HUREWICZ SETS OF REALS WITHOUT PERFECT SUBSETS

DUŠAN REPOVŠ, BOAZ TSABAN, AND LYUBOMYR ZDOMSKYY

(Communicated by Julia Knight)

Abstract. We show that even for subsets $X$ of the real line that do not contain perfect sets, the Hurewicz property does not imply the property $S_1(\Gamma, \Gamma)$, asserting that for each countable family of open $\gamma$-covers of $X$, there is a choice function whose image is a $\gamma$-cover of $X$. This settles a problem of Just, Miller, Scheepers, and Szeptycki. Our main result also answers a question of Bartoszyński and the second author, and implies that for $C_p(X)$, the conjunction of Sakai’s strong countable fan tightness and the Reznichenko property does not imply Arhangel’skii’s property $\alpha_2$.

1. Introduction

By a set of reals we mean a separable, zero-dimensional, and metrizable space (such spaces are homeomorphic to subsets of the real line $\mathbb{R}$). Fix a set of reals $X$. Let $\mathcal{O}$ denote the collection of all open covers of $X$. An open cover $\mathcal{U}$ of $X$ is a $\gamma$-cover of $X$ if it is infinite and for each $x \in X$, $x$ is a member of all but finitely many members of $\mathcal{U}$. Let $\Gamma$ denote the collection of all open $\gamma$-covers of $X$. Motivated by Menger’s work, Hurewicz [6] introduced the Hurewicz property $U_{\text{fin}}(\mathcal{O}, \Gamma)$:

For each sequence $\{\mathcal{U}_n\}_{n \in \mathbb{N}}$ of members of $\mathcal{O}$ that do not contain a finite subcover, there exist finite sets $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\{\bigcup \mathcal{F}_n : n \in \mathbb{N}\} \in \Gamma$.

Every $\sigma$-compact space satisfies $U_{\text{fin}}(\mathcal{O}, \Gamma)$, but the converse fails [7] [2].

Let $\mathcal{A}$ and $\mathcal{B}$ be any two families. Motivated by works of Rothberger, Scheepers introduced the following prototype of properties [12]:

$S_1(\mathcal{A}, \mathcal{B})$: For each sequence $\{\mathcal{U}_n\}_{n \in \mathbb{N}}$ of members of $\mathcal{A}$, there exist members $U_n \in \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\{U_n : n \in \mathbb{N}\} \in \mathcal{B}$.

It is easy to see that $U_{\text{fin}}(\mathcal{O}, \Gamma) = U_{\text{fin}}(\Gamma, \Gamma)$, and therefore $S_1(\Gamma, \Gamma)$ implies $U_{\text{fin}}(\mathcal{O}, \Gamma)$ [12]. However, a set of reals satisfying $S_1(\Gamma, \Gamma)$ cannot contain perfect subsets [7]. It follows that, for example, $\mathbb{R}$ satisfies $U_{\text{fin}}(\mathcal{O}, \Gamma)$ but not $S_1(\Gamma, \Gamma)$. In the fundamental paper [7], we are asked whether there are nontrivial examples showing that $U_{\text{fin}}(\mathcal{O}, \Gamma)$ does not imply $S_1(\Gamma, \Gamma)$.
Problem 1.1 (Just, Miller, Scheepers, Szeptycki [7]). Let $X$ be a set of reals that does not contain a perfect set, but that does have the Hurewicz property. Does $X$ then satisfy $S_1(\Gamma, \Gamma)$?

We give a negative answer that also yields a new result concerning function spaces.

2. The main theorem

We prove a stronger assertion than what is needed to settle Problem 1.1; this will be useful for the next section. Let $C_\Gamma$ denote the collection of all clopen $\gamma$-covers of $X$. Clearly, $S_1(\Gamma, \Gamma)$ implies $S_1(C_\Gamma, C_\Gamma)$.

The hypothesis in the following theorem is a consequence of the Continuum Hypothesis. See [3] for a survey of the involved cardinals.

Theorem 2.1. Assume that $b = c$. There exists a set of reals $X$ such that:

1. $X$ does not contain a perfect set;
2. all finite powers of $X$ have the Hurewicz property $U_{\text{fin}}(O, \Gamma)$;
3. no set of reals containing $X$ satisfies $S_1(C_\Gamma, C_\Gamma)$.

Theorem 2.1 is proved in three steps. The first step is analogous to Theorem 4.2 of [5] and will be used to show that the constructed set is not contained in a set of reals satisfying $S_1(C_\Gamma, C_\Gamma)$. We say that a convergent sequence $\{x_n\}_{n \in \mathbb{N}}$ is nontrivial if $\lim_{n} x_n \notin \{x_n : n \in \mathbb{N}\}$.

Lemma 2.2. Let $X$ be a subspace of a zero-dimensional metrizable space $Y$ satisfying $S_1(C_\Gamma, C_\Gamma)$, and let $\{x^m_n\}_{n \in \mathbb{N}}$, $m \in \mathbb{N}$, be nontrivial convergent sequences in $X$. Then there are a countable closed cover $\{F_k : k \in \mathbb{N}\}$ of $X$ and an infinite $A \subseteq \mathbb{N}$, such that $F_k \cap \{x^m_n : n \in A\}$ is finite for all $k, m$.

Proof. Let $d$ be a metric on $Y$ that generates its topology. For each $m$, do the following. Let $x^m_n = \lim_{n} x^m_n$, and for each $n$ take a clopen neighborhood $C^m_n$ of $x^m_n$ in $Y$, whose diameter is smaller than $d(x^m_n, x_m)/2$. For each $m, n$, set

$$U^m_n = Y \setminus (C^m_n \cup C^m_{n+1} \cup \cdots \cup C^m_{n+m}).$$

For each $m$, $\{U^m_n : n \in \mathbb{N}\}$ is a clopen $\gamma$-cover of $Y$. Apply $S_1(C_\Gamma, C_\Gamma)$ to get $f \in \mathbb{N}^\mathbb{N}$ such that $\{U^m_{f(m)} : m \in \mathbb{N}\}$ is a (clopen) $\gamma$-cover of $Y$. As $U^m_{f(m)} \subseteq Y \setminus C^0_{f(m)}$ for each $m$, we have that the image $A$ of $f$ is infinite.

For each $k$, let $F_k = \bigcap_{i \geq k} U^m_{f(i)}$. \{F_k : k \in \mathbb{N}\} is a closed ($\gamma$-)cover of $Y$. Fix $k$ and $m$. If $n$ is large enough and $n \in A$, then $n = f(i)$ with $i \geq m, k$. As $x^m_n = x^m_{f(i)} \in C^m_{f(i)}$ and $i \geq m$, $x^m_n \notin U^m_{f(i)}$. As $i \geq k$, $U^m_{f(i)} \supseteq F_k$, and therefore $x^m_n \notin F_k$.

To make sure that our constructed set does not contain a perfect set and that it satisfies the Hurewicz property in all finite powers, we will use the following. Let $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ be the one-point compactification of $\mathbb{N}$, and $\overline{\mathbb{N}}^\mathbb{N}$ be the collection of all nondecreasing elements $f$ of $\overline{\mathbb{N}}$ (endowed with the Tychonoff product topology) such that $f(n) < f(n+1)$ whenever $f(n) < \infty$. $\overline{\mathbb{N}}^\mathbb{N}$ is homeomorphic to the Cantor space (see [13] for an explicit homeomorphism) and can therefore be viewed as a set of reals.

\footnote{It is an open problem whether the converse implication holds [3, 11].}
Let $S$ be the family of all nondecreasing finite sequences in $\mathbb{N}$. For $s \in S$, $|s|$ denotes its length. For each $s \in S$, define $q_s \in \mathbb{N}^{|s|}$ by $q_s(n) = s(n)$ if $n < |s|$, and $q_s(n) = \infty$ otherwise. Let $Q$ be the collection of all these elements $q_s$. $Q$ is dense in $\mathbb{N}^{|s|}$.

For a set $D$ and $f, g \in \mathbb{N}^D$, $f \leq^* g$ means: $f(d) \leq g(d)$ for all but finitely many $d \in D$. A b-scale is an unbounded (with respect to $\leq^*$) set $\{f_\alpha : \alpha < b\} \subseteq \mathbb{N}^\omega$ of increasing functions, such that $f_\alpha \leq^* f_\beta$ whenever $\alpha < \beta$.

**Theorem 2.3** (Bartoszyński-Tsaban [2]). Let $X \subseteq \mathbb{N}^\omega$ be a union of a b-scale and $Q$. Then $X$ contains no perfect subset, and all finite powers of $H$ satisfy the Hurewicz property $U_{\text{fin}}(O, \Gamma)$.

For each $s \in S$, $\{q_s \cdot n\}_{n \in \mathbb{N}}$ (where $\cdot$ denotes a concatenation of sequences) is a nontrivial convergent sequence in $\mathbb{N}^{|s|} \setminus Q$ and

$$\lim_{n \to \infty} q_s \cdot n = q_s.$$

The following will be used in our construction.

**Lemma 2.4.** Let $X$ be a closed subspace of $\mathbb{N}^\omega$. If $X \cap \{q_s \cdot n : n \in \mathbb{N}\}$ is infinite for each $s \in S$, then there exists $\phi : S \to \mathbb{N}$ such that for all $x \in X$ and all $n \geq 2$, $x(n) \geq \phi(x \upharpoonright n) \leq \phi(x \upharpoonright (n + 1)).$

**Proof.** For each $s \in S$, let $k(s)$ be such that $q_{s \cdot k} \in \mathbb{N}^\omega \setminus X$ for all $k \geq k(s)$. As $X$ is closed in $\mathbb{N}^\omega$, for each $k \geq k(s)$ there is $m(s, k)$ such that

$$\left\{ z \in \mathbb{N}^\omega : z \upharpoonright (|s| + 1) = s \cdot k, z(|s| + 1) > m(s, k) \right\} \cap X = \emptyset.$$

(Note that $\{ z \in \mathbb{N}^\omega : z \upharpoonright (|s| + 1) = s \cdot k, z(|s| + 1) > m \}, m \in \mathbb{N}$, is a neighborhood base at $q_{s \cdot k}$.) Define $\phi : S \to \mathbb{N}$ by

$$\phi(s) = \max\{k(s), m(s \upharpoonright (|s| - 1)), s(|s| - 1)\}$$

when $|s| \geq 2$, and by $\phi(s) = 0$ when $|s| < 2$. Let $x \in X$ and $n \geq 2$. If $x(n) \geq \phi(x \upharpoonright n)$, then $x(n) \geq k(x \upharpoonright n)$; hence $x(n + 1) \leq m(x \upharpoonright n, x(n)) \leq \phi(x \upharpoonright (n + 1))$. □

It remains to prove the following.

**Proposition 2.5.** Assume that $b = \aleph_1$. There exists a b-scale $B = \{b_\alpha : \alpha < b\}$ such that for each closed cover $\{F_n : n \in \mathbb{N}\}$ of $B \cup Q$ and each infinite set $A \subseteq \mathbb{N}$, there are $n$ and $s \in S$ such that $F_n \cap \{q_{s \cdot k} : k \in A\}$ is infinite.

**Proof.** Let $\{A_\alpha : \alpha < \aleph_1\}$ be an enumeration of all infinite subsets of $\mathbb{N}$, such that for each infinite $A \subseteq \mathbb{N}$, there are $\aleph_1$ many $\alpha < \aleph_1$ with $A_\alpha = A$.

As $b = \aleph_1$, there is a (standard) scale in $\mathbb{N}^{\aleph_1}$, that is, a family $\{\phi_\alpha : \alpha < \aleph_1\} \subseteq \mathbb{N}^{\aleph_1}$ such that:

1. For each $\phi \in \mathbb{N}^{\aleph_1}$, there is $\beta < \aleph_1$ such that $\phi \leq^* \phi_\beta$;
2. For all $\alpha < \beta < \aleph_1$, $\phi_\alpha \leq^* \phi_\beta$.

\footnote{Strictly speaking, $q_{s \cdot n} \notin \mathbb{N}^\omega$ when $n < s(|s| - 1)$, but since we are dealing with convergent sequences, we can ignore the first few elements.}
For an infinite $A \subseteq \mathbb{N}$, let $\overline{A} = A \cup \{\infty\}$ and

\[ \overline{A}^{\mathbb{N}} = \{ x \in \mathbb{N}^{\mathbb{N}} : x(n) \in A \text{ for all } n \}. \]

The order isomorphism between $\mathbb{N} \cup \{\infty\}$ and $\mathbb{N} \cup \{\infty\}$ induces an order isomorphism $\Psi_A : \overline{A}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$.

By induction on $\alpha < \beta = \epsilon$, construct a $\beta$-scale $B = \{ b_\alpha : \alpha < \epsilon \}$ such that for each $\alpha < \epsilon$, $b_\alpha \in (A_\alpha)^{\mathbb{N}}$, and

\[ \Psi_A(b_\alpha)(n) > \phi_\alpha(\Psi_A(b_\alpha) \upharpoonright n) \]

for all $n \geq 2$.

We claim that $X = B \cup Q$ is as required. Indeed, let $A$ be an infinite subset of $\mathbb{N}$. Take an increasing enumeration $\{ q^k : k \in \mathbb{N} \}$ for each $\alpha < \epsilon$, $b_\beta \in \overline{A}^{\mathbb{N}}$. Set $c_\alpha = \Psi_A(b_\beta)$, and $C = \{ c_\alpha : \alpha < \epsilon \}$. By the construction of the functions $b_\alpha$,

\[ c_\alpha(n) > \phi_\beta(c_\alpha \upharpoonright n) \geq \phi_\alpha(c_\alpha \upharpoonright n) \]

for all but finitely many $n$.

Let $\{ K_m : m \in \mathbb{N} \}$ be a closed cover of $C \cup Q$. Then there are $m$ and $s \in S$ such that $K_m \cap \{ q^k : k \in \mathbb{N} \}$ is infinite: Otherwise, by Lemma 2.3, for each $m$ there is $\psi_m \in \mathbb{N}^S$ such that for all $x \in K_m$ and $n \geq 2$, $x(n) \geq \psi_m(x \upharpoonright n)$ implies $x(n + 1) \leq \psi_m(x \upharpoonright n)$. Let $\alpha < \epsilon$ be such that for each $m$, $\phi_\alpha(s) \geq \psi_m(s)$ for all but finitely many $s \in S$. It is easy to verify that $c_\alpha \notin K_m$ for all $m$, a contradiction.

Now consider any closed cover $\{ F_m : m \in \mathbb{N} \}$ of $B \cup Q$ and set $K_m = \Psi_A(F_m \cap \overline{A}^{\mathbb{N}})$. Let $s \in S$ and $m$ be such that $K_m \cap \{ q^k : k \in \mathbb{N} \}$ is infinite. Then for $\hat{s} \in S$ such that $\hat{s}(i)$ is the $s(i)$-th element of $A$ for each $i < |s|$, we have that $F_m \cap \{ q^k : k \in A \}$ is infinite. \qed

This completes the proof of Theorem 2.1. The following corollary of Theorem 2.1 answers in the negative Problem 15(1) of Bartoszyński and the second author [4].

**Corollary 2.6.** The union of a $\beta$-scale and $Q$ need not satisfy $S_1(\Gamma, \Gamma)$. \qed

3. **Reformulation for spaces of continuous functions**

Let $Y$ be a (not necessarily metrizable) topological space. For $y \in Y$ and $A \subseteq Y$, write $\lim A = y$ if $A$ is countable and an (any) enumeration of $A$ converges nontrivially to $y$. Let $\Gamma_y = \{ A \subseteq Y : \lim A = y \}$. $Y$ has the Arhangel’skii property $\alpha_2$ [11] if $S_1(\Gamma_y, \Gamma_y)$ holds for all $y \in Y$.

Fix a set of reals $X$. $C_p(X)$ is the subspace of the Tychonoff product $\mathbb{R}^X$ consisting of the continuous functions. It was recently discovered, independently by Bukovský and Haleš [3] and by Sakai [11], that $C_p(X)$ has the property $\alpha_2$ if, and only if, $X$ satisfies $S_1(\Gamma_1, \Gamma_1)$.

Many additional connections of this type are studied in the literature. For families $\mathcal{A}$ and $\mathcal{B}$, consider the following prototype [14].

$S_{\text{fin}}(\mathcal{A}, \mathcal{B})$: For each sequence $\{ U_n \}_{n \in \mathbb{N}}$ of members of $\mathcal{A}$, there exist finite subsets $F_n \subseteq U_n$, $n \in \mathbb{N}$, such that $\bigcup_{n} F_n \in \mathcal{B}$. 
For a topological space \( Y \) and \( y \in Y \), let \( \Omega_y = \{ A \subseteq Y : y \in \overline{A} \setminus A \} \). \( Y \) has the Arhangel’skiǐ countable fan tightness \([1]\) if \( \mathsf{S}_{\text{fin}}(y, \Omega_y) \) holds for each \( y \in Y \). \( Y \) has the Reznichenko property if for each \( y \in Y \) and each \( A \in \Omega_y \), there are pairwise disjoint finite sets \( F_n \subseteq A, n \in \mathbb{N} \), such that each neighborhood \( U \) of \( y \) intersects \( F_n \) for all but finitely many \( n \).

For sets of reals \( X \), \( C_p(X) \) has countable fan tightness and the Reznichenko property if, and only if, all finite powers of \( X \) have the Hurewicz property \( U_{\text{fin}}(O, \Gamma) \) \([9]\). Thus, Theorem \([2, 3]\) can be reformulated as follows.

**Theorem 3.1.** Assume that \( b = c \). There exists a set of reals \( X \) without perfect subsets such that \( C_p(X) \) has countable fan tightness and the Reznichenko property, but does not have the Arhangel’skiǐ property \( \alpha_2 \).

A topological space \( Y \) has the Sakai strong countable fan tightness if \( \mathsf{S}_1(\Omega_y, \Omega_y) \) holds for each \( y \in Y \). Sakai proved that for sets of reals, \( C_p(X) \) has strong countable fan tightness if, and only if, all finite powers of \( X \) satisfy \( \mathsf{S}_1(O, O) \) \([10]\). For sets of reals \( X \), \( C_p(X) \) has strong countable fan tightness and the Reznichenko property if, and only if, all finite powers of \( X \) satisfy \( U_{\text{fin}}(O, \Gamma) \) as well as \( \mathsf{S}_1(O, O) \) \([8]\).

If \( b \leq \text{cov}(\mathcal{M}) \) and \( X \) is a union of a \( b \)-scale and \( Q \), then all finite powers of \( X \) satisfy \( U_{\text{fin}}(O, \Gamma) \) as well as \( \mathsf{S}_1(O, O) \) \([2]\). As the Continuum Hypothesis (or just Martin’s Axiom) implies that \( b = \text{cov}(\mathcal{M}) = c \), we have the following.

**Corollary 3.2.** Even for \( C_p(X) \) where \( X \) is a set of reals, the conjunction of strong countable fan tightness and the Reznichenko property does not imply the Arhangel’skiǐ property \( \alpha_2 \).

### 4. Concluding Remarks and Open Problems

Our results are consistency results. What is not settled is whether the answers to the problems addressed in this paper are undecidable.

**Problem 4.1.** Is it consistent that all sets of reals that have the Hurewicz property \( U_{\text{fin}}(O, \Gamma) \) but have no perfect subsets satisfy \( \mathsf{S}_1(\Gamma, \Gamma) \)?

**Problem 4.2.** Is it consistent that each union of a \( b \)-scale and \( Q \) satisfies:

1. \( \mathsf{S}_1(\Gamma, \Gamma) \)?
2. \( \mathsf{S}_1(\Gamma, \Gamma) \) in all finite powers?

**Problem 4.3.** Is it consistent that for each set of reals \( X \), if \( C_p(X) \) has both strong countable fan tightness and the Reznichenko property, then \( C_p(X) \) has the Arhangel’skiǐ property \( \alpha_2 \)?

### References


**Institute of Mathematics, Physics and Mechanics and Faculty of Education, University of Ljubljana, P.O.B. 2964, Ljubljana, Slovenia 1001**

E-mail address: dusan.repovs@guest.arnes.si

**Department of Mathematics, Bar-Ilan University, Ramat-Gan 52900, Israel; and Department of Mathematics, Weizmann Institute of Science, Rehovot 76100, Israel**

E-mail address: tsaban@math.biu.ac.il

**Kurt Gödel Research Center for Mathematical Logic, Währinger Str. 25, A-1090 Vienna, Austria**

E-mail address: lzdomsky@gmail.com