mathcal{F}_{m_n}:n\in\mathbb{N}\}\in\mathscr{B}(X)$, and $|\mathcal{F}_n|\le g(n)$ for all n. □

Thus, for each $f\in\mathbb{N}^\mathbb{N}$ with $\limsup_n f(n)=\infty$, $\mathsf{U}_f(\mathscr{A},\mathscr{B})=\mathsf{U}_{\mathrm{id}}(\mathscr{A},\mathscr{B})$, where id is the identity function, $\mathrm{id}(n)=n$ for all n. We henceforth use the notation

$$\mathsf{U}_n(\mathscr{A},\mathscr{B})$$

for $\mathsf{U}_{\mathrm{id}}(\mathscr{A},\mathscr{B})$.

Our proof of Theorem 2.12 shows the following.

Theorem 3.3. *For each $\mathfrak{b}$-scale B, $B\cup[\mathbb{N}]^{<\infty}$ satisfies $\mathsf{U}_n(\Gamma,\Gamma)$.*

Proof. In the proof of Theorem 2.12 we show that $B\cup[\mathbb{N}]^{<\infty}$ satisfies $\mathsf{U}_{n+1}(\Gamma,\Gamma)$. By Lemma 3.2, this is the same as $\mathsf{U}_n(\Gamma,\Gamma)$. □

We will soon show that $\mathsf{U}_n(\Gamma,\Gamma)$ is strictly stronger than $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$.

A cover $\mathcal{U}$ of X is *multifinite* [22] if there exists a partition of $\mathcal{U}$ into infinitely many finite covers of X. Let $\mathscr{A}$ be a family of covers of X. $\daleth(\mathscr{A})$ is the family of all covers $\mathcal{U}$ of X such that: Either $\mathcal{U}$ is multifinite, or there exists a partition $\mathcal{P}$ of $\mathcal{U}$ into finite sets such that $\{\bigcup\mathcal{F}:\mathcal{F}\in\mathcal{P}\}\setminus\{X\}\in\mathscr{A}$ [19].

The special case $\daleth(\Gamma)$ was first studied by Kočinac and Scheepers [13], where it was proved that $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)=\mathsf{S}_{\mathrm{fin}}(\Omega,\daleth(\Gamma))$. Additional results of this type are available in Babinkostova-Kočinac-Scheepers [1], and in general form in Samet-Scheepers-Tsaban [19].

Theorem 3.4 (Samet, et al. [19])**.** $\mathsf{U}_{\mathrm{fin}}(\Gamma,\daleth(\Gamma))=\mathsf{S}_{\mathrm{fin}}(\Gamma,\daleth(\Gamma))$.

Theorem 3.5. $\mathsf{U}_n(\Gamma,\Gamma)$ *implies* $\mathsf{S}_1(\Gamma,\daleth(\Gamma))$.

Proof. We prove the following, stronger statement: Assume that X satisfies $\mathsf{U}_n(\Gamma,\Gamma)$, and let $s(n)=1+\dots+n=(n+1)n/2$. For all $\mathcal{U}_1,\mathcal{U}_2,\dots\in\Gamma(X)$, there are $U_1\in\mathcal{U}_1,U_2\in\mathcal{U}_2,\dots$, such that for each $x\in X$, $x\in\bigcup_{k=s(n)}^{s(n+1)}U_k$ for all but finitely many n.

Let $\mathcal{U}_1,\mathcal{U}_2,\dots\in\Gamma(X)$. We may assume that for each n, $\mathcal{U}_{n+1}$ refines $\mathcal{U}_n$. Apply $\mathsf{U}_n(\Gamma,\Gamma)$ to $\mathcal{U}_{s(1)},\mathcal{U}_{s(2)},\dots$ to obtain $U_1\in\mathcal{U}_{s(1)},U_2,U_3\in\mathcal{U}_{s(2)},\dots$, such that for each $x\in X$, $x\in\bigcup_{k=s(n)+1}^{s(n+1)}U_k$ for all but finitely many n. For each n and each $k=s(n)+1,\dots,s(n+1)$, replace U_k by an equal or larger set from $\mathcal{U}_k$. □

Remark 3.6. The statement at the beginning of the last proof is in fact a characterization of $\mathsf{U}_n(\Gamma,\Gamma)$.

Remark 3.7. In general, if every pair of elements of $\mathscr{A}$ has a joint refinement in $\mathscr{A}$, and $\mathscr{B}$ is finitely thick in the sense of [22], then $\mathsf{U}_n(\mathscr{A},\mathscr{B})$ implies $\mathsf{S}_1(\mathscr{A},\daleth(\mathscr{B}))$.

In particular, when $\mathscr{B}=\mathrm{O}$, $\daleth(\mathscr{B})=\mathrm{O}$, and thus $\mathsf{U}_n(\mathscr{A},\mathrm{O})=\mathsf{S}_1(\mathscr{A},\mathrm{O})$. For example, $\mathsf{U}_n(\Gamma,\mathrm{O})=\mathsf{S}_1(\Gamma,\mathrm{O})$.

Thus, the Bartoszyński-Shelah Theorem tells that for each $\mathfrak{b}$-scale B, $B\cup[\mathbb{N}]^{<\infty}$ satisfies $\mathsf{S}_{\mathrm{fin}}(\Gamma,\daleth(\Gamma))$, whereas Theorem 3.3 tells that it indeed satisfies $\mathsf{S}_1(\Gamma,\daleth(\Gamma))$. As $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$ does not even imply $\mathsf{S}_1(\Gamma,\mathrm{O})$ (Lemma 1.16), we have that $\mathsf{U}_n(\Gamma,\Gamma)$ is strictly stronger than $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$.

Theorem 3.8 (Tsaban-Zdomskyy [17])**.** *Assume the Continuum Hypothesis (or just $\mathfrak{b}=\mathfrak{c}$). There is a $\mathfrak{b}$-scale B such that no set of reals containing $B\cup[\mathbb{N}]^{<\infty}$ satisfies $\mathsf{S}_1(\Gamma,\Gamma)$.*

By Theorems 3.3 and 3.8, $\mathsf{U}_n(\Gamma,\Gamma)\neq\mathsf{S}_1(\Gamma,\Gamma)$. Thus, $\mathsf{U}_n(\Gamma,\Gamma)$ is strictly in between $\mathsf{S}_1(\Gamma,\Gamma)$ and $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$.

A natural refinement of the Problem 9, solved in Theorem 3.8, is the following.

Problem 3.9 (Zdomskyy)**.** *Is there a set of reals X without perfect subsets, such that X satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$ but not $\mathsf{U}_n(\Gamma,\Gamma)$?*

4. A visit at the border of ZFC

By Lemma 3.2, there are only the following kinds of (strongly) Hurewicz properties: $\mathsf{U}_{\mathrm{fin}}(\Gamma,\Gamma)$, $\mathsf{U}_n(\Gamma,\Gamma)$, and $\mathsf{U}_c(\Gamma,\Gamma)$, for constants $c\in\mathbb{N}$. For $c=1$, $\mathsf{U}_c(\Gamma,\Gamma)=\mathsf{S}_1(\Gamma,\Gamma)$, and thus by the results of the previous section, at least three of these properties are distinct. (We consider properties *distinct* if they are not provably equivalent.)

By Theorem 2.11, $\mathsf{U}_1(\Gamma,\Gamma)$ may be trivial. The next strongest property is $\mathsf{U}_2(\Gamma,\Gamma)$. We prove that it is not trivial.

Definition 4.1. Let $s,a\in[\mathbb{N}]^{\infty}$. *$s$ slaloms*[8] a if $a\cap[s(n),s(n+1))\neq\emptyset$ for all but finitely many n. *s slaloms* a set $Y\subseteq[\mathbb{N}]^{\infty}$ if it slaloms each $a\in Y$.

By Lemma 2.13, a set $Y\subseteq[\mathbb{N}]^{\infty}$ is bounded if, and only if, there is s which slaloms Y.

Definition 4.2. A *slalom $\mathfrak{b}$-scale* is an unbounded set $\{b_\alpha:\alpha<\mathfrak{b}\}\subseteq[\mathbb{N}]^{\infty}$, such that b_β slaloms b_α for all $\alpha<\beta<\mathfrak{b}$.

By Lemma 2.13, we have the following.

Lemma 4.3. *There are slalom $\mathfrak{b}$-scales.* □

We are now ready to prove the main result of this paper.

Theorem 4.4. *For each slalom $\mathfrak{b}$-scale B, $B\cup[\mathbb{N}]^{<\infty}$ satisfies $\mathsf{U}_2(\Gamma,\Gamma)$.*

Proof. Let $B=\{b_\alpha:\alpha<\mathfrak{b}\}$ be a slalom $\mathfrak{b}$-scale. Let $\mathcal{U}_1,\mathcal{U}_2,\dots\in\Gamma(B\cup[\mathbb{N}]^{<\infty})$.

For each n, take $a_n\in[\mathbb{N}]^{\infty}$ and distinct $U_1^n,U_2^n,\dots$ for $\mathcal{U}_n$ as in Lemma 2.14. We may assume that $a_n(1)=1$. Let $a\in[\mathbb{N}]^{\infty}$ slalom $\{a_n:n\in\mathbb{N}\}$. As B is unbounded, there is by Lemma 2.13 $\alpha<\mathfrak{b}$, such that $I=\{m:[a(m),a(m+3))\cap b_\alpha=\emptyset\}$ is infinite. (Otherwise, $\{a(3n):n\in\mathbb{N}\}$ would slalom B.) For each n, let

$$I_n=\{m\geq n:[a_n(m),a_n(m+2))\cap b_\alpha=\emptyset\}.$$

As a slaloms a_n, I_n is infinite, and therefore $\{U_m^n:m\in I_n\}\in\Gamma(B\cup[\mathbb{N}]^{<\infty})$.

As $|\{x_\beta:\beta<\alpha\}|<\mathfrak{b}$, $\{x_\beta:\beta<\alpha\}$ satisfies $\mathsf{S}_1(\Gamma,\Gamma)$ (Lemma 2.7), and thus, there are $m_n\in I_n$, $n\in\mathbb{N}$, such that $\{U_{m_n}^n:n\in\mathbb{N}\}\in\Gamma(\{x_\beta:\beta<\alpha\})$. We claim that

$$\{U_{m_n}^n\cup U_{m_n+1}^n:n\in\mathbb{N}\}\in\Gamma(B\cup[\mathbb{N}]^{<\infty}).$$

[8]Short for "is a slalom for".

If $x \in [\mathbb{N}]^{<\infty}$, then for each large enough n, $x \cap [a_n(m_n), a_n(m_n+1)) = \emptyset$ (because $m_n \geq n$), and thus $x \in U^n_{m_n}$. For $\beta < \alpha$, $b_\beta \in U^n_{m_n}$ for all large enough n, by the choice of m_n.

For $\beta \geq \alpha$ (that's the interesting case), we have the following: Let $m_n \in I_n$, and let k be such that

$$b_\alpha(k) < a_n(m_n) < a_n(m_n+2) \leq b_\alpha(k+1).$$

If n is large, then k is large, and as b_β slaloms b_α, there is i such that

$$b_\beta(i) \leq b_\alpha(k) < a_n(m_n) < a_n(m_n+2) \leq b_\alpha(k+1) < b_\beta(i+2).$$

There are two possibilities for $a_n(m_n+1)$: If $a_n(m_n+1) \leq b_\beta(i+1)$, then $[a_n(m_n), a_n(m_n+1)) \cap b_\beta = \emptyset$, and thus $b_\beta \in U^n_{m_n}$. Otherwise, $a_n(m_n+1) > b_\beta(i+1)$, and thus $[a_n(m_n+1), a_n(m_n+2)) \cap b_\beta = \emptyset$. Therefore, $b_\beta \in U^n_{m_n+1}$ in this case. □

Theorem 4.5. *Assume the Continuum Hypothesis (or just $\mathfrak{b} = \mathfrak{c}$). There is a slalom $\mathfrak{b}$-scale B such that $B \cup [\mathbb{N}]^{<\infty}$ satisfies $\mathsf{U}_2(\Gamma, \Gamma)$, but no set of reals containing $B \cup [\mathbb{N}]^{<\infty}$ satisfies $\mathsf{S}_1(\Gamma, \Gamma)$.*

Proof. Consider the proof of Theorem 3.8, given in [17]. We need only make sure that in Proposition 2.5 of [17], B can be constructed in a way that it is a *slalom* $\mathfrak{b}$-scale. This should be taken care of in the second paragraph of page 2518.

At step $\alpha < \mathfrak{b}$ of this construction, we are given a set Y with $|Y| = |\alpha| < \mathfrak{b}$, and a set $a_\alpha \in [\mathbb{N}]^\infty$. Take an infinite $b_\alpha \subseteq a_\alpha$ such that b_α slaloms Y. (E.g., take a slalom b for Y, and then define $b_\alpha \subseteq a_\alpha$ by induction on n, such that for each n, $|b \cap [b_\alpha(n), b_\alpha(n+1))| \geq 2$.) By induction on n, thin out b_α such that it satisfies the displayed inequality there for all n. b_α remains a slalom for Y.

Theorem 4.4 guarantees that $B \cup [\mathbb{N}]^{<\infty}$ satisfies $\mathsf{U}_2(\Gamma, \Gamma)$. □

By Theorem 2.11, it is consistent that $\mathsf{S}_1(\Gamma, \Gamma)$ is trivial, whereas by Theorem 4.4, $\mathsf{U}_2(\Gamma, \Gamma)$ is never trivial. The following remains open.

Conjecture 4.6. *$\mathsf{U}_2(\Gamma, \Gamma)$ is strictly stronger than $\mathsf{U}_n(\Gamma, \Gamma)$.*

5. The Hurewicz Problem

In the same 1927 paper Hurewicz asked the following.

Problem 5.1 (Hurewicz [11]). *Is there a metric space satisfying $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$, but not $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$?*

In a footnote added at the proof stage (the same one mentioned before Theorem 1.2), Hurewicz quotes the following, which solves his problem if the Continuum Hypothesis is assumed.

Theorem 5.2 (Sierpiński). *Every Luzin set satisfies $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$, but not $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$.*

Proof. Let L be a Luzin set. We have already proved that L satisfies $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$ (Theorem 1.2). It remains to show that L does not satisfy $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$.

As L contains no perfect sets, $\mathbb{R} \setminus L$ is dense in $\mathbb{R}$. Fix a countable dense $D \subseteq \mathbb{R} \setminus L$. $\mathbb{R} \setminus D$ is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$,[9] which in turn is homeomorphic to $[\mathbb{N}]^\infty$ (e.g., using continued fractions).

[9] D is order-isomorphic to $\mathbb{Q}$. An order isomorphism $f : D \to \mathbb{Q}$ extends uniquely to and order isomorphism $f : \mathbb{R} \to \mathbb{R}$ by setting $f(r) = \sup\{f(d) : d < r\}$. The restriction of f to $\mathbb{R} \setminus D$ is a homeomorphism.

As $L \subseteq \mathbb{R} \setminus D$, we may assume that $L \subseteq [\mathbb{N}]^\infty$.[10] By Lemma 2.6, it suffices to show that L is unbounded. For each $b \in [\mathbb{N}]^\infty$, the set

$$\{a \in [\mathbb{N}]^\infty : a \leq^* b\} = \bigcup_{n\in\mathbb{N}} \{a \in [\mathbb{N}]^\infty : (\forall m \geq n)\ a(m) \leq b(m)\},$$

with each $\{a \in [\mathbb{N}]^\infty : (\forall m \geq n)\ a(m) \leq b(m)\}$ nowhere dense. Thus, $\{a \in [\mathbb{N}]^\infty : a \leq^* b\}$ is meager, and therefore does not contain L. $\square$

Hurewicz's problem remained, however, open until the end of 2002.

Theorem 5.3 (Chaber-Pol [6]). *There is a set of reals satisfying* $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$ *but not* $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$.

Chaber and Pol's proof is topological and uses a technique due to Michael. The following combinatorial proof contains the essence of their proof.

Proof of Theorem 5.3. The proof is dichotomic. If $\mathfrak{b} < \mathfrak{d}$, then any unbounded $B \subseteq [\mathbb{N}]^\infty$ of cardinality $\mathfrak{b}$ satisfies $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$ (Lemma 1.5) but not $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$ (Lemma 2.6).

Lemma 5.4. *For each $s \in [\mathbb{N}]^\infty$, there is $a \in [\mathbb{N}]^\infty$ such that: $a^{\mathsf{c}} = \mathbb{N} \setminus a \in [\mathbb{N}]^\infty$, $a \not\leq^* s$, and $a^{\mathsf{c}} \not\leq^* s$.*

Proof. Let $m_1 > s(1)$. For each $n > 1$, let $m_n > s(m_{n-1})$. Let $a = \bigcup_n [m_{2n-1}, m_{2n})$. For each n:

$$\begin{aligned} a(m_{2n}) &\geq m_{2n+1} > s(m_{2n}); \\ a^{\mathsf{c}}(m_{2n-1}) &\geq m_{2n} > s(m_{2n-1}). \quad \square \end{aligned}$$

So, assume that $\mathfrak{b} = \mathfrak{d}$. Fix a scale $\{s_\alpha : \alpha < \mathfrak{d}\} \subseteq [\mathbb{N}]^\infty$. For each $\alpha < \mathfrak{d}$, use Lemma 5.4 to pick $a_\alpha \in [\mathbb{N}]^\infty$ such that:

(1) $a_\alpha^{\mathsf{c}} = \mathbb{N} \setminus a_\alpha$ is infinite;
(2) $a_\alpha \not\leq^* s_\alpha$; and
(3) $a_\alpha^{\mathsf{c}} \not\leq^* s_\alpha$.

Let $A = \{a_\alpha : \alpha < \mathfrak{d}\}$. For $b \in [\mathbb{N}]^\infty$, let $\alpha < \mathfrak{d}$ be such that $b <^* s_\alpha$. Then $\{\beta : a_\beta \leq^* b\} \subseteq \alpha$. As in the proof of Theorem 1.7, this implies that A is $\mathfrak{d}$-concentrated on $[\mathbb{N}]^{<\infty}$, and thus $A \cup [\mathbb{N}]^{<\infty}$ satisfies $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$ (indeed, $\mathsf{S}_1(\Gamma,\mathrm{O})$ – Corollary 1.14).

On the other hand, $A \cup [\mathbb{N}]^{<\infty}$ is homeomorphic to $Y = \{x^{\mathsf{c}} : x \in A \cup [\mathbb{N}]^{<\infty}\}$, which is an unbounded subset of $[\mathbb{N}]^\infty$ (by item (3) of the construction). By Lemma 2.6, Y (and therefore $A \cup [\mathbb{N}]^{<\infty}$) does not satisfy $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$. $\square$

The advantage of the last proof is its simplicity. However, it does not provide an explicit example, and in the case $\mathfrak{b} < \mathfrak{d}$ gives a trivial example, i.e., one of cardinality smaller than $\mathrm{non}(\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O}))$. We conclude with an explicit solution.

Theorem 5.5 (Tsaban-Zdomskyy [23]). *There is a set of reals of cardinality $\mathfrak{d}$, satisfying* $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$ *(indeed,* $\mathsf{S}_1(\Gamma,\mathrm{O})$*), but not* $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$.

---

[10] If L is a Luzin set in a topological space X and $f : X \to Y$ is a homeomorphism, then $f[L]$ is a Luzin set in Y, since "being meager" is preserved by homeomorphisms.

Our original proof uses in its crucial step a topological argument. Here, we give a more combinatorial argument, based on a (slightly amended) lemma of Mildenberger.

A set $Y \subseteq [\mathbb{N}]^\infty$ is *groupwise dense* if:

(1) $a \subseteq^* y \in Y$ implies $a \in Y$; and
(2) For each $a \in [\mathbb{N}]^\infty$, there is an infinite $I \subseteq \mathbb{N}$ such that $\bigcup_{n\in I}[a(n), a(n+1)) \in Y$.

For Y satisfying (1), Y is groupwise dense if, and only if, Y is nonmeager [4].

Proof of Theorem 5.5. Fix a dominating set $\{d_\alpha : \alpha < \mathfrak{d}\}$. Define $a_\alpha \in [\mathbb{N}]^\infty$ by induction on $\alpha < \mathfrak{d}$. Step α: Let $Y = \{d_\beta, a_\beta : \beta < \alpha\}$. $|Y| < \mathfrak{d}$.

The following is proved by Mildenberger as part of the proof of [15, Theorem 2.2], except that we eliminate the "next" function from her argument.

Lemma 5.6 (Mildenberger [15])**.** *For each $Y \subseteq [\mathbb{N}]^\infty$ with $|Y| < \mathfrak{d}$, $G = \{a \in [\mathbb{N}]^\infty : (\forall y \in Y)\ a \not\leq^* y\}$ is groupwise dense.*

Proof. Clearly, G satisfies (1) of the definition of groupwise density. We verify (2).

We may assume that Y is closed under maxima of finite subsets. Let $g \in [\mathbb{N}]^\infty$ be a witness that Y is not dominating. Then the family of all sets $\{n : y(n) < g(n)\}$, $y \in Y$, can be extended to a nonprincipal ultrafilter $\mathcal{U}$.

Let $a \in [\mathbb{N}]^\infty$. By thinning out a, we may assume that $g(a(n)) < a(n+1)$ for all n. For $i = 0, 1, 2$, let

$$a_i = \bigcup_{n\in\mathbb{N}} [a(3n+i), a(3n+i+1)).$$

Then there is i such that $a_i \in \mathcal{U}$. We claim that $a_{i+2 \bmod 3} \in G$. Let $y \in Y$. For each k in the infinite set $\{n : y(n) < g(n)\} \cap a_i$, let n be such that $k \in [a(3n+i), a(3n+i+1))$. Then

$$y(k) < g(k) < g(a(3n+i+1)) < a(3n+i+2) \leq a_{i+2 \bmod 3}(k),$$

because $a(3n+i+2)$ is the first element of $a_{i+2 \bmod 3}$ greater or equal to k, and $a_{i+2 \bmod 3}(k) \geq k$. □

Let $G = \{a \in [\mathbb{N}]^\infty : (\forall y \in Y)\ a \not\leq^* y\}$. As G is groupwise dense, there is $a_\alpha \in G$ such that a_α^{c} is infinite and $a_\alpha^{\mathsf{c}} \not\leq^* d_\alpha$. To see this, take an interval partition as in the proof of Lemma 5.4. Then there is an infinite subfamily of the even intervals, whose union a_α is in G. For each n such that $[m_{2n-1}, m_{2n}) \subseteq a_\alpha$, $a^{\mathsf{c}}(m_{2n-1}) \geq m_{2n} > s(m_{2n-1})$.[11]

Thus, there is

$$a_\alpha \in \{a \in [\mathbb{N}]^\infty : (\forall y \in Y)\ a \not\leq^* y\} \setminus \{a \in [\mathbb{N}]^\infty : a^{\mathsf{c}} \leq^* d_\alpha\}.$$

Continue exactly as in the above proof of Theorem 5.3. □

Chaber and Pol's Theorem in [6] is actually stronger than Theorem 5.3 above, and establishes the existence of a set of reals X such that X does not satisfy $\mathsf{U}_{\text{fin}}(\mathrm{O}, \Gamma)$,[12] but all finite powers of X satisfy $\mathsf{S}_{\text{fin}}(\mathrm{O}, \mathrm{O})$.

[11]Alternatively, note that $\{a : a^{\mathsf{c}} \leq^* d_\alpha\}$ is homeomorphic to the meager set $\{a : a \leq^* d_\alpha\}$, and thus cannot contain a groupwise dense (i.e., nonmeager) set.

[12]And thus neither any finite power of X, since X is a continuous image of X^k for each k.

Their proof shows that if $\mathfrak{b} = \mathfrak{d}$, then there is such an example of cardinality $\mathfrak{d}$. The assumption "$\mathfrak{b} = \mathfrak{d}$" was weakened to "$\mathfrak{d}$ is regular" by Tsaban and Zdomskyy [23], but the following remains open.

Problem 5.7. *Is there, provably in ZFC, a nontrivial (i.e., one of cardinality at least $\mathfrak{d}$) example of a set of reals such that X does not satisfy $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$, but all finite powers of X satisfy $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$?*

In other words, the question whether there is a nondichotomic proof of Chaber and Pol's full theorem remains open.

Acknowledgments. We thank Gabor Lukacs, Lyubomyr Zdomskyy and the referee for their useful comments, which lead to improvements in the presentation of this paper.

References

[1] L. Babinkostova, L. Kočinac, and M. Scheepers, *Combinatorics of open covers (VIII)*, Topology and its Applications **140** (2004), 15–32.

[2] T. Bartoszyński and S. Shelah, *Continuous images of sets of reals*, Topology and its Applications **116** (2001), 243–253.

[3] T. Bartoszyński and B. Tsaban, *Hereditary topological diagonalizations and the Menger-Hurewicz Conjectures*, Proceedings of the American Mathematical Society **134** (2006), 605–615.

[4] A. Blass, *Combinatorial cardinal characteristics of the continuum*, in: **Handbook of Set Theory** (M. Foreman, A. Kanamori, and M. Magidor, eds.), Kluwer Academic Publishers, Dordrecht, to appear. `http://www.math.lsa.umich.edu/~ablass/hbk.pdf`

[5] L. Bukovský and K. Ciesielski, *Spaces on which every pointwise convergent series of continuous functions converges pseudo-normally*, Proceedings of the American Mathematical Society **133**, 605–611.

[6] J. Chaber and R. Pol, *A remark on Fremlin-Miller theorem concerning the Menger property and Michael concentrated sets*, unpublished note (October 2002).

[7] D. Fremlin and A. Miller, *On some properties of Hurewicz, Menger and Rothberger*, Fundamenta Mathematica **129** (1988), 17–33.

[8] F. Galvin and A. Miller, *γ-sets and other singular sets of real numbers*, Topology and its Applications **17** (1984), 145–155.

[9] S. Garcia-Ferreira and A. Tamariz-Mascarua, *Some generalizations of rapid ultrafilters and Id-fan tightness*, Tsukuba Journal of Mathematics **19** (1995), 173–185.

[10] W. Hurewicz, *Über eine Verallgemeinerung des Borelschen Theorems*, Mathematische Zeitschrift **24** (1925), 401–421.

[11] W. Hurewicz, *Über Folgen stetiger Funktionen*, Fundamenta Mathematicae **9** (1927), 193–204.

[12] W. Just, A. Miller, M. Scheepers, and P. Szeptycki, *The combinatorics of open covers II*, Topology and its Applications **73** (1996), 241–266.

[13] L. Kočinac and M. Scheepers, *Combinatorics of open covers (VII): Groupability*, Fundamenta Mathematicae **179** (2003), 131–155.

[14] K. Menger, *Einige Überdeckungssätze der Punktmengenlehre*, Sitzungsberichte der Wiener Akademie **133** (1924), 421–444.

[15] H. Mildenberger, *Groupwise dense families*, Archive for Mathematical Logic **40** (2001), 93–112.

[16] A. Miller and B. Tsaban, *Point-cofinite covers in Laver's model*, Proceedings of the American Mathematical Society **138** (2010), 3313–3321.

[17] D. Repovš, B. Tsaban, and L. Zdomskyy, *Hurewicz sets of reals without perfect subsets*, Proceedings of the American Mathematical Society **136** (2008), 2515–2520.

[18] M. Sakai, *The sequence selection properties of $C_p(X)$*, Topology and its Applications **154** (2007), 552–560.

[19] N. Samet, M. Scheepers, and B. Tsaban, *Partition relations for Hurewicz-type selection hypotheses*, Topology and its Applications **156** (2009), 616–623.

[20] M. Scheepers, *Combinatorics of open covers I: Ramsey theory*, Topology and its Applications **69** (1996), 31–62.
[21] M. Scheepers, *$C_p(X)$ and Arhangel'skiĭ's α_i spaces*, Topology and its Applications **89** (1998), 265–275.
[22] B. Tsaban, *Strong γ-sets and other singular spaces*, Topology and its Applications **153** (2005), 620–639.
[23] B. Tsaban and L. Zdomskyy, *Scales, fields, and a problem of Hurewicz*, Journal of the European Mathematical Society **10** (2008), 837–866.
[24] J. Valueva, *On some fan-tightness type properties*, Commentationes Mathematicae Universitatis Carolinae **39** (1998), 415–421.
[25] J. Valueva, *A remark on combinatorics of open covers and C_p-spaces*, Questions and Ansers in General Topology **16** (1998), 183–187.

Appendix A. $\mathsf{S}_f(\mathscr{A},\mathscr{B})$

Properties closely related to our $\mathsf{U}_f(\mathscr{A},\mathscr{B})$ were considered in the literature. Consider, for each $f \in \mathbb{N}^\mathbb{N}$, the following selection hypothesis.

$\mathsf{S}_f(\mathscr{A},\mathscr{B})$: For all $\mathcal{U}_1,\mathcal{U}_2,\dots \in \mathscr{A}$, there are finite $\mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2,\dots$ such that such that $|\mathcal{F}_n| \le f(n)$ for all n, and $\bigcup_n \mathcal{F}_n \in \mathscr{B}$.

In [9, 5] it is proved that for each $f \in \mathbb{N}^\mathbb{N}$, $\mathsf{S}_f(\mathrm{O},\mathrm{O}) = \mathsf{S}_1(\mathrm{O},\mathrm{O})$. Indeed, by Remark 3.7 we have that for all $\mathscr{A}$,

$$\mathsf{S}_f(\mathscr{A},\mathrm{O}) = \mathsf{U}_n(\mathscr{A},\mathrm{O}) = \mathsf{S}_1(\mathscr{A},\mathrm{O}).$$

A family $\mathscr{B}$ of open covers of X is *finitely thick* [22] if:

(1) If $\mathcal{U} \in \mathscr{B}$ and for each $U \in \mathcal{U}$:
$\mathcal{F}_U$ is a finite nonempty family of open sets such that for each $V \in \mathcal{F}_U$, $U \subseteq V \neq X$,
then $\bigcup_{U\in\mathcal{U}} \mathcal{F}_U \in \mathscr{B}$.
(2) If $\mathcal{U} \in \mathscr{B}$ and $\mathcal{V} = \mathcal{U} \cup \mathcal{F}$ where $\mathcal{F}$ is finite and $X \notin \mathcal{F}$, then $\mathcal{V} \in \mathscr{B}$.[13]

Many families of "rich" covers considered in the literature, including O,Ω,Γ [20, 12], are finitely thick. Also, for each of these families, each pair of elements has a joint refinement in the same family.

The case $\mathscr{A} = \mathscr{B} = \Omega$ of the following theorem was proved in [9, 25].

Theorem A.1. *Assume that each pair of elements of $\mathscr{A}$ has a joint refinement in $\mathscr{A}$, and $\mathscr{B}$ is finitely thick. For each $f \in \mathbb{N}^\mathbb{N}$, $\mathsf{S}_f(\mathscr{A},\mathscr{B}) = \mathsf{S}_1(\mathscr{A},\mathscr{B})$.*

Proof. As $1 \le f(n)$ for all n, $\mathsf{S}_1(\mathscr{A},\mathscr{B})$ implies $\mathsf{S}_f(\mathscr{A},\mathscr{B})$. To prove the remaining implication, assume that X satisfies $\mathsf{S}_f(\mathscr{A},\mathscr{B})$.

Let $\mathcal{U}_1,\mathcal{U}_2,\dots \in \mathscr{A}(X)$. Let $s(n) = f(1)+f(2)+\dots+f(n)$ for all n. For each n, take $\mathcal{V}_n \in \mathscr{A}(X)$ refining $\mathcal{U}_1,\dots,\mathcal{U}_{s(n)}$.

Apply $\mathsf{S}_f(\mathscr{A},\mathscr{B})$ to the sequence $\mathcal{V}_1,\mathcal{V}_2,\dots$, to obtain $\mathcal{F}_1 \subseteq \mathcal{V}_1, \mathcal{F}_2 \subseteq \mathcal{V}_2,\dots$, such that $|\mathcal{F}_n| \le f(n)$ for all n, and $\bigcup_n \mathcal{F}_n \in \mathscr{B}(X)$.

Fix n. For each $k \in \{s(n-1)+1,\dots,s(n)\}$, pick $U_k \in \mathcal{U}_k$ such that each member of $\mathcal{F}_n$ is contained in some U_k. As $\mathscr{B}$ is finitely thick, $\{U_k : k \in \mathbb{N}\} \in \mathscr{B}(X)$. □

Thus, in our context, the scheme $\mathsf{S}_f(\mathscr{A},\mathscr{B})$ does not introduce new properties. As we have seen in the present paper, this is not the case for $\mathsf{U}_f(\mathscr{A},\mathscr{B})$.

Department of Mathematics, Bar-Ilan University, Ramat-Gan 52900, Israel
E-mail address: tsaban@math.biu.ac.il
URL: http://www.cs.biu.ac.il/~tsaban

[13] We will not use Item (2) of the definition of *finitely thick* here.