Let $S_1(\Gamma, \Gamma)$ be the statement: For each sequence of point-cofinite open covers, one can pick one element from each cover and obtain a point-cofinite cover. $\mathfrak{b}$ is the minimal cardinality of a set of reals not satisfying $S_1(\Gamma, \Gamma)$. We prove the following assertions:

1. If there is an unbounded tower, then there are sets of reals of cardinality $\mathfrak{b}$ satisfying $S_1(\Gamma, \Gamma)$.

2. It is consistent that all sets of reals satisfying $S_1(\Gamma, \Gamma)$ have cardinality smaller than $\mathfrak{b}$.

These results can also be formulated as dealing with Arhangel’skí́ı’s property $\alpha_2$ for spaces of continuous real-valued functions.

The main technical result is that in Laver’s model, each set of reals of cardinality $\mathfrak{b}$ has an unbounded Borel image in the Baire space $\omega^\omega$.

1. **Background**

Let $P$ be a nontrivial property of sets of reals. The critical cardinality of $P$, denoted non$(P)$, is the minimal cardinality of a set of reals not satisfying $P$. A natural question is whether there is a set of reals of cardinality at least non$(P)$, which satisfies $P$, i.e., a nontrivial example.

We consider the following property. Let $X$ be a set of reals. $\mathcal{U}$ is a point-cofinite cover of $X$ if $\mathcal{U}$ is infinite, and for each $x \in X$, $\{U \in \mathcal{U} : x \in U\}$ is a cofinite subset of $\mathcal{U}$. Having $X$ fixed in the background, let $\Gamma$ be the family of all point-cofinite open covers of $X$. The following properties were introduced by Hurewicz [8], Tsaban [19], and Scheepers [15], respectively.

$U_{\text{fin}}(\Gamma, \Gamma)$: For all $U_0, U_1, \cdots \in \Gamma$, none containing a finite subcover, there are finite $F_0 \subseteq U_0, F_1 \subseteq U_1, \cdots$ such that $\bigcup\{F_n : n \in \omega\} \in \Gamma$.

$U_2(\Gamma, \Gamma)$: For all $U_0, U_1, \cdots \in \Gamma$, there are $F_0 \subseteq U_0, F_1 \subseteq U_1, \cdots$ such that $|F_n| = 2$ for all $n$, and $\bigcup\{F_n : n \in \omega\} \in \Gamma$.

$S_1(\Gamma, \Gamma)$: For all $U_0, U_1, \cdots \in \Gamma$, there are $U_0 \in U_0, U_1 \in U_1, \cdots$ such that $\{U_n : n \in \omega\} \in \Gamma$.

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\[1\text{Historically, point-cofinite covers were named } \gamma\text{-covers, since they are related to a property numbered } \gamma \text{ in a list from } \alpha \text{ to } \epsilon \text{ in the seminal paper [7] of Gerlits and Nagy.}\]

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Clearly, $S_1(\Gamma, \Gamma)$ implies $U_2(\Gamma, \Gamma)$, which in turn implies $U_{\text{fin}}(\Gamma, \Gamma)$. None of these implications is reversible in ZFC [19]. The critical cardinality of all three properties is $b$ [9].

Bartoszyński and Shelah [1] proved that there are, provably in ZFC, totally imperfect sets of reals of cardinality $b$ satisfying the Hurewicz property $U_{\text{fin}}(\Gamma, \Gamma)$. Tsaban proved the same assertion for $U_2(\Gamma, \Gamma)$ [19]. These sets satisfy $U_{\text{fin}}(\Gamma, \Gamma)$ in all finite powers [2].

We show that in order to obtain similar results for $S_1(\Gamma, \Gamma)$, hypotheses beyond ZFC are necessary.

2. Constructions

We show that certain weak (but not provable in ZFC) hypotheses suffice to have nontrivial $S_1(\Gamma, \Gamma)$ sets, even ones which possess this property in all finite powers.

Definition 2.1. A tower of cardinality $\kappa$ is a set $T \subseteq [\omega]^{\omega}$ which can be enumerated bijectively as $\{x_\alpha : \alpha < \kappa\}$, such that for all $\alpha < \beta < \kappa$, $x_\beta \subseteq^* x_\alpha$.

A set $T \subseteq [\omega]^{\omega}$ is unbounded if the set of its enumeration functions is unbounded; i.e., for any $g \in \omega^{\omega}$ there is an $x \in T$ such that for infinitely many $n$, $g(n)$ is less than the $n$-th element of $x$.

Scheepers [16] proved that if $t = b$, then there is a set of reals of cardinality $b$ satisfying $S_1(\Gamma, \Gamma)$. If $t = b$, then there is an unbounded tower of cardinality $b$, but the latter assumption is weaker.

Lemma 2.2 (folklore). If $b < d$, then there is an unbounded tower of cardinality $b$.

Proof. Let $B = \{b_\alpha : \alpha < b\} \subseteq \omega^{\omega}$ be a $b$-scale; that is, each $b_\alpha$ is increasing, $b_\alpha \leq^* b_\beta$ for all $\alpha < \beta < b$, and $B$ is unbounded.

As $|B| < d$, $B$ is not dominating. Let $g \in \omega^{\omega}$ exemplify that. For each $\alpha < b$, let $x_\alpha = \{n : b_\alpha(n) \leq g(n)\}$. Then $T = \{x_\alpha : \alpha < b\}$ is an unbounded tower: Clearly, $x_\beta \subseteq^* x_\alpha$ for $\alpha < \beta$. Assume that $T$ is bounded, and let $f \in \omega^{\omega}$ exemplify that. For each $\alpha$, writing $x_\alpha(n)$ for the $n$-th element of $x_\alpha$:

$$b_\alpha(n) \leq b_\alpha(x_\alpha(n)) \leq g(x_\alpha(n)) \leq g(f(n))$$

for all but finitely many $n$. Thus, $g \circ f$ shows that $B$ is bounded, a contradiction. □

Theorem 2.3. If there is an unbounded tower (of any cardinality), then there is a set of reals $X$ of cardinality $b$ that satisfies $S_1(\Gamma, \Gamma)$.

Theorem 2.3 follows from Propositions 2.4 and 2.5.

Proposition 2.4. If there is an unbounded tower, then there is one of cardinality $b$.

Proof. By Lemma 2.2 it remains to consider the case $b = d$. Let $T$ be an unbounded tower of cardinality $\kappa$. Let $\{f_\alpha : \alpha < b\} \subseteq \omega^{\omega}$ be dominating. For each $\alpha < b$, pick $x_\alpha \in T$ which is not bounded by $f_\alpha$. $\{x_\alpha : \alpha < b\}$ is unbounded, being unbounded in a dominating family. □

Blass’s survey [4] is a good reference for the definitions and details of the special cardinals mentioned in this paper.
Define a topology on $P(\omega)$ by identifying $P(\omega)$ with the Cantor space $2^\omega$, via characteristic functions. Scheepers’s mentioned proof actually establishes the following result, to which we give an alternative proof.

**Proposition 2.5** (essentially, Scheepers [16]). For each unbounded tower $T$ of cardinality $\mathfrak{b}$, $T \cup [\omega]^{<\omega}$ satisfies $S_1(\Gamma, \Gamma)$.

**Proof.** Let $T = \{x_\alpha : \alpha < \mathfrak{b}\}$ be an unbounded tower of cardinality $\mathfrak{b}$. For each $\alpha$, let $X_\alpha = \{x_\beta : \beta < \alpha\} \cup [\omega]^{<\omega}$. Let $\mathcal{U}_0, \mathcal{U}_1, \ldots$ be point-cofinite open covers of $X_\mathfrak{b} = T \cup [\omega]^{<\omega}$. We may assume that each $\mathcal{U}_n$ is countable and that $\mathcal{U}_i \cap \mathcal{U}_j = \emptyset$ whenever $i \neq j$.

By the proof of Lemma 1.2 of [6], for each $k$ there are distinct $U^k_0, U^k_1, \ldots \in \mathcal{U}_k$, and an increasing sequence $m^k_0 < m^k_1 < \ldots$, such that for each $n$ and $k$,

$$\{x \subseteq \omega : x \cap (m^k_n, m^k_{n+1}) = \emptyset\} \subseteq U^k_n.$$  

As $T$ is unbounded, there is $\alpha < \mathfrak{b}$ such that for each $k$, $I_k = \{n : x_\alpha \cap (m^k_n, m^k_{n+1}) = \emptyset\}$ is infinite.

For each $k$, $\{U^k_n : n \in \omega\}$ is an infinite subset of $\mathcal{U}_k$, and thus a point-cofinite cover of $X_\alpha$. As $|X_\alpha| < \mathfrak{b}$, there is $f \in \omega^\omega$ such that

$$\forall x \in X_\alpha \exists k_0 \forall k \geq k_0 \forall n > f(k) \; x \in U^k_n.$$  

For each $k$, pick $n_k \in I_k$ such that $n_k > f(k)$.

We claim that $\{U^k_{n_k} : k \in \omega\}$ is a point-cofinite cover of $X_\mathfrak{b}$: If $x \in X_\alpha$, then $x \in U^k_{n_k}$ for all but finitely many $k$, because $n_k > f(k)$ for all $k$. If $x = x_\beta$, $\beta \geq \alpha$, then $x \subseteq^* x_\alpha$. For each large enough $k$, $m^k_{n_k}$ is large enough, so that $x \cap (m^k_{n_k}, m^k_{n_k+1}) \subseteq x_\alpha \cap (m^k_{n_k}, m^k_{n_k+1}) = \emptyset$, and thus $x \in U^k_{n_k}$. \hfill $\square$

**Remark 2.6.** Zdomskyy points out that for the proof to go through, it suffices that $\{x_\alpha : \alpha < \mathfrak{b}\}$ is such that there is an unbounded $\{y_\alpha : \alpha < \mathfrak{b}\} \subseteq [\omega]^{<\omega}$ such that for each $\alpha$, $x_\alpha$ is a pseudointersection of $\{y_\beta : \beta < \alpha\}$. We do not know whether the assertion mentioned here is weaker than the existence of an unbounded tower.

We now turn to nontrivial examples of sets satisfying $S_1(\Gamma, \Gamma)$ in all finite powers. In general, $S_1(\Gamma, \Gamma)$ is not preserved by taking finite powers [9], and we use a slightly stronger hypothesis in our construction.

**Definition 2.7.** Let $\mathfrak{b}_0$ be the additivity number of $S_1(\Gamma, \Gamma)$, that is, the minimum cardinality of a family $\mathcal{F}$ of sets of reals, each satisfying $S_1(\Gamma, \Gamma)$, such that the union of all members of $\mathcal{F}$ does not satisfy $S_1(\Gamma, \Gamma)$.

$t \leq \mathfrak{h}$, and Scheepers proved that $\mathfrak{h} \leq \mathfrak{b}_0 \leq \mathfrak{b}$ [17]. It follows from Theorem 3.6 that, consistently, $\mathfrak{b} < \mathfrak{b}_0 = \mathfrak{b}$. It is open whether $\mathfrak{b}_0 = \mathfrak{b}$ is provable. If $t = \mathfrak{b}$ or $\mathfrak{h} = \mathfrak{b} < \mathfrak{b}_0$, then there is an unbounded tower of cardinality $\mathfrak{b}_0$.

**Theorem 2.8.** For each unbounded tower $T$ of cardinality $\mathfrak{b}_0$, all finite powers of $T \cup [\omega]^{<\omega}$ satisfy $S_1(\Gamma, \Gamma)$.

**Proof.** We say that $\mathcal{U}$ is an $\omega$-cover of $X$ if no member of $\mathcal{U}$ contains $X$ as a subset, but each finite subset of $X$ is contained in some member of $\mathcal{U}$. We need a multidimensional version of Lemma 1.2 of [6].

**Lemma 2.9.** Assume that $[\omega]^{<\omega} \subseteq X \subseteq P(\omega)$, and let $e \in \omega$. For each open $\omega$-cover $\mathcal{U}$ of $X^e$, there are $m_0 < m_1 < \ldots$ and $U_0, U_1, \ldots \in \mathcal{U}$, such that for all $x_0, \ldots, x_{e-1} \subseteq \omega$, $(x_0, \ldots, x_{e-1}) \in U_n$ whenever $x_i \cap (m_n, m_{n+1}) = \emptyset$ for all $i < e$. 
Proof. As $\mathcal{U}$ is an open $\omega$-cover of $X^e$, there is an open $\omega$-cover $\mathcal{V}$ of $X$ such that \{${V}^e : V \in \mathcal{V}$\} refines $\mathcal{U}$\footnote{Choosing infinitely many elements from each cover, instead of one, can be done by adding to the given sequence of covers all cofinite subsets of the given covers.}.

Let $m_0 = 0$. For each $n \geq 0$: Assume that $V_0, \ldots, V_{n-1} \in \mathcal{V}$ are given, and $U_0, \ldots, U_{n-1} \in \mathcal{U}$ are such that $V_i^e \subseteq U_i$ for all $i < n$. Fix a finite $F \subseteq X$ such that $F^e$ is not contained in any of the sets $U_0, \ldots, U_{n-1}$. As $\mathcal{V}$ is an $\omega$-cover of $X$, there is $V_n \in \mathcal{V}$ such that $F \cup \mathcal{P}\{0, \ldots, m_n\} \subseteq V_n$. Take $U_n \in \mathcal{U}$ such that $V_n^e \subseteq U_n$. Then $U_n \notin \{U_0, \ldots, U_{n-1}\}$. As $V_n$ is open, for each $s \subseteq \{0, \ldots, m_n\}$ there is $k_s$ such that for each $x \in P(\omega)$ with $x \cap \{0, \ldots, k_s - 1\} = s$, $x \in V_n$. Let $m_{n+1} = \max\{k_s : s \subseteq \{0, \ldots, m_n\}\}$.

If $x_i \cap (m_n, m_{n+1}) = \emptyset$ for all $i < e$, then $(x_0, \ldots, x_{e-1}) \in V_{m+1}^e \subseteq U_n$. \hfill \qed

The assumption in the theorem that there is an unbounded tower of cardinality $b_0$ implies that $b_0 = b$. The proof is by induction on the power $e$ of $T \cup [\omega]^{<\omega}$. The case $e = 1$ follows from Theorem 2.5.

Let $\mathcal{U}_0, \mathcal{U}_1, \ldots \in \Gamma((T \cup [\omega]^{<\omega})^e)$. We may assume that these covers are countable. As in the proof of Theorem 2.5 (this time using Lemma 2.9), there are for each $k$, $m_0^k < m_1^k < \ldots$ and $U_0^k, \ldots \in \mathcal{U}_e$ (so that $\{U_n^k : n \in \omega\} \in \Gamma((T \cup [\omega]^{<\omega})^e)$), such that for all $y_0, \ldots, y_{e-1} \in \omega$, $(y_0, \ldots, y_{e-1}) \in U_n^k$ whenever $y_i \cap (m_n^k, m_{n+1}^k) = \emptyset$ for all $i < e$.

Let $\alpha_0$ be such that $X^e_{\alpha_0}$ is not contained in any member of $\bigcup_k \mathcal{U}_\kappa$. As $T$ is unbounded, there is $\alpha$ such that $\alpha_0 < \alpha < b$, and for each $k$, $I_k = \{n : x_\alpha \cap (m_n^k, m_{n+1}^k) = \emptyset\}$ is infinite.

Let $Y = \{ x_\beta : \beta \geq \alpha \}$. $(T \cup [\omega]^{<\omega})^e \setminus Y^e$ is a union of fewer than $b_0$ homeomorphic copies of $(T \cup [\omega]^{<\omega})^{e-1}$. By the induction hypothesis, $(T \cup [\omega]^{<\omega})^{e-1}$ satisfies $S_1(\Gamma, \Gamma)$, and therefore so does $(T \cup [\omega]^{<\omega})^e \setminus Y^e$. For each $k$, $\{U_n^k : n \in I_k\}$ is a point-cofinite cover of $(T \cup [\omega]^{<\omega})^e \setminus Y^e$, and thus there are infinite $J_0 \subseteq I_0, J_1 \subseteq I_1, \ldots$, such that $\{\bigcap_{n \in J_k} U_n^k : k \in \omega\}$ is a point-cofinite cover of $(T \cup [\omega]^{<\omega})^e \setminus Y^e$\footnote{Choosing infinitely many elements from each cover, instead of one, can be done by adding to the given sequence of covers all cofinite subsets of the given covers.}.

For each $k$, pick $n_k \in J_k$ such that: $m_{n_k}^k > m_{n_k}^{k-1}$, $x_\alpha \cap (m_{n_k}^k, m_{n_k}^{k+1}) = \emptyset$, and $U_{n_k} \notin \{U_{n_0}, \ldots, U_{n_k}^k\}$.

$\{U_n^k : k \in \omega\} \in \Gamma((T \cup [\omega]^{<\omega})^e)$: if $x \in (T \cup [\omega]^{<\omega})^e \setminus Y^e$, then $x \in U_n^k$ for all but finitely many $k$. If $x = (x_{\beta_0}, \ldots, x_{\beta_{e-1}}) \in Y$, then $\beta_0, \ldots, \beta_{e-1} \geq \alpha$, and thus $x_{\beta_0}, \ldots, x_{\beta_{e-1}} \subseteq x_\alpha$. For each large enough $k$, $m_{n_k}^k$ is large enough, so that $x_\alpha \cap (m_{n_k}^k, m_{n_k}^{k+1}) \subseteq x_\alpha \cap (m_{n_k}^k, m_{n_k}^{k+1}) = \emptyset$ for all $i < e$, and thus $x \in U_{n_k}$. \hfill \qed

There is an additional way to obtain nontrivial $S_1(\Gamma, \Gamma)$ sets: The hypothesis $b = \text{cov}([\mathcal{N}]) = \text{cof}([\mathcal{N}])$ provides $b$-Sierpiński sets, and $b$-Sierpiński sets satisfy $S_1(\Gamma, \Gamma)$, even for $\text{Borel}$ point-cofinite covers. Details are available in \cite{15}.

We record the following consequence of Theorem 2.3 for later use.

**Corollary 2.10.** For each unbounded tower $T$ of cardinality $b$, $T \cup [\omega]^{<\omega}$ satisfies $S_1(\Gamma, \Gamma)$ for open covers, but not for Borel covers.

**Proof.** The latter property is hereditary for subsets \cite{18}. By a theorem of Hurewicz, a set of reals satisfies $U_{\text{fin}}(\Gamma, \Gamma)$ if and only if each continuous image of $X$ in $\omega^\omega$ is bounded. It follows that the set $T \subseteq T \cup [\omega]^{<\omega}$ does not even satisfy $U_{\text{fin}}(\Gamma, \Gamma)$. \hfill \qed
3. A consistency result

By the results of the previous section, we have the following.

**Lemma 3.1.** Assume that every set of reals with property $S_1(\Gamma, \Gamma)$ has cardinality $< b$, and $\mathcal{M} = P_2$. Then $\mathcal{M} = \mathcal{M} < b = \aleph_2$.

**Proof.** As there is no unbounded tower, we have that $t < b = \aleph_2$. As $c = \aleph_2$, $\aleph_1 = t < b = \aleph_2$. Since there are no $b$-Sierpiński sets and $b = \text{cof}(\aleph) = c$, $\text{cof}(\aleph) < b$. \hfill \Box

In Laver’s model [11], $\aleph_1 = t = \text{cof}(\aleph) < b = \aleph_2$. We will show that, indeed, $S_1(\Gamma, \Gamma)$ is trivial there. Laver’s model was constructed to realize Borel’s Conjecture, asserting that “strong measure zero” is trivial. In some sense, $S_1(\Gamma, \Gamma)$ is a dual of strong measure zero. For example, the canonical examples of $S_1(\Gamma, \Gamma)$ sets are Sierpiński sets, a measure-theoretic object, whereas the canonical examples of strong measure zero sets are Luzin sets, a Baire category theoretic object. More about that can be seen in [18].

The main technical result of this paper is the following.

**Theorem 3.2.** In the Laver model, if $X \subseteq 2^\omega$ has cardinality $b$, then there is a Borel map $f : 2^\omega \to \omega^\omega$ such that $f[X]$ is unbounded.

**Proof.** The notation in this proof is as in Laver [11]. We will use the following slightly simplified version of Lemma 14 of [11].

**Lemma 3.3 (Laver).** Let $\mathbb{P}_{\omega_2}$ be the countable support iteration of Laver forcing, $p \in \mathbb{P}_{\omega_2}$, and let $\dot{a}$ be a $\mathbb{P}_{\omega_2}$-name such that $p \Vdash \dot{a} \in 2^\omega$.

Then there is a condition $q$ stronger than $p$ and finite $U_s \subseteq 2^\omega$ for each $s \in q(0)$ extending the root of $q(0)$ such that for all such $s$ and all $n$:

$$q(0)_s \upharpoonright 1, 2^\omega) \Vdash \exists u \in U_s, u | n = \dot{a} | n$$

for all but finitely many immediate successors $t$ of $s$ in $q(0)$.

Assume that $X \subseteq 2^\omega$ has no unbounded Borel image in $\mathcal{M}[G_{\omega_2}]$, i.e., Laver’s model. For every code $u \in 2^\omega$ for a Borel function $f : 2^\omega \to \omega^\omega$ there exists $g \in \omega^\omega$ such that for every $x \in X$ we have that $f(x) \leq^* g$.

By a standard Löwenheim-Skolem argument (see Theorem 4.5 on page 281 of [3] or section 4 on page 580 of [12]), we may find $\alpha < \omega_2$ such that for every code $u \in \mathcal{M}[G_{\alpha}]$ there is an upper bound $g \in \mathcal{M}[G_{\alpha}]$. By the arguments employed by Laver [11 Lemmata 10 and 11], we may assume that $\mathcal{M}[G_{\alpha}]$ is the ground model $\mathcal{M}$.

Since the continuum hypothesis holds in $\mathcal{M}$ and $|X| = b = \aleph_2$, there are $p \in G_{\omega_2}$ and $\dot{a}$ such that $p \Vdash \dot{a} \in X$ and $\dot{a} \notin \mathcal{M}$.

Work in the ground model $\mathcal{M}$.

Let $q \leq p$ be as in Lemma 3.3. Define $Q = \{s \in q(0) : \text{root}(q(0)) \subseteq s\}$ and let $U_s, s \in Q$, be the finite sets from the lemma. Let $U = \bigcup_{s \in Q} U_s$. Define a Borel map $f : 2^\omega \to \omega^Q$ so that for every $x \in 2^\omega \setminus U$ and for each $s \in Q$: If
we may assume that $(Dow)$ Lemma 3.5 The main technical result in [5] is the following.

suffices to show that it is $\forall \in H$ that $\forall H \subseteq \omega \cap M$ and $r \leq q$ such that $r \forces f(\hat{a}) \leq^* \hat{g}$.

Since $p$ forced that $a$ is not in the ground model, it cannot be that $a$ is in $U$. We may extend $r(0)$ if necessary so that if $s = \text{root}(r(0))$, then $r \forces f(\hat{a})(s) \leq \hat{g}(s)$.

But this is a contradiction to Lemma 3.3 since for all but finitely many $t \in r(0)$ which are immediate extensions of $s$:

$$r(0), ^\ast q \forces [1, \omega_2) \forces f(\hat{a})(s) > \hat{g}(s).$$

In [20], Tsaban and Zdomskyy prove that $T_h$ for Borel covers is equivalent to the $Ko\v{c}n$ac property $S_{\text{cof}}(\Gamma, \Gamma)$ [10], asserting that for all $U_0, U_1, \ldots \in \Gamma$, there are cofinite subsets $V_0 \subseteq U_0, V_1 \subseteq U_1, \ldots$ such that $\bigcup_n V_n \in \Gamma$. The main result of [5] can be reformulated as follows.

**Theorem 3.4 (Dow [5]).** In Laver's model, $S_1(\Gamma, \Gamma)$ implies $S_{\text{cof}}(\Gamma, \Gamma)$.

For the reader's convenience, we give Dow's proof, adapted to the present notation.

**Proof.** A family $\mathcal{H} \subseteq [\omega]^\omega$ is $\omega$-splitting if for each countable $A \subseteq [\omega]^\omega$, there is $H \in \mathcal{H}$ which splits each element of $A$, i.e.,

$$|A \cap H| = |A \setminus H| = \omega$$

for all $A \in A$.

The main technical result in [5] is the following.

**Lemma 3.5 (Dow).** In Laver’s model, each $\omega$-splitting family contains an $\omega$-splitting family of cardinality $< b$.

Assume that $X$ satisfies $S_1(\Gamma, \Gamma)$. Let $U_0, U_1, \ldots$ be open point-cofinite countable covers of $X$. We may assume that $U_i \cap U_j = \emptyset$ whenever $i \neq j$. Put $U = \bigcup_{n<\omega} U_n$. We identify $U$ with $\omega$, its cardinality.

Define $\mathcal{H} \subseteq [U]^\omega$ as follows. For $H \in [U]^\omega$, put $H \in \mathcal{H}$ if and only if there exists $V \in [U]^\omega$, a point-cofinite cover of $X$, such that $H \cap U_n \subseteq^* V$ for all $n$. We claim that $H$ is an $\omega$-splitting family. As $\mathcal{H}$ is closed under taking infinite subsets, it suffices to show that it is $\omega$-hitting; i.e., for any countable $A \subseteq [U]^\omega$ there exists $H \in \mathcal{H}$ which intersects each $A \in A$. (It is enough to intersect each $A \in A$, since we may assume that $A$ is closed under taking cofinite subsets.)

Let $A \subseteq [U]^\omega$ be countable. For each $n$, choose sets $U_{n,m} \subseteq [U_n]^\omega$, $m \in \omega$, such that for each $A \in A$, if $A \cap U_n$ is infinite, then $U_{n,m} \subseteq A$ for some $m$. Apply the $S_1(\Gamma, \Gamma)$ to the family $\{U_{n,m} : n, m \in \omega\}$ to obtain a point-cofinite $V \subseteq U$ such that $V \cap U_{n,m}$ is nonempty for all $n, m$.

Next, choose finite subsets $F_n \subseteq U_n$, $n \in \omega$, such that for each $A \in A$ with $A \cap U_n$ finite for all $n$, then $A \subseteq^* \bigcup_n F_n$. Take $H = V \cup \bigcup_n F_n$. Then $H$ is in $\mathcal{H}$ and meets each $A \in A$. This shows that $\mathcal{H}$ is an $\omega$-splitting family.

By Lemma 3.5, there is an $\omega$-splitting $\mathcal{H}' \subseteq \mathcal{H}$ of cardinality $< b$. For each $H \in \mathcal{H}'$, let $V_H$ witness that $H$ is in $\mathcal{H}$; i.e., $V_H \subseteq U$ is a point-cofinite cover of $X$ and $H \cap U_n \subseteq^* V_H$ for all $n$.

---

4To see why, replace each $U_n$ by $U_n \setminus \bigcup_{i<n} U_i$ and discard the finite ones. It suffices to show that $S_{\text{cof}}(\Gamma, \Gamma)$ applies to those that are left.
By the definition of $b$, we may find finite $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n \in \omega$, such that for each $H \in \mathcal{H}'$,

$$H \cap \mathcal{U}_n \subseteq \mathcal{V}_H \cup \mathcal{F}_n$$

for all but finitely many $n$. We claim that $\mathcal{W} = \bigcup_n \mathcal{U}_n \setminus \mathcal{F}_n$ is point-cofinite. Suppose it is not. Then there is $x \in X$ such that for infinitely many $n$, there is $U_n \in \mathcal{U}_n \setminus \mathcal{F}_n$ with $x \notin U_n$. Let $H \in \mathcal{H}'$ contain infinitely many of these $U_n$. By the above inclusion, all but finitely many of these $U_n$ are in $\mathcal{V}_H$. This contradicts the fact that $\mathcal{V}_H$ is point-cofinite. \hfill $\Box$

We therefore have the following.

**Theorem 3.6.** In Laver’s model, each set of reals $X$ satisfying $S_1(\Gamma, \Gamma)$ has cardinality less than $b$.

**Proof.** By Dow’s Theorem, $S_1(\Gamma, \Gamma)$ implies $S_{\text{cof}}(\Gamma, \Gamma)$, which in turn implies $S_1(\Gamma, \Gamma)$ for Borel covers \cite{20}. The latter property is equivalent to having all Borel images in $\omega^\omega$ bounded \cite{18}. Apply Theorem 3.2. \hfill $\Box$

Thus, it is consistent that strong measure zero and $S_1(\Gamma, \Gamma)$ are both trivial.

The proof of Dow’s Theorem \cite{15} becomes more natural after replacing, in Lemma 3.5 “$\omega$-splitting” by “$\omega$-hitting”. This is possible, due to the following fact (cf. Remark 4 of \cite{5}).

**Proposition 3.7.** For each infinite cardinal $\kappa$, the following are equivalent:

1. Each $\omega$-splitting family contains an $\omega$-splitting family of cardinality $< \kappa$.
2. Each $\omega$-hitting family contains an $\omega$-splitting family of cardinality $< \kappa$.

**Proof.** (1 $\Rightarrow$ 2) Suppose $A$ is an $\omega$-hitting family. Let $B = \bigcup_{A \in A} [A]^\omega$. Then $B$ is $\omega$-splitting. By (1) there exists $C \subseteq B$ of size $< \kappa$ which is $\omega$-splitting. Choose $D \subseteq A$ of size $< \kappa$ such that for every $C \in C$ there exists $D \in D$ with $C \subseteq D$. Then $D$ is $\omega$-hitting.

(2 $\Rightarrow$ 1) Suppose $A$ is an $\omega$-splitting family. For each $A \subseteq \omega$ define

$$A^* = \{2n : n \in A\} \cup \{2n + 1 : n \in A\}.$$ 

Then the family $A^* = \{A^* : A \in A\}$ is $\omega$-hitting. To see this, suppose that $B$ is countable. Without loss we may assume that $B = B_0 \cup B_1$, where each element of $B_0$ is a subset of the evens and each element of $B_1$ is a subset of the odds. For $B \in B_0$ let $C_B = \{n : 2n \in B\}$ and for $B \in B_1$ let $C_B = \{n : 2n + 1 \in B\}$. Now put

$$C = \{C_B : B \in B\}.$$ 

Since $A$ is $\omega$-splitting there is $A \in A$ which splits $C$. If $B \in B_0$, then $A \cap C_B$ infinite implies $B \cap A^*$ infinite. If $B \in B_1$, then $A \cap C_B$ infinite implies $B \cap A^*$ infinite.

By (2) there exists $A_0 \subseteq A$ of cardinality $< \kappa$ such that $A_0^*$ is $\omega$-hitting. We claim that $A_0$ is $\omega$-splitting. Given any $B \subseteq \omega$ let $B' = \{2n : n \in B\}$ and let $B'' = \{2n + 1 : n \in B\}$. Given $B \subseteq [\omega]^\omega$ countable, there exists $A \in A_0$ such that $A^*$ hits each $B'$ and $B''$ for $B \in B$. But this implies that $A$ splits $B$. \hfill $\Box$

4. Applications to Arhangel’skii’s $\alpha_i$ spaces

Let $Y$ be a general (not necessarily metrizable) topological space. We say that a countably infinite set $A \subseteq Y$ converges to a point $y \in Y$ if each (equivalently, some) bijective enumeration of $A$ converges to $y$. The following concepts are due
Corollary 4.5. If there is an unbounded tower, then there is a set of reals $X$ such that $C(X)$ is an $\omega_2$ space but not $\alpha_1$.

Essentially, Corollary 4.3 is a special case of Corollary 2.10 whereas Corollary 4.5 is equivalent to Corollary 2.10.
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REFERENCES


