Order one invariants of immersions of surfaces into 3-space

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Abstract. We classify all order one invariants of immersions of a closed orientable surface F into $\mathbb{R}^3$, with values in an arbitrary Abelian group $\mathbb{G}$. We show that for any F and $\mathbb{G}$ and any regular homotopy class $\mathcal{A}$ of immersions of F into $\mathbb{R}^3$, the group of all order one invariants on $\mathcal{A}$ is isomorphic to $\mathbb{G}^{\aleph_0} \oplus \mathbb{B} \oplus \mathbb{B}$ where $\mathbb{G}^{\aleph_0}$ is the group of all functions from a set of cardinality $\aleph_0$ into $\mathbb{G}$ and $\mathbb{B} = \{x \in \mathbb{G} : 2x = 0\}$.

Our work includes foundations for the study of finite order invariants of immersions of a closed orientable surface into $\mathbb{R}^3$, analogous to chord diagrams and the 1-term and 4-term relations of knot theory.

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1. Introduction

The notion of finite order invariants arose in knot theory. It is extended here to the setting of stable immersions of a closed orientable surface into $\mathbb{R}^3$. We give a general analysis of finite order invariants, finally classifying all order one invariants (Theorem 5.1). A small subclass of order one invariants has previously been studied, namely, the “local invariants”, studied in [MB], [G], [N1], [N3].

The structure of the paper is as follows: In Section 2 we describe the self-intersection appearing in immersions which correspond to the codimension 1 (abbreviated codim 1) strata in the space of immersions; they will be the basis for our definition of finite order invariants. In Section 3 finite order invariants are defined, without referring to a globally defined co-orientation for the strata in the space of immersions; indeed it is shown that such continuous choice of co-orientation does not exist. We construct the combinatorial object on which order n invariants are defined up to order $n - 1$ invariants. We then obtain a first relation satisfied by invariants of order n , which is related to the non-existence of co-orientations. In Section 4 we obtain the relations on invariants arising from local 2-parameter families of immersions. In Section 5 we classify all order one invariants by analyzing essential loops in the space of immersions. In Section

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6 we use our classification of order one invariants to produce an infinite list of equations which are satisfied by any eversion of the sphere. This extends a list of four such equations appearing in [G]. In Section 7 it is shown that for invariants of order $n > 1$ with values in $\mathbb{G} = \mathbb{Z}/2$, the relations we have obtained are insufficient. In that the analysis of [N2] is also completed. Finally it is shown that for any Abelian group $\mathbb{G}$ there exist invariants of any order n with values in $\mathbb{G}$. The problem of classifying all invariants of order $n > 1$ remains open.

2. Self intersection

Let F be a closed orientable surface and denote by $Imm(F, \mathbb{R}^3)$ the space of all immersions of F into $\mathbb{R}^3$, with the C^1 topology. The strongest equivalence between immersions $i : F \rightarrow \mathbb{R}^3$ that we consider is the following:

Definition 2.1. *A regular homotopy $H_t : F \rightarrow \mathbb{R}^3$ will be called of type A if it is of the form $H_t = U_t \circ i \circ V_t$ where $i : F \rightarrow \mathbb{R}^3$ is an immersion and $U_t : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $V_t : F \rightarrow F$ are isotopies. Two immersions $i, j : F \rightarrow \mathbb{R}^3$ will be called A-equivalent if there is a regular homotopy of type A between them.*

The division of $Imm(F, \mathbb{R}^3)$ into A-equivalence classes, forms a stratification of this space. An immersion $i : F \rightarrow \mathbb{R}^3$ is called *stable* if it is an interior point of its A-equivalence class. This clearly holds iff the class itself is open, and those are the strata of codim 0. The codimension of the stratum to which a given immersion belongs depends on the self intersection appearing in that immersion, see [HK] for local analysis. Parts of our work will also be local: given an immersion $i : F \rightarrow \mathbb{R}^3$ and a self intersection point of i , we will look at all deformations of i which move F only in a small neighborhood of that intersection point. The *local stratification* will then mean the stratification of this smaller space of immersions and *local codimension* of the local strata will again refer to this smaller space. We will take a close look at the local stratification up to codim 2. The codim 0 strata are, as mentioned, the A-equivalence classes of stable immersions. Their self intersection is everywhere in general position i.e. locally equivalent to the intersection of two or three coordinate planes in $\mathbb{R}^3$. In this section we will discuss the codim 1 strata. The codim 2 strata will be discussed in Section 4.

The codim 1 strata are divided into four types which (following [G]) we call: E, H, T, Q . In the notation of [HK] they are respectively $A_0^2|A_1^+$, $A_0^2|A_1^-$, $A_0^3|A_1$, A_0^4 . The four types may be demonstrated by the following local representatives, where formulae in 3-space defining the different sheets involved in the self intersection, are given. Representatives of the codim 1 strata are obtained from the formulae below by setting $\lambda = 0$. Letting λ vary, we obtain a 1-parameter family of immersions which is transverse to the given codim 1 stratum.

E : $z = 0, \quad z = x^2 + y^2 + \lambda$. See Figure 3, ignoring the vertical plane.

H : $z = 0, \quad z = x^2 - y^2 + \lambda$. See Figure 4, ignoring the vertical plane.

T : $z = 0$, $y = 0$, $z = y + x^2 + \lambda$. See Figure 5, ignoring the vertical plane $x = 0$.

Q : $z = 0$, $y = 0$, $x = 0$, $z = x + y + \lambda$. This is simply four planes passing through one point, any three of which are in general position.

A self intersection of local codim 1 (namely E , H , T or Q) will be called a CE point (after the first and last letters of “co-dimension one”). By CE point we will also refer to the point in $\mathbb{R}^3$ where this self-intersection takes place. Once an orientation is chosen for the surface, the above four types of self intersection split into twelve types, as we explain below. A choice of one of the two sides of the local codim 1 stratum at a given point of the stratum, is represented by the choice of $\lambda < 0$ or $\lambda > 0$ in the formulae above. We will refer to such a choice as a *co-orientation for the configuration of the self intersection*. A completely different notion of co-orientation that we will encounter is the following: If $i : F \rightarrow \mathbb{R}^3$ is an immersion and F is oriented then its orientation and that of $\mathbb{R}^3$ induce a co-orientation for $i(F)$ in $\mathbb{R}^3$. To avoid confusion between the two notions, we will use the term “co-orientation” only for the former. For the latter we will speak of the “preferred side of $i(U)$ in $\mathbb{R}^3$ ” wherever $U \subseteq F$ is embedded by i .

We now present the twelve types of self intersection in the oriented setting and specify a co-orientation wherever possible:

Types E and T : the configuration of the self intersection at the two sides of the stratum is distinct even with no orientation on the surface, namely, for $\lambda < 0$ there is an additional 2-sphere in the image of the immersion, and we choose this side ($\lambda < 0$) as our positive side for the co-orientation. Now, for type E (respectively type T) this additional 2-sphere is made of two 2-cells (respectively three 2-cells). If the surface is oriented then we may distinguish three (respectively four) different types of E (respectively T) self intersections, which we denote by E^a (respectively T^a) where $0 \leq a \leq 2$ (respectively $0 \leq a \leq 3$) denotes the number of such 2-cells for which the little 3-cell bounded by the 2-sphere lies on their non-preferred side in $\mathbb{R}^3$.

Types H and Q : The configuration of the self intersection on the two sides of these strata are indistinguishable without an orientation on the surface. For oriented surfaces we proceed as follows: Let V denote the region between the sheets of the surface which appears once we move away from the stratum (i.e. once $\lambda \neq 0$). For type Q this is a simplex defined by our four sheets. For type H this region is not bounded by the local configuration, but may still be defined, e.g. for $\lambda > 0$ in the formula for H , V will be a region consisting of points close to the origin and satisfying $0 \leq z \leq x^2 - y^2 + \lambda$.

We may now count the number a of sheets for which V is on their preferred (or non-preferred) side. On the other side of the stratum this number will be $2 - a$ for type H and $4 - a$ for type Q . The un-ordered pair of numbers $\{a, 2 - a\}$ serves to split the H configuration into two types $\{2, 0\}$ and $\{1, 1\}$ which we name respectively H^2 and H^1 . Similarly the un-ordered pair $\{a, 4 - a\}$ splits Q into three

types $\{4, 0\}$, $\{3, 1\}$, $\{2, 2\}$ which we name respectively Q^4 , Q^3 , Q^2 . Furthermore, in case this is a pair of *distinct* numbers, i.e. for H^2 , Q^4 , Q^3 , then it may be used to define a co-orientation for that configuration. For H^2 we choose as the positive side of the stratum, that side where both sheets have V on their preferred side. For Q^a , $a = 4, 3$, we choose as the positive side of the stratum, that where a sheets have V on their *non*-preferred side. Note that our choice for H and Q uses an opposite convention. Our convention for Q resembles that which we used for E and T . The reason we made an opposite choice for H is only to avoid some minus signs in the sequel.

Finally, the two sides of the H^1 and Q^2 strata are indistinguishable even when the surface is oriented. This fact is expressed in the following proposition:

Proposition 2.2. *Let B be a ball neighborhood in $\mathbb{R}^3$ of a CE of type H^1 or Q^2 . There exists an orientation preserving diffeomorphism $h : B \rightarrow B$ which maps the configuration onto itself (permuting the sheets), being orientation preserving on each sheet, but reversing the co-orientation of the configuration (i.e. motion of the configuration into one side of the stratum corresponds under h to motion into the other side).*

We will show that this local symmetry of the H^1 and Q^2 configurations implies a global one-sidedness of the H^1 and Q^2 strata in $\text{Imm}(F, \mathbb{R}^3)$, i.e. there is no global continuous choice of co-orientation for the H^1 and Q^2 strata, (see Remark 3.7 below).

We have seen that for an oriented surface there are twelve different types of CEs. We have attached a symbol to each type and have chosen a co-orientation for all but two of them. This set of twelve symbols will be denoted $\mathcal{C}$. For the sake of convenience we extend $\mathcal{C}$ to a set $\tilde{\mathcal{C}}$ of fifteen symbols, adjoining the symbols H^0 , Q^0 , Q^1 . The new symbol H^a (respectively Q^a) will denote the same configuration as H^{2-a} (respectively Q^{4-a}) only with opposite co-orientation. The co-orientation attached to the new symbols is consistent with that attached to the old ones, namely for any H^a , $a \neq 1$ (respectively Q^a , $a \neq 2$) the positive side of the stratum is where a sheets have V on their preferred side (respectively non-preferred side).

3. Finite order invariants

We fix once and for all a closed oriented surface F and a regular homotopy class $\mathcal{A}$ of immersions of F into $\mathbb{R}^3$. We denote by $I_n \subseteq \mathcal{A}$ ($n \geq 0$) the space of all immersions in $\mathcal{A}$ which have precisely n CE points (the self intersection being elsewhere stable). In particular, I_0 is the space of all stable immersions in $\mathcal{A}$.

Let $\mathbb{G}$ be any Abelian group and let $f : I_0 \rightarrow \mathbb{G}$ be an invariant, i.e. a function which is constant on each connected component of I_0 . Given an immersion $i \in I_n$, a *temporary co-orientation* for i is a choice of co-orientation at each of the n CE

points $p_1, \dots, p_n$ of i . Given a temporary co-orientation $\mathfrak{T}$ for i , define $f^{\mathfrak{T}}(i)$ as follows: Let $i_1, \dots, i_{2^n}$ be the 2^n immersions in I_0 obtained from i by slightly deforming it in the 2^n possible ways. We define:

$$f^{\mathfrak{T}}(i) = \sum_{m=1}^{2^n} \epsilon_1^m \cdots \epsilon_n^m f(i_m)$$

where ϵ_k^m is 1 or -1 according to whether in order to obtain i_m we have deformed the configuration of i at p_k positively or negatively, with respect to $\mathfrak{T}$. The following is clear:

Lemma 3.1. *If $i \in I_n$ and $\mathfrak{T}_1, \mathfrak{T}_2$ are two temporary co-orientations for i which differ at precisely one CE of i then $f^{\mathfrak{T}_1}(i) = -f^{\mathfrak{T}_2}(i)$.*

By Lemma 3.1, the statement $f^{\mathfrak{T}}(i) = 0$ is independent of the temporary co-orientation $\mathfrak{T}$ and we may simply write $f(i) = 0$.

Definition 3.2. *An invariant $f : I_0 \rightarrow \mathbb{G}$ will be called of finite order if there is an n such that $f(i) = 0$ for all $i \in I_{n+1}$. The minimal such n will be called the order of f .*

Let $i : F \rightarrow \mathbb{R}^3$ be an immersion having a CE located at $p \in \mathbb{R}^3$. As in [N2], we define the degree $d_p(i)$ of the CE at p as follows: Let B be a tiny ball in $\mathbb{R}^3$ centered at p . $i^{-1}(B)$ is a union of some (two, three or four) discs in F which pass p . Let $\hat{i} : F \rightarrow \partial B$ be the map obtained from i as follows: On $F - i^{-1}(B)$ we define $\hat{i}$ by radial projection (centered at p). Now, if D is one of the discs in $i^{-1}(B)$ then $i(D)$ cuts ∂B into two hemispheres. $\hat{i}$ is defined to map D onto the hemisphere which lies on the preferred side of $i(D)$ in $\mathbb{R}^3$. (Recall that F is oriented). Finally we define $d_p(i)$ as the degree of the map $\hat{i} : F \rightarrow \partial B$ (the orientation on ∂B being that induced from B).

Let $C_p(i)$ be the expression R_m^a where $R^a \in \mathcal{C}$ is the symbol describing the configuration of the CE of i at p (one of the twelve symbols) and $m = d_p(i)$. Let $\mathcal{C}_n$ denote the set of all *un-ordered* n -tuples of expressions R_m^a with $R^a \in \mathcal{C}$, $m \in \mathbb{Z}$. Finally, we define a map $C : I_n \rightarrow \mathcal{C}_n$ as follows: If $i \in I_n$ with CEs located at $p_1, \dots, p_n \in \mathbb{R}^3$, then we define $C(i) \in \mathcal{C}_n$ to be the un-ordered n -tuple $[C_{p_1}(i), \dots, C_{p_n}(i)]$. It is easy to see that $C : I_n \rightarrow \mathcal{C}_n$ is surjective.

Definition 3.3. *A regular homotopy $H_t : F \rightarrow \mathbb{R}^3$ between immersions $i, j \in I_n$ will be called of type B if it is of the following form: If $B_1, \dots, B_n \subseteq \mathbb{R}^3$ are little balls centered at the n CE points of i and $U = i^{-1}(\bigcup_k B_k)$ then H_t fixes U and moves $F - U$ within $\mathbb{R}^3 - \bigcup_k B_k$.*

Two immersions $i, j \in I_n$ will be called AB equivalent if there is a regular homotopy H_t between i and j which is alternatingly of type A and B. Such a regular homotopy will be called an AB equivalence.

A proof of the following proposition appears in [N2] for immersions including only quadruple points. The proof for the general case is identical, we include it here for completeness.

Proposition 3.4. *Let $i, j \in I_n$, then i and j are AB equivalent iff $C(i) = C(j)$.*

Proof. If i and j are AB equivalent then clearly $C(i) = C(j)$. For the converse, assume $C(i) = C(j)$. One can order the CE's of i and j , respectively $p'_1, \dots, p'_n$ and $p_1, \dots, p_n$, such that $C_{p'_k}(i) = C_{p_k}(j)$, $k = 1, \dots, n$. This means in particular, that if $B'_1, \dots, B'_n$ and $B_1, \dots, B_n$ are neighborhoods of the p'_k 's and p_k 's respectively, then for each k there is an orientation preserving diffeomorphism from B'_k to B_k which takes each sheet of $i(F) \cap B'_k$ orientation preservingly onto the corresponding sheet of $j(F) \cap B_k$. These diffeomorphisms may all be realized by one ambient isotopy $U_t : \mathbb{R}^3 \rightarrow \mathbb{R}^3$. There is then an isotopy $V_t : F \rightarrow F$ such that the final immersion i' of the regular homotopy $U_t \circ i \circ V_t$ satisfies that i' and j have the same n CE points $p_1, \dots, p_n \in \mathbb{R}^3$, $i'^{-1}(\bigcup_k B_k) = j^{-1}(\bigcup_k B_k)$ which we name U , and $i'|_U = j|_U$. Also $d_{p_k}(i') = d_{p_k}(j)$ for $k = 1, \dots, n$. Now U is a union of some discs $D_1, \dots, D_r$. We construct the following handle decomposition of F . $D_1, \dots, D_r$ will be the 0-handles. If g is the genus of F we will have 1-handles $h_1, \dots, h_{2g+r-1}$ as follows: $h_1, \dots, h_{2g}$ will each have both ends glued to D_1 such that $F - (D_1 \cup h_1 \cup \dots \cup h_{2g})$ will be a disc containing $D_2, \dots, D_r$. Then for $k = 1, \dots, r-1$, h_{2g+k} will have one end glued to D_k and the other to D_{k+1} . The complement of all 0- and 1-handles is again one disc, which will be the unique 2-handle. We will now construct a regular homotopy of the form $i' \circ V'_t$ ($V'_t : F \rightarrow F$ an isotopy) from i' to an immersion i'' which will have the property that the restrictions of i'' and j to all 1-handles, are regularly homotopic keeping all 0-handles fixed. Since i' and j are regularly homotopic (recall $i, j \in I_n \subseteq \mathcal{A}$), this is already true for $h_1, \dots, h_{2g}$. Now take h_{2g+1} . If $i'|_{h_{2g+1}}$ and $j|_{h_{2g+1}}$ are not regularly homotopic keeping D_1 and D_2 fixed, then V'_t performs one full rotation of D_2 , creating a Dehn twist in a thin annulus around D_2 in F . h_{2g+1} will now satisfy the needed property. Note also that this rotation of D_2 moves only h_{2g+1} and h_{2g+2} , keeping all other 0- and 1-handles fixed. We continue this way along the chain of 1-handles, rotating D_{k+1} if necessary for the sake of h_{2g+k} . For $k < r-1$ this will also move h_{2g+k+1} , but we never need to move 1-handles that have previously been taken care of. Also $d_{p_k}(i'') = d_{p_k}(i) = d_{p_k}(j)$ for all $k = 1, \dots, n$. We now perform a regular homotopy H_t on the union of 0- and 1-handles which fixes the 0-handles, and regularly homotopes each 1-handle h , from $i''|_h$ to $j|_h$, avoiding $\bigcup_k B_k$. This is possible by the construction of i'' . Denote our 2-handle by D . So far we have constructed H_t only on $F - D$. By means of [S], H_t may be extended to D , still avoiding $\bigcup_k B_k$, arriving at an immersion i''' . And so, we are left with regularly homotoping $i'''|_D$ to $j|_D$ (relative ∂D). Since $d_{p_k}(i''') = d_{p_k}(j)$ for all $k = 1, \dots, n$, these maps are homotopic in $\mathbb{R}^3 - \bigcup_k B_k$. It then follows from the Smale-Hirsch Theorem ([H]), that they are also *regularly* homotopic in $\mathbb{R}^3 - \bigcup_k B_k$ (since the obstruction to that would lie in $\pi_2(SO_3) = 0$).

The regular homotopy from i to i'' was of type A, and that from i'' to j was of type B. $\square$

Proposition 3.5. *Let $f : I_0 \rightarrow \mathbb{G}$ be an invariant of order n . Let $i \in I_n$ be an immersion with CEs $p_1, \dots, p_n$ and assume the configuration of i at p_1 is of type H^1 or Q^2 . Let $\mathfrak{T}$ be a temporary co-orientation for i . Then the element $a = f^{\mathfrak{T}}(i) \in \mathbb{G}$ satisfies $a = -a$ (or equivalently $2a = 0$).*

Proof. For $k = 1, \dots, n$ let B_k be a small neighborhood of p_k in $\mathbb{R}^3$. Let $h : B_1 \rightarrow B_1$ be the diffeomorphism given by Proposition 2.2. We use h on B_1 and the identity map on $B_2, \dots, B_n$, in the proof of Proposition 3.4, getting an AB equivalence H_t from i to itself which reverses the co-orientation of the configuration in B_1 and fixes F in $B_2, \dots, B_n$. Let $\mathfrak{T}_t$ be the temporary co-orientation for H_t which is obtained by continuously carrying $\mathfrak{T}$ along the regular homotopy. If at some time t_0 , H_t passes through an $(n+1)$ th CE at $p \in \mathbb{R}^3$, we choose an arbitrary co-orientation for H_{t_0} at p . Together with $\mathfrak{T}_{t_0}$ this gives a temporary co-orientation $\mathfrak{T}'$ for the $n+1$ CEs of H_{t_0} . Since f is of order n , $f^{\mathfrak{T}'}(H_{t_0}) = 0$ which means that $f^{\mathfrak{T}_{t_0-\epsilon}}(H_{t_0-\epsilon}) = f^{\mathfrak{T}_{t_0+\epsilon}}(H_{t_0+\epsilon})$. Finally we arrive back at i but with opposite co-orientation than the original one at p_1 and the same co-orientation at $p_2, \dots, p_n$.

By the string of equalities $f^{\mathfrak{T}_{t_0-\epsilon}}(H_{t_0-\epsilon}) = f^{\mathfrak{T}_{t_0+\epsilon}}(H_{t_0+\epsilon})$ on the one hand and by Lemma 3.1 on the other hand, we finally get $f^{\mathfrak{T}}(i) = -f^{\mathfrak{T}}(i)$. $\square$

Corollary 3.6. *Let $f : I_0 \rightarrow \mathbb{G}$ be an invariant of order n . Let $i \in I_n$ be an immersion with at least one CE of type H^1 or Q^2 . Then $f^{\mathfrak{T}}(i)$ is independent of the temporary co-orientation $\mathfrak{T}$.*

Proof. This follows from Lemma 3.1 and Proposition 3.5 $\square$

Remark 3.7. We have seen in the proof of Proposition 3.5 that if $i \in I_n$ has a CE of type H^1 or Q^2 then there is an AB equivalence from i to itself such that if we follow that CE along the AB equivalence, then it returns to itself but with opposite co-orientation. If one seeks a globally defined co-orientation, it is a minimal requirement that it be continuously defined along AB equivalences. So we see that such globally defined co-orientation does not exist for CEs of type H^1 and Q^2 .

If f is an invariant of order n then Corollary 3.6 enables us to extend f to I_n as follows: We choose once and for all a *permanent* co-orientation for the ten configurations which allow it; in fact, we choose those co-orientations given in Section 2 above. Now, if $i \in I_n$ and all n CEs of i are not of configuration H^1 or Q^2 then we define $f(i)$ using our permanent co-orientation for each of the CEs. If at least one of the CEs of i is of configuration H^1 or Q^2 then by Corollary 3.6 $f(i)$ is well defined, independent of a temporary co-orientation. We will assume from now on without mention that any f of order n is extended to I_n in this way. (Note that if f is of order n then we are not extending f to I_k for $0 < k < n$).

Proposition 3.8. *Let f be an invariant of order n and $i, j \in I_n$. If $C(i) = C(j)$ then $f(i) = f(j)$.*

Proof. By Proposition 3.4 i and j are AB equivalent. As in the proof of Proposition 3.5, f is unchanged whenever we pass an $(n + 1)$ th CE and so $f(i) = f(j)$. $\square$

By Proposition 3.8 and since $C : I_n \rightarrow \mathcal{C}_n$ is surjective, any order n invariant f induces a well defined function $u(f) : \mathcal{C}_n \rightarrow \mathbb{G}$. Now $u(f) = u(g)$ iff f and g differ by an invariant of order at most $n - 1$. And so if V_n denotes the group of all invariants on $\mathcal{A}$ of order at most n then $f \mapsto u(f)$ induces an injection $u : V_n / V_{n-1} \rightarrow \mathcal{C}_n^*$ where $\mathcal{C}_n^*$ is the group of all functions from $\mathcal{C}_n$ to $\mathbb{G}$. Finding the image of u for all n would give a classification of all finite order invariants. A first relation that must be satisfied by the image of u follows from Proposition 3.5 above, namely:

Corollary 3.9. *If $g \in \mathcal{C}_n^*$ is in the image of u and if $x \in \mathcal{C}_n$ involves the symbol H^1 or Q^2 then $2g(x) = 0$.*

The purpose of the next section is to specify a subgroup $\Delta_n \subseteq \mathcal{C}_n^*$ which contains the image of u . Δ_n will be the subgroup of $\mathcal{C}_n^*$ which is determined by the relation of Corollary 3.9 above and by all relations obtained by looking at local 2-parameter families of immersions.

In Section 5 we will show that $u : V_1 / V_0 \rightarrow \Delta_1$ is surjective. By this we classify all order one invariants (Theorem 5.1). We will show in Section 7 that for $\mathbb{G} = \mathbb{Z}/2$ and $n > 1$, $u : V_n / V_{n-1} \rightarrow \Delta_n$ is *not* surjective. This we do by demonstrating one particular function in Δ_n which is not attained by u . We will also show that for any $\mathbb{G}$ there exist invariants of any order n .

Question 3.10. What is the image of u for $n > 1$? (given F , $\mathcal{A}$ and $\mathbb{G}$).

We now explain our analogy with knot theory referred to in the introduction. The space I_0 on which we have defined our invariants is the space of all stable immersions of F into $\mathbb{R}^3$. For the circle S^1 , a stable immersion into $\mathbb{R}^3$ is necessarily an embedding i.e. a knot, thus our finite order invariants of immersions are analogous to those of knots. The set $\mathcal{C}_n$ that we have defined above is analogous to the set of all chord diagrams of order n . The analogy is made clear by Propositions 3.4 and 3.8 above. Δ_n which we define below will be analogous to the space of functions on chord diagrams which satisfy the 1-term and 4-term relations. (More precisely, our definition of Δ_n will only use 2-parameter families of immersions, whereas the 1-term relation of knot theory is produced by an essential loop in $Imm(S^1, \mathbb{R}^3)$; but by Remark 5.2 below, the analogous essential loops in our setting produce no additional relations). An analogue of our relation of Corollary 3.9 above, does not exist in knot theory, since the strata of knot theory are globally two sided. Note also that the relation of Corollary 3.9 is obtained from a loop in $I_n \cup I_{n+1}$ whereas all our other relations, (as the relations of knot theory), will

be obtained from loops in $I_{n-1} \cup I_n$. Finally, the analogue for knots of our Question 3.10 has been fully answered in [K] for $\mathbb{G} = \mathbb{Q}$, namely, the analogue of $u : V_n/V_{n-1} \rightarrow \Delta_n$ is surjective for all n .

At this point we would like to further justify our definition of finite order invariants (Definition 3.2). For an invariant to be of order n , its values in a neighborhood of any immersion with $n + 1$ CEs must satisfy a certain restriction; whereas, no restriction is imposed on the invariant in a neighborhood of other, more general immersions of codim $n + 1$. If one wishes to impose such additional restrictions, then one must first note that in general, the stratification of $Imm(F, \mathbb{R}^3)$ in a neighborhood of an immersion of codim $n + 1$ does not resemble that of an immersion with $n + 1$ CEs. And so additional restrictions related to general codim $n + 1$ immersions would require separate definitions suited to the structure of the stratification in each case. Furthermore, if we wish to follow the example set by knot theory, we should note the following: Consider an immersion of S^1 in $\mathbb{R}^3$ which has a triple point, i.e. self intersection equivalent to the intersection of the three coordinate axes in $\mathbb{R}^3$. This is an immersion of codim 3 and the stratification of $Imm(S^1, \mathbb{R}^3)$ in a neighborhood of such immersion happens to be identical to that in a neighborhood of an immersion with three double points, namely, three hypersurfaces intersecting in general position. So, one might want to impose further restrictions on knot invariants of order n using codim $n + 1$ immersions which include such triple point and $n - 2$ double points, in a way identical to that which uses immersions with $n + 1$ double points. Such additional restriction would diminish the space of invariants of order at most n . (In fact, it is easy to show that the space of knot invariants of order at most n with this restricted definition is equal to the space of invariants of order at most $n - 1$ with the usual definition.) We conclude that the option of adding another restriction naturally presents itself in the setting of knot theory, but knot theorists have chosen *not* to do so. By this choice they have established the meaning of finite order invariants as being a restriction related to simplest (i.e. codim 1) singularities occurring in several distinct places, and *not* to other, more complicated singularities.

4. Local analysis

Let $i \in \mathcal{A}$ be an immersion with a self intersection of local codim 2 at p and $n - 1$ additional self-intersections of local codim 1 (i.e. CEs) at $p_1, \dots, p_{n-1}$. We look at a 2-parameter family of immersions which moves F only in a neighborhood of p , such that the immersion i corresponds to parameters $(0, 0)$ and such that this 2-parameter family is transverse to the local codim 2 stratum at i . In this 2-parameter family of immersions we look at a loop which encircles the point of intersection with the codim 2 strata, i.e. a circle around the origin in the parameter plane. This circle crosses the local codim 1 strata some r times. Between each two intersections we have an immersion in I_{n-1} with the same $n - 1$ CEs, at $p_1, \dots, p_{n-1}$. At

each intersection with the local codim 1 strata, an n th CE is added, obtaining an immersion in I_n . Let $i_1, \dots, i_r$ be the r immersions in I_n so obtained and let ϵ_k , $k = 1, \dots, r$ be 1 or -1 according to whether we are passing the n th CE of i_k in the direction of its permanent co-orientation, if it has one. If the CE is of type H^1 or Q^2 then ϵ_k is arbitrarily chosen. Now let $f : I_0 \rightarrow \mathbb{G}$ be an invariant of order n , then it is easy to see that $\sum_{k=1}^r \epsilon_k f(i_k) = 0$. (Note that by Corollary 3.9 the arbitrariness of the choice of ϵ_k for types H^1 and Q^2 has no effect). Looking now at $u : V_n/V_{n-1} \rightarrow \mathcal{C}_n^*$ we obtain relations that must be satisfied by a function in $\mathcal{C}_n^*$ in order for it to lie in the image of u . In this section we will find all relations on $\mathcal{C}_n^*$ obtained in this way. The relations on a $g \in \mathcal{C}_n^*$ will be written as relations on the symbols R_m^a , e.g. $0 = T_m^a - T_m^{3-a}$ will stand for the set of all relations of the form $0 = g([T_m^a, R_{2d_2}^{a_2}, \dots, R_{nd_n}^{a_n}]) - g([T_m^{3-a}, R_{2d_2}^{a_2}, \dots, R_{nd_n}^{a_n}])$ with arbitrary $R_{2d_2}^{a_2}, \dots, R_{nd_n}^{a_n}$. Thus the relations of Corollary 3.9 are written as $0 = 2H_m^1$ and $0 = 2Q_m^2$ (for all m).

In this section we will use our extended set of symbols $\tilde{\mathcal{C}}$ and we denote by $\tilde{\mathcal{C}}_n$ the set of all un-ordered n -tuples of expressions of the form R_m^a where $R^a \in \tilde{\mathcal{C}}$. Recalling Lemma 3.1, we identify $\mathcal{C}_n^*$ with the group of functions on $\tilde{\mathcal{C}}_n$ which satisfy the relations $H_m^0 = -H_m^2$, $Q_m^0 = -Q_m^4$, $Q_m^1 = -Q_m^3$ (these represent relations in the above sense).

We now look at local 2-parameter families of immersions which are transverse to the various local codim 2 strata. These may be divided into six types (see [HK]) which we will name after the types of CEs appearing in the 2-parameter family: EH , TT , ET , HT , TQ , QQ . In the notation of [HK] they are respectively: $A_0^2|A_2$, $A_0^3|A_2$, $(A_0^2|A_1^+)(A_0)$, $(A_0^2|A_1^-)(A_0)$, $(A_0^3|A_1)(A_0)$, A_0^5 .

For each of the first five types we give the following:

1. Formula for a local representative.
2. Sketch of the configuration for some value (λ_1, λ_2) of the parameters.
3. Diagram of the 2 dimensional parameter space, where intersection with the codim 1 strata is depicted, including their co-orientations (this is called a bifurcation diagram).
4. The relation arising.

Note that Corollary 3.9 takes care of the cases when there is actually no co-orientation.

For these five types, the bifurcation diagram is obtained from the sketch and formula in a straight forward manner. Whenever the plane $x = 0$ appears in a configuration below, we assume (by rotating the configuration if necessary) that its preferred side is $x > 0$. The integer m in terms of which the degrees of the CEs are given, is the degree of the central codim 2 immersion at its codim 2 self intersection. (We have originally defined degree only for CEs, but the same definition applies to any self-intersection). We then go on to type QQ ; it requires special analysis which will be done in detail.

 Fig. 1. EH configuration

$$EH: \quad z = 0, \quad z = y^2 + x^3 + \lambda_1 x + \lambda_2.$$

$$0 = E_m^a - H_m^a \quad (1)$$

 Fig. 2. TT configuration

$$TT: \quad z = 0, \quad y = 0, \quad z = y + x^3 + \lambda_1 x + \lambda_2.$$

$$0 = T_m^a - T_m^{3-a} \quad (2)$$

 Fig. 3. ET configuration

$$ET: \quad z = 0, \quad x = 0, \quad z = (x - \lambda_1)^2 + y^2 + \lambda_2.$$

$$0 = T_m^a - T_m^{a+1} - E_{m-1}^a + E_m^a \quad (3)$$

Fig. 4. *HT* configuration

$$HT: \quad z = 0, \quad x = 0, \quad z = (x - \lambda_1)^2 - y^2 + \lambda_2.$$

$$0 = -T_m^{a+1} + T_m^a - H_{m-1}^a + H_m^a \quad (4)$$

Fig. 5. *TQ* configuration

$$TQ: \quad z = 0, \quad y = 0, \quad x = 0, \quad z = y + (x - \lambda_1)^2 + \lambda_2.$$

$$0 = Q_m^a - Q_m^{a+1} - T_{m-1}^a + T_m^a \quad (5)$$

QQ: This configuration is a quintuple point, i.e. five sheets passing through a point, any three of which are in general position. We construct a 2-parameter family of quintuples of oriented planes which represents a local 2-parameter family of immersions which is transverse to the local quintuple point stratum. Let $P_1, \dots, P_5$ be five oriented planes through the origin in $\mathbb{R}^3$ any three of which are in general position. For $k = 1, \dots, 5$ let $u_k \in \mathbb{R}^3$ be the unit vector which is perpendicular to P_k and pointing into the preferred side of P_k in $\mathbb{R}^3$. Any three of

the vectors $u_1, \dots, u_5$ are independent. A vector $(\lambda_1, \dots, \lambda_5) \in \mathbb{R}^5$ will represent the quintuple of planes $P_1^{\lambda_1}, \dots, P_5^{\lambda_5}$ in $\mathbb{R}^3$ where $P_k^\lambda = \{x \in \mathbb{R}^3 : x \cdot u_k = \lambda\}$. (In particular, for each k : $P_k^0 = P_k$).

If $\lambda \in \mathbb{R}$ and $v \in \mathbb{R}^3$ then $P_k^{\lambda+v \cdot u_k}$ is the translate of the plane P_k^λ by the vector v . Let $V \subseteq \mathbb{R}^5$ be the 3-dimensional subspace defined by $V = \{(v \cdot u_1, \dots, v \cdot u_5) : v \in \mathbb{R}^3\}$ then $(\lambda_1, \dots, \lambda_5) \in V$ iff $P_1^{\lambda_1}, \dots, P_5^{\lambda_5}$ all meet at a point. Let $U = V^\perp \subseteq \mathbb{R}^5$ then U represents a local 2-parameter family of immersions which is transverse to the local codim 2 stratum of quintuple points. If A is the 5×3 matrix whose rows are $u_1, \dots, u_5$ then the columns of A span V and so $U = V^\perp$ is the left kernel of A i.e. $U = \{(\lambda_1, \dots, \lambda_5) : \lambda_1 u_1 + \dots + \lambda_5 u_5 = 0\}$.

We now find the points in U which represent configurations involving a CE, which in this case must be a quadruple point. Let $l_k = U \cap (V + \mathbb{R}\epsilon_k)$ where $\epsilon_1, \dots, \epsilon_5$ is the standard basis of $\mathbb{R}^5$ (i.e. $\epsilon_1 = (1, 0, 0, 0, 0)$ etc.). Then l_k represents all configurations in U where the planes $\{P_j : j \neq k\}$ meet at a quadruple point; indeed, an element of l_k is an element of U which is obtained from an element of V i.e. a quintuple point, by adding $r\epsilon_k$ for some r i.e. pushing away the plane P_k .

For a given l_k we would like to determine the configuration of the quadruple point represented by it. Take say l_5 . We must look at the configuration of the four planes $P_1^{\lambda_1}, \dots, P_4^{\lambda_4}$ given by points $(\lambda_1, \dots, \lambda_5)$ on the two sides of l_5 in U , and see how many of the planes $P_1^{\lambda_1}, \dots, P_4^{\lambda_4}$ have the simplex created by them, on their non-preferred side. It is enough to check one point on each side of l_5 in U and as we shall see, it will be most convenient to look at the points of $l_5^\perp$ (where $\perp$ here means the orthogonal complement in U). Now $l_5^\perp = \epsilon_5^\perp \cap U$ (the $\perp$ on the right is the orthogonal complement in $\mathbb{R}^5$); i.e. $l_5^\perp = \{(\lambda_1, \dots, \lambda_4, 0) \in \mathbb{R}^5 : \lambda_1 u_1 + \dots + \lambda_4 u_4 = 0\}$. For such points we can determine the configuration of the simplex using the following lemma:

Lemma 4.1. *Let $v_1, \dots, v_4 \in \mathbb{R}^3$ be four vectors such that any three of them are independent and $v_1 + \dots + v_4 = 0$. If $0 < \mu_1, \dots, 0 < \mu_4 \in \mathbb{R}$ then the origin $0 \in \mathbb{R}^3$ lies in the interior of the simplex determined by the four planes $\{x \in \mathbb{R}^3 : x \cdot v_k = \mu_k\}$.*

Proof. Since $0 \cdot v_k = 0 < \mu_k$, the domain determined by the four planes in which the origin lies is: $D = \{x \in \mathbb{R}^3 : x \cdot v_k \leq \mu_k \text{ for all } 1 \leq k \leq 4\}$. This is a convex domain in $\mathbb{R}^3$. If it is not the simplex determined by the four planes then it is unbounded and so there is a ray based at 0 which is contained in D , i.e. there is a vector $v \neq 0$ such that $rv \cdot v_k \leq \mu_k$ for every $1 \leq k \leq 4$ and any $r > 0$. It follows that for each k : $v \cdot v_k \leq 0$. If for some k , $v \cdot v_k < 0$ then $v \cdot (v_1 + \dots + v_4) < 0$ contradicting $v_1 + \dots + v_4 = 0$. So $v \cdot v_k = 0$ for all k , contradicting the fact that any three of $v_1, \dots, v_4$ are independent. $\square$

Back to our $l_5^\perp$, we use Lemma 4.1 with $v_k = \lambda_k u_k$ and $\mu_k = (\lambda_k)^2, k = 1, \dots, 4$, (note that $\lambda_1, \dots, \lambda_4$ are all non-zero since each three $u_{k's}$ are independent)

obtaining that 0 lies in the interior of the simplex determined by the equations $x \cdot \lambda_k u_k = (\lambda_k)^2$ ($k = 1, \dots, 4$) which is the same as $x \cdot u_k = \lambda_k$ i.e. the planes $P_1^{\lambda_1}, \dots, P_4^{\lambda_4}$. Now, the vector u_k , when based at $P_k^{\lambda_k}$, points into the preferred side of $P_k^{\lambda_k}$ in $\mathbb{R}^3$. On the other hand, u_k points away from the side where the origin lies (which is the side where the simplex lies) iff $\lambda_k > 0$. We conclude that if $(\lambda_1, \dots, \lambda_4, 0) \in l_5^\perp$ then the number of faces of the simplex determined by $P_1^{\lambda_1}, \dots, P_4^{\lambda_4}$ which have the simplex on their non-preferred side, is precisely the number of positive numbers among $\lambda_1, \dots, \lambda_4$.

The origin of U splits $l_5^\perp$ into two half lines. Clearly the number p of plus signs is constant on such a half-line and is $4 - p$ on the other half line. So for given quintuple point, our task is to find the number p of plus signs in each of the ten half lines of the $l_k^\perp$ s. Now $l_k^\perp$ has $\lambda_k = 0$, so it partitions U into the domains where $\lambda_k > 0$ and $\lambda_k < 0$. We thus use it to determine the sign of λ_k in the half-lines of $l_j^\perp$ for $j \neq k$. Examples of such analysis appear in Figure 6, which we now explain: Each of the short thick lines in a diagram represents an $l_k^\perp$. The arrows on each $l_k^\perp$ point to the side of it in U where λ_k is positive. To determine the number p of positive λ_j s corresponding to a given half line of $l_k^\perp$ we need to count for how many $l_j^\perp$ s ($j \neq k$) this half line lies on their positive side, i.e. the side designated by the arrows. This is the number appearing in the circle located on the given half $l_k^\perp$ in the diagram (at the tip of the short thick line). Finally the longer thinner lines are the l_k s themselves (each drawn perpendicular to the corresponding $l_k^\perp$). The pair of numbers at each tip of l_k is simply copied from the corresponding sides of $l_k^\perp$. When passing an l_k , this pair of numbers (which is of the form $p, 4 - p$) tells us the type of quadruple point we are passing, and the co-orientation with which we are passing it, namely, if say we use symbols only in $\mathcal{C}$ then the side chosen by our permanent co-orientation is the side where the larger number of the pair appears. (As before, Corollary 3.9 takes care of the case when there is no co-orientation).

Finally we need to determine the degrees $d_p(i)$ at each of the ten quadruple points. Let m be the degree of the quintuple point and look say at l_1 . We claim that the half of l_1 which has $\lambda_1 > 0$ represents quadruple points with degree m whereas the half with $\lambda_1 < 0$ represents quadruple points with degree $m - 1$. To establish this we need to show that a quadruple point represented by a point in the half of l_1 with $\lambda_1 > 0$ is obtained from a translate of our quintuple point by pushing P_1 into its preferred side i.e. the side pointed at by u_1 . Recall that a point $(\lambda_1, \dots, \lambda_5) \in l_1$ is of the form $v + r\epsilon_1$ where v is perpendicular to U and represents a common translation of all five planes. So $P_1^{\lambda_1}$ is obtained by pushing P_1 away from the quintuple point represented by v , and this push is into the preferred side of P_1 iff $r > 0$. It remains to notice that the sign of λ_1 is the same as that of r , since $(\lambda_1, \dots, \lambda_5)$ is the orthogonal projection of $(r, 0, 0, 0, 0)$ to U . The integers m or $m - 1$ appearing at the tips of each l_k in the diagrams are the degrees. (Recall that the arrows on $l_k^\perp$ point to the side where $\lambda_k > 0$).

 Fig. 6. QQ configuration

Once we have all this information registered at the tips of the l_k s, we can read off the diagram the relation determined by the given quintuple point. We claim that for any configuration of lines and arrows as above, eight of the terms cancel, always leaving the same relation:

$$Q_m^2 = Q_{m-1}^2 \quad (6)$$

We offer two ways of seeing this: The first way is to verify that the four diagrams appearing in Figures 6 are in fact all essentially distinct ways for choosing such distribution of arrows. Then explicitly write out the relation obtained by each of them. We only remark that there is no loss of generality by the fact that the diagrams are sketched with equal angles between the l_k s, since all that is relevant

to us is the cyclic ordering of the $l_k^\perp$ s and not the common cyclic ordering of all $l_1, \dots, l_5, l_1^\perp, \dots, l_5^\perp$.

A second way to see that the same relation $Q_m^2 = Q_{m-1}^2$ is always obtained is as follows: Since we have five arrows pointing clockwise and five counter-clockwise, there must be two consecutive arrows which are pointing toward each other. These half $l_k^\perp$ s must have the same number at their tip, since they are on the same side of any other $l_j^\perp$ and are both on the positive side of each other (i.e. the side designated by the arrow). We now look at the corresponding positive half l_k s. They both have degree m and will both have the same number copied next to them from the half $l_k^\perp$, but written on the opposite side. So their contribution to the relation is precisely the negative of each other. The same will be true for the negative halves of these l_k s (only now with degrees $m-1$). We now erase these two lines from the diagram and argue by (a two step) induction that the formal property analogous to the one we are proving, holds for the remaining three line diagram (checking that it holds for a one line diagram). It remains to observe that once we re-insert the two erased lines, they jointly add precisely 1 to the numbers at the tips of all other short thick lines and so the property continues to hold for the five line diagram.

We denote by $\Delta_n = \Delta_n(\mathbb{G})$ the subgroup of $\mathcal{C}_n^*$ satisfying relations 1–6 above and the relation of Corollary 3.9, namely: $0 = 2H_m^1 = 2Q_m^2$. Recall also that since we are identifying $\mathcal{C}_n^*$ with a set of functions on $\tilde{\mathcal{C}}_n$, we also have the relations $H_m^0 = -H_m^2$ and $Q_m^a = -Q_m^{4-a}$ ($a = 0, 1$).

Now that we have obtained our relations, we return to our original $\mathcal{C}_n$ with only twelve symbols i.e. we dispose of the redundant symbols H^0 , Q^0 , Q^1 . Let $\mathbb{B} \subseteq \mathbb{G}$ be defined by $\mathbb{B} = \{x \in \mathbb{G} : 2x = 0\}$. After some simplification, the relations defining Δ_n may finally be presented as follows:

- $E_m^2 = -E_m^0 = H_m^2$, $E_m^1 = H_m^1$.
- $T_m^0 = T_m^3$, $T_m^1 = T_m^2$.
- $H_m^1 = H_{m-1}^1 \in \mathbb{B}$.
- $Q_m^2 = Q_{m-1}^2 \in \mathbb{B}$.
- $H_m^2 - H_{m-1}^2 = T_m^3 - T_m^2$.
- $Q_m^4 - Q_m^3 = T_m^3 - T_{m-1}^3$, $Q_m^3 - Q_m^2 = T_m^2 - T_{m-1}^2$.

5. Order one invariants

In this section we will show that the injection $u : V_1/V_0 \rightarrow \Delta_1$ is surjective. Let us first give an explicit presentation of Δ_1 . We see from the presentation of the relations appearing in the end of the previous section, that a function $g \in \Delta_1$ may be assigned arbitrary values in $\mathbb{G}$ for the symbols $\{T_m^a\}_{a=2,3, m \in \mathbb{Z}}$, H_0^2 and arbitrary values in $\mathbb{B}$ for the two symbols H_0^1 , Q_0^2 . Once this is done then the value of g on all other symbols is uniquely determined, namely:

1. $H_m^1 = H_0^1$ for all m .
2. $H_m^2 = H_0^2 + \sum_{k=1}^m (T_k^3 - T_k^2)$ for $m \geq 0$.
3. $H_m^2 = H_0^2 - \sum_{k=m+1}^0 (T_k^3 - T_k^2)$ for $m < 0$.
4. $E_m^0 = -H_m^2$, $E_m^1 = H_m^1$, $E_m^2 = H_m^2$ for all m .
5. $T_m^0 = T_m^3$, $T_m^1 = T_m^2$ for all m .
6. $Q_m^2 = Q_0^2$ for all m .
7. $Q_m^3 = Q_m^2 + T_m^2 - T_{m-1}^2$ for all m .
8. $Q_m^4 = Q_m^3 + T_m^3 - T_{m-1}^3$ for all m .

Let X denote the set of symbols $\{T_m^a\}_{a=2,3,m \in \mathbb{Z}} \cup \{H_0^2\}$ then we have obtained that $\Delta_1(\mathbb{G}) \cong \mathbb{G}^X \oplus \mathbb{B} \oplus \mathbb{B}$ where $\mathbb{G}^X$ denotes the group of all functions from X to $\mathbb{G}$. We can also present $\Delta_1(\mathbb{G})$ through a universal object as follows: We define a universal Abelian group $\mathbb{G}_U$ by the Abelian group presentation $\mathbb{G}_U = \langle \{t_m^a\}_{a=2,3,m \in \mathbb{Z}}, h_0^2, h_0^1, q_0^2 \mid 2h_0^1 = 2q_0^2 = 0 \rangle$. Then we define the universal element $g_U \in \Delta_1(\mathbb{G}_U)$ by $g_U(T_m^a) = t_m^a$ ($a = 2, 3$), $g_U(H_0^2) = h_0^2$, $g_U(H_0^1) = h_0^1$, $g_U(Q_0^2) = q_0^2$ and the value of g_U on the other symbols of $\mathcal{C}_1$ is determined by formulae 1–8 above, so indeed $g_U \in \Delta_1(\mathbb{G}_U)$. Then for arbitrary Abelian group $\mathbb{G}$ we have $\Delta_1(\mathbb{G}) \cong \text{Hom}(\mathbb{G}_U, \mathbb{G})$ where the isomorphism maps a homomorphism $\phi \in \text{Hom}(\mathbb{G}_U, \mathbb{G})$ to the function $\phi \circ g_U \in \Delta_1(\mathbb{G})$. We will show that there is an order 1 invariant $f_U : I_0 \rightarrow \mathbb{G}_U$ such that its extension to I_1 induces g_U on $\mathcal{C}_1$, i.e. $u(f_U) = g_U$. It will follow that for any group $\mathbb{G}$, $u : V_1/V_0 \rightarrow \Delta_1(\mathbb{G})$ is surjective, since if $g \in \Delta_1(\mathbb{G})$ and $g = \phi \circ g_U$ where $\phi \in \text{Hom}(\mathbb{G}_U, \mathbb{G})$ then $u(\phi \circ f_U) = g$.

We choose a base immersion $i_0 \in I_0$ once and for all. Given any $i \in I_0$ we take a generic regular homotopy from i_0 to i i.e. a path in $\mathcal{A}$ from i_0 to i transverse with respect to the global stratification. The value of f_U on i will be the sum with signs, of the values of g_U on the CEs that we pass along our path from i_0 to i , where the signs are determined by whether we are passing the given CE in the direction of its co-orientation. (Note that by definition of $\mathbb{G}_U$, whenever there is no co-orientation then the element we are adding is of order 2). We must show that this sum is independent of our choice of path, or equivalently that it is 0 along any closed path. First we observe that it is 0 on null-homotopic loops. Indeed, by slightly deforming the null-homotopy, we may assume it too is transverse with respect to the stratification. We may then break the null-homotopy into small pieces and clearly the value accumulated when going around our loop is the sum of the values going around each little piece. For appropriate partition there will be four types of such pieces according to their intersection with the strata, this intersection may

1. be empty,
2. be an arc of I_1 ,
3. be two intersecting arcs of I_1 (their intersection point being in I_2),
4. contain a point of local codim 2.

In all four cases the sum going around the little piece is 0. For cases 1 and 2 this is trivial. For case 3 this is true since local motion near one of the CEs does not effect the symbol R_m^a attached to the other CE. For case 4 this is true since $g_U \in \Delta_1(\mathbb{G}_U)$.

Once we have established that this sum in $\mathbb{G}_U$ is 0 for any null-homotopic loop, we have a well defined homomorphism $\pi_1(\mathcal{A}) \rightarrow \mathbb{G}_U$ and we must show that this homomorphism is 0. We first show this for the case $F = S^2$ in which case there is only one regular homotopy class, which we still name $\mathcal{A}$. Now $\pi_1(\mathcal{A}) = \mathbb{Z} \oplus \mathbb{Z}/2$ where the unique order 2 element is represented by one full rotation of S^2 in $\mathbb{R}^3$. Such rotation passes through no CEs at all, and so gives 0 as needed. Let $K \subseteq \pi_1(\mathcal{A})$ be the subgroup generated by this order 2 element in $\pi_1(\mathcal{A})$ and let $P = P(S^2) = \pi_1(\mathcal{A})/K$, then we are left with showing that the map induced on $P (\cong \mathbb{Z})$ is 0.

If $H_t : S^2 \rightarrow \mathbb{R}^3$ is a regular homotopy, then attaching to H_t one or more of the superscripts $\mathbb{R}^3$, S^2 and T will denote respectively, composition from the left with the antipodal map of $\mathbb{R}^3$, composition from the right with the antipodal map of S^2 and reversal of time. Note that these three operations on H_t commute with each other. Let $p : S^2 \rightarrow \mathbb{R}P^2$ be the double covering and let $i : \mathbb{R}P^2 \rightarrow \mathbb{R}^3$ be some immersion. Let $s = i \circ p$ then $s : S^2 \rightarrow \mathbb{R}^3$ is an immersion satisfying $s(-x) = s(x)$. Let H_t be a regular homotopy from the inclusion $e : S^2 \rightarrow \mathbb{R}^3$ to s . Let $G_t = H_t * (H_t^{S^2, T})$ where $*$ denotes concatenation from left to right, then G_t is a regular homotopy from e to $-e$ i.e. an eversion of the sphere. Also $G_t^{S^2, T} = G_t$. We now define $F_t = G_t * G_t^{\mathbb{R}^3} = G_t * (G_t^{S^2, T})^{\mathbb{R}^3} = G_t * (G_t^{S^2, \mathbb{R}^3})^T$. From the presentation of F_t as $G_t * G_t^{\mathbb{R}^3}$ we can see that there are isotopies $U_t : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ and $V_t : S^2 \rightarrow S^2$ such that $U_t \circ F_t \circ V_t$ satisfies the conditions of [N1] Proposition 2.1. It follows that F_t represents an odd power of the generator of P . We also have $F_t = G_t * (G_t^{S^2, \mathbb{R}^3})^T$ and let G'_t be a generic regular homotopy which is a slight perturbation of G_t (G_t was highly non-generic), then $F'_t = G'_t * (G'^{S^2, \mathbb{R}^3}_t)^T$ will still represent an odd power of the generator of P (though it will not be precisely equal to $G'_t * G'^{\mathbb{R}^3}_t$). We now observe that reversing the orientations of both S^2 and $\mathbb{R}^3$ simultaneously, preserves all our definitions of types, degrees and co-orientations of CEs and so $G'^{S^2, \mathbb{R}^3}_t$ produces precisely the same element in $\mathbb{G}_U$ as G'_t and so $(G'^{S^2, \mathbb{R}^3}_t)^T$ produces the negative of that element. So we get that the value on the loop F'_t is 0. Since F'_t represents an odd power of the generator of P and since in $\mathbb{G}_U$ an odd multiple of any non-zero element is still non-zero, we get that the value is 0 on the generator of P and so on all P . This completes the proof for the case $F = S^2$.

Now let F be of higher genus and let $\mathcal{A}$ be a given regular homotopy class. Again $\pi_1(\mathcal{A}) \cong \mathbb{Z} \oplus \mathbb{Z}/2$. Again our claim is clear for $K \subseteq \pi_1(\mathcal{A})$ which is defined as above and so again it is enough to look at the map induced on $P = P(F, \mathcal{A}) = \pi_1(\mathcal{A})/K$.

Looking back for a moment at S^2 let H_t be a loop representing a generator of $P(S^2)$ and we may assume H_t fixes a given disc $D \subseteq S^2$. Choose a point x on a ray perpendicular to the fixed image of D in $\mathbb{R}^3$, x being far enough so that the image of H_t never passes x . Now change the constant embedding of D to be one with a thin “thorn” pulled out of D and embedded along this ray, reaching x . The new loop H'_t obtained in this way also represents the generator of $P(S^2)$ and has the property that a given point in the fixed disc D (namely, the tip of the thorn) does not participate in any self intersections throughout H'_t . Since we have already proved our result for S^2 , we know that the symbols encountered along the loop H'_t add up to 0 in the group $\mathbb{G}_U$. Now take a tiny immersion of F in I_0 , located near the tip of our fixed thorn, and connect sum it with H'_t (for all t), obtaining a loop $F_t : F \rightarrow \mathbb{R}^3$ in $\mathcal{A}$. It follows from the proof of [N1] Theorem 3.4 that this loop $F_t : F \rightarrow \mathbb{R}^3$ represents a generator of $P(F, \mathcal{A})$. Since the tip of the thorn was far away from any self intersections occurring in H'_t , the tiny fixed immersion of F will not cause any new CEs and will not affect the degrees of the CEs of H'_t and so will not alter the sum of symbols attained by H'_t , which we know is 0 in $\mathbb{G}_U$. We conclude that f_U is well defined; it is evident that f_U is of order 1 and $u(f_U) = g_U$. This completes the proof of our main result:

Theorem 5.1. *For any closed orientable surface F , regular homotopy class $\mathcal{A}$ of immersions of F into $\mathbb{R}^3$ and Abelian group $\mathbb{G}$, the injection $u : V_1/V_0 \rightarrow \Delta_1$ is surjective.*

Remark 5.2. If $i \in I_{n-1}$ and $x \in i(F)$ is not a self intersection point then let B be a small neighborhood of x in $\mathbb{R}^3$ and $D = i^{-1}(B) \subseteq F$. By grafting a generator of $P(S^2)$ as above, we may move D in B as to produce a loop in $I_{n-1} \cup I_n \subseteq \mathcal{A}$ which generates $P(F, \mathcal{A})$. A priori such a loop could have added another relation to those with which we have defined Δ_n , but it follows from the work of this section that such loop does not add any new relation. (This is in contrast with knot theory where the analogous loop produces the 1-term relation. Recall that we are viewing knots in the space of immersions rather than in the space of all maps).

6. Eversions of the sphere

We present an application of Theorem 5.1. Let $e : S^2 \rightarrow \mathbb{R}^3$ denote the inclusion. An eversion of the sphere is a regular homotopy $H_t : S^2 \rightarrow \mathbb{R}^3$ between e and $-e$. In [MB] it has first been established that the number of CEs of type Q occurring in any generic eversion of the sphere, is odd.

Given a generic eversion of the sphere, and given a class v of CEs, we denote by $N(v)$ the number of CEs of type v occurring during this eversion, counted with signs if all CEs of v have a co-orientation, and counted mod 2 if v includes CEs of type H^1 or Q^2 . We may thus write the result of [MB] as follows: For any eversion of the sphere,

$$(1) N(Q) = 1 \pmod{2}.$$

In [G], three more such equations satisfied by any eversion of the sphere have been established, namely,

$$(2) N(T) = 0$$

$$(3) N(H^2) + N(E^2) - N(E^0) = -1$$

$$(4) N(H^1) + N(E^1) = 0 \pmod{2}.$$

We now show how to deduce these four equations from our work, and then establish an infinite list of new such equations satisfied by any eversion of the sphere.

The Morin eversion described in [M] has 14 CEs. By direct inspection they are: (a $\pm$ sign appears where there is no co-orientation)

$$E_1^0 + E_2^2 + T_2^2 \pm H_2^1 \pm H_2^1 + T_2^1 - Q_2^4 - H_1^2 - T_1^2 \pm H_0^1 \pm H_0^1 - T_1^1 - E_0^0 - E_1^2.$$

Via the relations defining Δ_1 this simplifies to:

$$Q_0^2 - H_1^2.$$

And so any order one invariant f on $Imm(S^2, \mathbb{R}^3)$ based at e , (i.e. having $f(e) = 0$) will satisfy:

$$f(-e) = u(f)(Q_0^2) - u(f)(H_1^2).$$

Recall that we have denoted $X = \{T_m^a\}_{a=2,3,m \in \mathbb{Z}} \cup \{H_0^2\}$ and let us now denote $Y = X \cup \{H_0^1, Q_0^2\}$. By Theorem 5.1 and the opening paragraph of Section 5, an order one invariant f on $Imm(S^2, \mathbb{R}^3)$, with values in an Abelian group $\mathbb{G}$ and based at e , may be defined by choosing arbitrary values for $u(f)(x)$ in $\mathbb{G}$ for all symbols $x \in X$ and arbitrary values in $\mathbb{B} = \{a \in \mathbb{G} : 2a = 0\}$ for H_0^1 and Q_0^2 . Then the values on $\mathcal{C}_1 - Y$ are determined by formulae 1–8 of Section 5.

Let f_1 be the $\mathbb{Z}/2$ valued invariant with $g_1 = u(f_1)$ satisfying $g_1(Q_0^2) = 1 \in \mathbb{Z}/2$ and g_1 is 0 on all other symbols of Y . By 1–8 of Section 5 we then get $g_1(Q_m^a) = 1 \in \mathbb{Z}/2$ for any a, m and g_1 is 0 on all other symbols of $\mathcal{C}_1$. So for any eversion of the sphere, $N(Q) = f_1(-e) = g_1(Q_0^2) - g_1(H_1^2) = 1 \pmod{2}$ which is (1) above. Let f_2 be the $\mathbb{Z}$ valued invariant defined by $g_2(T_m^a) = 1$ for $a = 2, 3$ and all m and $g_2(H_0^2) = g_2(H_0^1) = g_2(Q_0^2) = 0$. By 1–8 of Section 5, $g_2(T_m^a) = 1$ for all a, m and g_2 is 0 on all other symbols in $\mathcal{C}_1$ and so for any eversion of the sphere $N(T) = f_2(-e) = g_2(Q_0^2) - g_2(H_1^2) = 0$ which is (2) above. In the same way define the $\mathbb{Z}$ valued invariant f_3 by $g_3(H_0^2) = 1$ and g_3 is 0 on the rest of Y . This leads to $g_3(H_m^2) = g_3(E_m^2) = 1$, $g_3(E_m^0) = -1$ for all m and g_3 is 0 on the rest of $\mathcal{C}_1$, getting for any eversion of the sphere $N(H^2) + N(E^2) - N(E^0) = f_3(-e) = g_3(Q_0^2) - g_3(H_1^2) = -1$ which is (3). Finally Let g_4 be the $\mathbb{Z}/2$ valued invariant defined by $g_4(H_0^1) = 1 \in \mathbb{Z}/2$ and g_4 is 0 on the rest of Y . This leads to $g_4(H_m^1) = g_4(E_m^1) = 1 \in \mathbb{Z}/2$ for all m and g_4 is 0 on the rest of $\mathcal{C}_1$, getting for any eversion of the sphere $N(H^1) + N(E^1) = f_4(-e) = g_4(Q_0^2) - g_4(H_1^2) = 0 \pmod{2}$ which is (4).

In the same way, we now produce an infinite list of new, finer equations, satisfied by any eversion of the sphere. For any pair b, r with $b \in \{2, 3\}$, $r \in \mathbb{Z}$ we define a $\mathbb{Z}$ valued invariant f_{br} by $g_{br}(T_r^b) = 1$ and if $r \leq 0$ then also $g_{br}(H_0^2) = (-1)^{b-1}$, and g_{br} is 0 on the rest of Y . This leads, by 1–8 of Section 5 to the following: $g_{br}(T_r^{3-b}) = g_{br}(T_r^b) = 1$, $g_{br}(H_m^2) = g_{br}(E_m^2) = -g_{br}(E_m^0) = (-1)^{b-1}$ for $m \geq r$, $g_{br}(Q_r^a) = 1$ for $a \geq b+1$, $g_{br}(Q_{r+1}^a) = -1$ for $a \geq b+1$, and g_{br} is 0 on the rest of $\mathcal{C}_1$. This gives for any eversion of the sphere $N(T_r^b) + N(T_r^{3-b}) + (-1)^{b-1} \left(N(H_{\geq r}^2) + N(E_{\geq r}^2) - N(E_{\geq r}^0) \right) + N(Q_r^{\geq b+1}) - N(Q_{r+1}^{\geq b+1}) = f_{br}(-e) = g_{br}(Q_0^2) - g_{br}(H_1^2)$, where e.g. $N(H_{\geq r}^2)$ means the number (with signs) of CEs occurring during the eversion which are of type H_m^2 with $m \geq r$. So finally:

Proposition 6.1. *For each pair b, r with $b \in \{2, 3\}$, $r \in \mathbb{Z}$, the following equation is satisfied by any eversion of the sphere:*

$$N(T_r^{b,3-b}) + (-1)^{b-1} \left(N(H_{\geq r}^2) + N(E_{\geq r}^2) - N(E_{\geq r}^0) \right) + N(Q_r^{\geq b+1}) - N(Q_{r+1}^{\geq b+1}) = \begin{cases} (-1)^b & r \leq 1 \\ 0 & r > 1 \end{cases}$$

We note that equations (2) and (3) above follow from Proposition 6.1. Indeed, given an eversion of the sphere $H_t : S^2 \rightarrow \mathbb{R}^3$, let k and K be the minimal and maximal degrees occurring in H_t respectively. For $r = \min(1, k-2)$ (and either b) Proposition 6.1 yields $N(H^2) + N(E^2) - N(E^0) = -1$ for H_t . Summing the equation of Proposition 6.1 over $k-1 \leq r \leq K$ and $b = 2, 3$ yields $N(T) = 0$ for H_t . (In fact $N(T) = 0$ is an obvious equation following from the fact that the number of triple points in both e and $-e$ is 0).

7. Higher order invariants

We now show that for $\mathbb{G} = \mathbb{Z}/2$ and $n > 1$ the map $u : V_n/V_{n-1} \rightarrow \Delta_n$ is not surjective. We define a function $g : \mathcal{C}_n \rightarrow \mathbb{Z}/2$ as follows: If $x \in \mathcal{C}_n$ includes at least one symbol which is not of type Q , then $g(x) = 0$ and if all symbols in x are of type Q then $g(x) = 1$. One verifies directly that g satisfies the relations appearing in the end of Section 4 and so $g \in \Delta_n(\mathbb{Z}_2)$. We will show that there is no order n invariant f such that $u(f) = g$. Let $f : I_0 \rightarrow \mathbb{Z}/2$ be an invariant of order n and assume $u(f) = g$. Since $\mathbb{G} = \mathbb{Z}/2$, there is no need for co-orientations for the extension of f and so f extends to I_k for any $k > 0$, in particular to I_{n-1} . Let $i, j \in I_{n-1}$ be two AB equivalent immersions such that all $n-1$ CEs of i and j are of type Q . If H_t is a (generic) AB equivalence between i and j then there are some finitely many times along H_t for which an n th CE appears. By definition of

g , the number mod 2 of such n th CEs which are of type Q is $f(i) - f(j) \in \mathbb{Z}/2$. In particular if $i = j$ then the number of such occurrences of an n th quadruple point is 0 mod 2.

Now take some $i \in I_{n-1}$ with $n - 1$ CEs located at $p_1, \dots, p_{n-1}$ all of type Q and furthermore the CE at p_1 is of type Q^2 . Let H_t be an AB equivalence from i to itself which fixes F in neighborhoods of $p_2, \dots, p_{n-1}$ and reverses the co-orientation of the CE at p_1 (as in the proof of Proposition 3.5). Still assuming the existence of an f such that $u(f) = g$, by the above discussion the number of quadruple point occurrences along H_t is 0 mod 2. Let $i' \in I_0$ be an immersion which is obtained from i by slightly deforming i in a neighborhood of each $p_k, k = 1, \dots, n - 1$. H_t induces a regular homotopy H'_t from i' to an immersion which differs from i' only in a neighborhood of p_1 and there are 0 mod 2 quadruple point occurrences along H'_t , precisely those occurring during H_t . We complete H'_t to a closed loop ending at i' causing one more quadruple point to occur, namely that at p_1 . And so we have constructed a loop in $I_0 \cup I_1$ (transverse to I_1) with 1 mod 2 quadruple point occurrences. This would imply (by our above discussion again) that the analogous order 1 invariant does not exist. But we have shown (Theorem 5.1) that for order 1, all invariants defined in this way do exist, a contradiction. (This order 1 invariant appeared as f_1 in Section 6.) We conclude that there is no order n invariant f such that $u(f) = g$ and so for $\mathbb{G} = \mathbb{Z}/2$ and $n > 1$ the map $u : V_n/V_{n-1} \rightarrow \Delta_n(\mathbb{Z}/2)$ is not surjective.

We remark that the order 1 invariant whose existence has been used in the above proof, namely that order 1 invariant which counts the number mod 2 of quadruple points occurring along generic regular homotopies, has been studied in [N1], [N3]. In [N1] its existence has been established for all closed surfaces (not only orientable) and also the existence of the analogous invariant in general 3-manifolds, under certain conditions. In [N3] an explicit formula has been given for the value of this invariant on embeddings.

Another remark: In [N2] the notion of a q -invariant has been defined. This is an invariant $f : I_0 \rightarrow \mathbb{Z}/2$ such that its extension to I_1 satisfies that $f(i) = 0$ if the CE of i is not a quadruple point. It follows that $f(i) = 0$ for any $i \in I_k$ ($k \geq 1$) which includes at least one CE which is not a quadruple point. If f is of order n then this means that $u(f)(x) = 0$ for any $x \in \mathcal{C}_n$ which includes at least one symbol which is not of type Q . It has been shown in [N2] and also follows from the relations defining Δ_n (end of Section 4), that if such f of order n exists, then $u(f)(x) = 1$ for any $x \in \mathcal{C}_n$ for which all symbols are of type Q . So what we have shown in this section is that q -invariants of order $n > 1$ do not exist, by this completing the work of [N2]. (The unique q -invariant of order 1 is the invariant discussed in the previous paragraph).

We conclude by showing that for any $\mathbb{G}$ there exist invariants of any order n . Let $p_n(x) = \binom{x+n}{n}$, then $p_n(x)$ is a polynomial of degree n satisfying the relation: $p_n(x+1) - p_n(x) = p_{n-1}(x+1)$. Now fix an $n \geq 0$ and an element $0 \neq g \in \mathbb{G}$

and define $f : I_0 \rightarrow \mathbb{G}$ by $f(i) = p_n(t(i))g$ where $t(i)$ denotes half the number of triple points occurring in i . (The number of triple points in a stable immersion of a closed orientable surface into $\mathbb{R}^3$ is known to be even, see [B]). One shows inductively that for $i \in I_k$, $f(i) = 0$ if at least one of the k CEs of i is not of type T and $f(i) = p_{n-k}(t(i) + k)g$ if all k CEs are of type T , where $t(i)$ is half the number of *stable* triple points occurring in i (i.e. we do not count the type T CEs as triple points) and where we use the permanent co-orientation of these type T CEs when computing $f(i)$. In particular $f(i) = g \neq 0$ for any $i \in I_n$ for which all n CEs are of type T and $f(i) = 0$ for all $i \in I_{n+1}$ and so f is an invariant of order precisely n .

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