# On Agent Types in Coalition Formation Problems

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## ABSTRACT

Coalitions and cooperation are key topics in multi–agent systems (MAS). They enable agents to achieve goals that they may not have been able to achieve independently. A range of previous studies have found that many problems in coalitional games tend to be computationally intractable - that is, the computational complexity grows rapidly as a function of the number of participating agents. However, these hardness results generally require that each agent is of a different *type*. Here, we observe that in many MAS settings, while the number of agents may grow, the number of different types of agents remains small. We formally define the notion of *agent types* in cooperative games. We then re-examine the computational complexity of the different coalition formation problems when assuming that the number of agent types is fixed. We show that most of the previously hard problems become polynomial when the number of agent types is fixed. We consider multiple different game formulations and representations (characteristic function with subadditive utilities, CRG, and graphical representations) and several different computational problems (including stability, core-emptiness, and Shapley value).

## Categories and Subject Descriptors

F.2 [**Theory of Computation**]: Analysis of Algorithms and Problem Complexity;
I.2.11[**Artificial Intelligence**]: Distributed Artificial Intelligence — *Multiagent Systems*

## General Terms

Algorithms, Theory

## Keywords

Coalition problems, Agent Types

## 1. INTRODUCTION

In multi–agent systems (MAS), where each agent has limited resources, coalitions of agents are a very powerful tool [1, 9, 10]. Coalitions enable agents to accomplish goals they may not have been able to accomplish independently. As such, understanding and predicting the dynamics of coalitions formation, e.g. which coalitions are more beneficial, stable and/or more likely to emerge, is a question of considerable interest in multi-agent settings.

A range of previous studies have shown that many of these problems are computationally intractable - that is, the computational complexity (most probably) grows nonpolynomially as a function of input size - in general, and the number of participating agents - in particular ([7, 11, 12]). However, a close analysis of these hardness results reveals that the proofs generally require that each agent be of a different *type* (intuitively, agents are of the same type if their contribution and utility are the same under identical circumstances). In practice, however, we observe that in many MAS settings this is not the case; in many cases, while the number of agents grows, the number of different types of agents remains small.

Accordingly, we re-examine the computational complexity of the different coalition problems, assuming that the number of different agent types is fixed. We analyze the structure of relevant coalitions under this assumption, and show that many of the problems that were proved hard under the general case, are polynomial when the number of agent types is fixed.

**Results and Contributions.** We first provide a formal definition of the notion of agent *types* in cooperative games (Section 2). In fact, we provide two notions: *strategic types* and *representational types*. The former defines types based on the the strategic power of the agents. The latter defines types based on the representation of the game. We consider the relation between the two notions, and some of their characteristics.

We then consider three different known game representations. First we consider the graphical representation of cooperative games, introduced by Deng and Papadimitriou [3] (Section 3). For this model [3] proved that the problem of coalition stability and the question whether the core is empty are both $\mathcal{NP}$-complete. We show that both problems are polynomial if the number of types is fixed. We also show that for this representation the two notions of types coincide.

Next, we consider Conitzer and Sandholm's concise representation of superadditive games in characteristic form [2] (Section 4). First we show that for this game representation it is $\mathcal{NP}$-hard to determine strategic types. Interestingly, if the number of types is fixed, then agents types can be nonetheless determined. We then consider coalition formation questions in this model. Conitzer & Sandholm [2] show that deciding whether the core is empty is $\mathcal{NP}$-hard.

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We show that when the number of agent types is fixed, the problem becomes polynomial. We also show that computing the Shapley value in this case is polynomial.

Finally, we consider the Coalition Resource Game (CRG) model [12] (Section 5). For this model we again prove that in the general case it is $\mathcal{NP}$-hard to determine strategic types. We then show that a host of problems that were previously proven to be hard in the general case, are tractable when the number of agent types is fixed. These problems include the question if a specific coalition is successful; if there exists a successful coalition; if a specific coalition is maximally successful; multiple questions regarding efficient utilization of resources; and more.

The significance of these results is two-fold. Firstly, since it is not uncommon that the number of agent types is small, it is important to know which problems are tractable in this case. Secondly, and more generally, we believe that these results exhibit the important point that computational hardness should not be considered as a universal impediment to any algorithmic solution to the problem. Real-life instances frequently have some additional structure that render them easier to solve than the general case, let alone the worst-case underlying the hardness result. We believe that uncovering such simplifying structures is no less important than providing general hardness results.

## 1.1 Definitions and Notations

**Cooperative Games.** We now provide some basic definitions and notations on cooperative games used throughout the paper. The reader is referred to [6] for a more complete exposition. We consider a game with a finite set $Ag$ of $n$ agents/players (we use the terms players and agents interchangeably). A coalition, $C$, is a (sub)set of agents, $C \subseteq Ag$. The *grand coalition* is the set of all agents, $Ag$. In a *cooperative game* each coalition is assigned a *value*. Formally,

DEFINITION 1.1. *A* cooperative game (with transferrable utilities) *is determined by the set of agents Ag and a characteristic* value function $v : 2^{Ag} \rightarrow R$, *assigning a value to each possible coalition.*

We note that the value function need not be provided explicitly. A *payoff vector*, $(p_1, ..., p_n)$, is an allocation of proceeds to the different agents. Given a payoff vector $p = (p_1, ..., p_n)$ and coalition $C$, we denote $p(C) = \sum_{i \in C} p_i$. In general, we mostly focus on allocating the proceeds of the grand coalition. A payoff vector is *efficient* if it allocates all the value of the grand coalition amongst the players, i.e. $p(Ag) = v(Ag)$. A payoff vector is *individually rational* if for any agent $i$, $p_i \geq v(\{i\})$ (otherwise the agent is better off working alone).

DEFINITION 1.2. *An* imputation *is a payoff vector that is efficient and individually rational.*

There are many possible payoff vectors and imputations for a game. However, under certain models of rational behavior of the agents, only some of them are possible or stable.

The notion of a *blocking coalition* is central in defining stability:

DEFINITION 1.3. *A coalition* $C$ blocks *payoff vector* $p = (p_1, ..., p_n)$ *if* $p(C) < v(C)$.

If $C$ blocks payoff vector $p$, then the members of $C$ would rather break-off and collectively gain $v(C) - p(C) > 0$. Thus, if $C$ blocks the payoff $p$, the agents of $C$ have an incentive to break off. Hence, the payoff vector is *unstable*. The core is a prominent solution concept focusing on such stability.

DEFINITION 1.4. *The core of a coalitional game is the set of all payoff vectors that are not blocked by any coalition.*

Another prominent solution concept is the Shapley value, which focuses on *fairness*, rather than on *stability*. The Shapley value defines a single payoff vector. The Shapley value of agent $i$ is defined as:

$$\phi(i) = \sum_{C \subseteq Ag \setminus \{i\}} \frac{|C|!(|Ag| - |C| - 1)!}{|Ag|!} (v(C \cup \{i\}) - v(C))$$

**Fixed Parameterized Tractability (FPT).** In this paper we consider the complexity of problems in the case that the number of agents types of fixed. This is closely related to the general theory of *parameterized complexity* in general, and *fixed parameter tractability (FPT)*, in particular (see [5]). It is out of the scope of this paper to introduce this theory. However, we do make a note when the solutions we provide are FPT.

# 2. DEFINING AGENT TYPES

Intuitively, agents are of the same type if they have identical characteristics in the game. An attempt to formalize this notion results in two different possible definitions.

**Strategic Types.** The first definition considers the strategic power of the players. In this respect, players are of the same type if they have the same strategic power in the game. Formally, we consider the marginal contribution of the players:

DEFINITION 2.1. *Agents* $i, j \in Ag$ *are* strategically equivalent *if for any coalition* $C$*, such that* $i, j \notin C$: $v(C \cup \{i\}) = v(C \cup \{j\})$.

The definition is identical to that of *symmetric players* in the Shapley value ( [6], page 436).

CLAIM 2.1. *Strategic equivalence of agents is an equivalence relation.*

PROOF. By definition the relation is reflexive and symmetric. It remains to show that it is transitive. Suppose $i_1, i_2$ are strategically equivalent and so are $i_2, i_3$. Consider a coalition $C$, $i_1, i_3 \notin C$. Consider two cases. If $i_2 \notin C$, then, by the equivalence of $i_1, i_2$, $v(C \cup \{i_1\}) = v(C \cup \{i_2\})$, and by the equivalence of $i_2, i_3$, $v(C \cup \{i_2\}) = v(C \cup \{i_3\})$. Hence $v(C \cup \{i_1\}) = v(C \cup \{i_3\})$. If $i_2 \in C$, let $C' = C - \{i_2\}$. Then, by the equivalence of $i_1, i_2$, $v((C' \cup \{i_3\}) \cup \{i_1\}) = v((C' \cup \{i_3\}) \cup \{i_2\})$, and by the equivalence of $i_2, i_3$, $v((C' \cup \{i_1\}) \cup \{i_2\}) = v((C' \cup \{i_1\}) \cup \{i_3\})$. Hence, $v(C \cup \{i_1\}) = v((C' \cup \{i_1\}) \cup \{i_2\}) = v((C' \cup \{i_3\}) \cup \{i_2\}) = v(C \cup \{i_3\})$. □

Therefore, strategic equivalence partitions the agents into equivalence classes, each of which we call a *strategic type*.

**Representational Types.** The definition of strategic types considers the true strategic power of the players. However, when presented with a game, even strategically equivalent players may be described differently. For example, consider a weighted threshold game with three players, 1, 2 and 3, with weights 30, 40 and 50, respectively. If the threshold is 60 then all three players, while described differently, are strategically equivalent. At times, one wants to consider the types of players as described in the game representation, rather than by their strategic power. Thus, we introduce the notion of *representational equivalence*, which formalizes the notion that two players are described identically. In order to do so, we note that when representing a game, players are necessarily associated with some identifier; most commonly a name or number, but possibly also a node in a graph, etc. Intuitively, players are representationally equivalent, if they only differ in their identifier. Formally,

DEFINITION 2.2. *Let $\Gamma$ be a game, and let $R(\Gamma)$ be the representation of $\Gamma$. Let $i_1, i_2$ be two players in the game, with identifiers $id_1$ and $id_2$, respectively. We say that players $i_1, i_2$ are* representationally equivalent *if giving agent $i_1$ the identifier $id_2$ and agent $i_2$ the identifier of $id_1$ would result in an identical representation, $R(\Gamma)$, for the game.*

As an example, suppose that we have three players, A(lice), B(ob), and C(arol). The values for the coalitions are: $v(\text{A,B}) = 1, v(\text{A,C}) = 2, v(\text{B,C}) = 2$ and $v(\text{A,B,C}) = 3$. Suppose that we use a representation that represents the game as a list of coalition values, listed in lexicographic order. Suppose we give A the identifier x, B - the identifier z, and C - the identifier y. Then, the representation of the game is: $(2, 1, 3, 2)$ (the lexicographic order is xy, xz, xyz, yz). Now, suppose that we exchange identifiers between A and B (A$\mapsto$z and B$\mapsto$x). Then the representation of the game remains the same - $(2, 1, 3, 2)$. Thus, A and B are representationally equivalent. If, however, we exchange between the identifiers of A and C (A$\mapsto$y and C$\mapsto$x), the representation of the game changes to $(2, 2, 3, 1)$. Thus, A and C are not equivalent.

It is immediate that representational equivalence is indeed an equivalence relation, which partitions the agents into equivalence classes. We call each such class a *representational type*. Clearly, if two agents are representationally equivalent they are also strategically equivalent. Thus, representational types form a refinement of strategic types (not necessarily strict).

**Coalition Types.** We extend the notion of types from single agents to coalitions.

DEFINITION 2.3. *We say that coalitions $C$, $C'$ are* equivalent (strategic or representational) *if for any agent type (strategic or representational, res.) both have the same number of members of the given type.*

We now establish that the value of a coalition is only determined by its type, not by the actual members.

LEMMA 2.1. *If $C$, $C'$ are equivalent (either strategically or representationally) then $v(C) = v(C')$.*

PROOF. We prove for strategic types. Since representational types are a refinement of strategic types, the lemma also holds for representational types.

Let $C = \{c_1, c_2, \ldots, c_t\}$ and $C' = \{c'_1, c'_2, \ldots, c'_t\}$. For each agent type, both coalitions have the same number of agents of the type. Thus, we may assume that they are ordered in a way such that for each $k = 1, \ldots, t$, $c_k$ and $c'_k$ are equivalent. For $k = 0 \ldots, t$, define a *hybrid* coalition that has its first $k$ elements from $C$ and the rest from $C'$, $C^{(k)} = \{c_1, \ldots, c_k, c'_{k+1}, \ldots, c'_t\}$. Then, $C = C^{(0)}$ and $C' = C^{(t)}$. For any $k$, $C^{(k-1)} = \{c_1, \ldots, c_{k-1}, c'_{k+1}, \ldots, c'_t\} \cup \{c'_k\}$ and $C^{(k)} = \{c_1, \ldots, c_{k-1}, c'_{k+1}, \ldots, c'_t\} \cup \{c_k\}$. Thus, the only difference between $C^{(k-1)}$ and $C^{(k)}$ is that one has $c_k$ and the other $c'_k$. However, $c_k$ and $c'_k$ are strategically equivalent. Thus, $v(C^{(k-1)}) = v(C^{(k)})$. Thus, all the $v(C^{(k)})$'s are equal, and $v(C) = v(C')$. □

**Algorithms.** By definition, there is a simple algorithm to determine if two agents are representationally equivalent: exchange their respective identifiers and examine the resulting representation. This is not necessarily the case for strategic equivalence. Later we will see examples for which it is computationally hard to determine if two agents are strategically equivalent.

**Notations.** We denote by $T$ the set of types (strategic or representational, depending on the context), and by $t$ the number of such types. We use lower case letters from the beginning of alphabet $(a, b, c, \ldots)$ to denote the agents types (either strategic or representational). For a type $a$, we denote by $A_a$ the agents of $Ag$ of type $a$, and $n_a$ the number of agents of this type.

# 3. THE GRAPHICAL COALITION GAME REPRESENTATION

Deng and Papadimitriou [3] introduced the graphical representation of coalition games, as follows. Consider an agents set $Ag$, and let $H = (Ag, E)$ a weighted undirected graph, with a weight function $w : E \rightarrow R^+$ on the edges. The nodes of the graph correspond to the agents of the game. The payoff to a coalition $C \subseteq Ag$ is the sum of the weights in the subgraph induced by $C$. Note that w.l.o.g. we may assume that $H$ is the complete graph, as non-existing edges can simply get weight zero. Given a graph $H$ and weight function $w$ (as above) we denote by $\Gamma = (H, w)$ the resulting cooperative game.

In [3] it was shown that the problems of determining whether an imputation is in the core and whether the core is empty are both $\mathcal{NP}$-complete. Here we show that when the number of agent types is fixed then both problems are polynomial.

## 3.1 Determining Agent Types

We first note that in graphical games, strategic types necessarily coincide with representational types. This is established by the following lemma.

LEMMA 3.1. *Let $\Gamma = (H, w)$ be a graphical coalition game. Then, agents $i_1, i_2$ are strategically equivalent iff they are representationally equivalent.*

PROOF. In the graphical representation, players are identified as nodes of the graph. Consider two agents $i_1, i_2$, corresponding to nodes $h_1$ and $h_2$, respectively. They are representationally equivalent iff for any other node $h_3$, $w(h_3, h_1) = w(h_3, h_1)$.

As noted above (Section 2), if $i_1, i_2$ are representationally equivalent they are also strategically equivalent. Conversely, suppose $i_1, i_2$ are strategically equivalent. In particular, for any singleton coalition $\{i_3\}$ $(i_3 \neq i_1, i_2)$, $v(\{i_3\} \cup \{i_1\}) = v(\{i_3\} \cup \{i_2\})$. However, $v(\{i_3\} \cup \{i_1\}) = w(h_3, h_1)$ and $v(\{i_3\} \cup \{i_2\}) = w(h_3, h_2)$. Thus, for every $h_3$, $w(h_3, h_1) = w(h_3, h_1)$. Thus, $i_1, i_2$ are representationally equivalent. □

Accordingly, in the case of graphical representation we do not differentiate between strategic and representational types. Note that since representational equivalence is easy to determine (see Section 2), we obtain:

COROLLARY 3.1. *For coalition games in graphical representation, it is polynomially tractable to determine the types of agents (strategic and representational).*

## 3.2 Core Non-Empty

The problem is defined as follows:

CORE–NON–EMPTY (CORE-EMPTY)
*Instance:* A cooperative game $\Gamma$.
*Question:* Is the core of $\Gamma$ non–empty?

THEOREM 3.1. *For cooperative games in graphical representation, if the number of agents types, $t$, is fixed, then* CORE–NON–EMPTY *is polynomially tractable (FPT).*

PROOF. In order to prove this we will first show that only specific types of imputations and specific types of blocking coalitions need be considered. This is established by the following two lemmata.

We say an imputation is *symmetric* if it grants all agents of the same type identical amounts. The following lemma immediately follows directly from the convexity of the core.

LEMMA 3.2. *If the core is non-empty then it contains a symmetric imputation.*

We say that coalition $C$ is *saturated* if for any type $a$, $C$ either contains all agents of the type or none.

LEMMA 3.3. *If a symmetric imputation $p$ is not in the core then it is blocked by some saturated coalition.*

PROOF. Let $p$ be a symmetric imputation not in the core. Then, by definition, it is blocked by some coalition $C$. Suppose that $C$ is not saturated. Then, in order to prove the lemma it is sufficient to show that if we add to $C$ a single agent of any type contained in $C$, the resulting coalition still blocks $p$. The lemma then follows by induction.

For type $a$ let $c_a$ be the number of agents of type $a$ in $C$. For each type $a$, let $p_a$ be the amount the imputation $p$ grants to agents of type $a$. Since $C$ blocks $p$, we have:

$$\begin{aligned} p(C) &= \sum_{a \in T} c_a p_a & (1) \\ &< \sum_{a,b \in T, a \neq b} c_a c_b w(a,b) + \sum_{a \in T} \frac{c_a(c_a-1)}{2} w(a,a) \\ &= v(C) \end{aligned}$$

For a type $a$ define *marginal contribution* of $a$ to $C$ as the additional value the players of type $a$ add to the value of $C$,

$$\begin{aligned} M(a,C) &= v(C) - v(C \setminus A_a) \\ &= \sum_{b \in T, b \neq a} c_a c_b w(a,b) + \frac{c_a(c_a-1)}{2} w(a,a) \end{aligned}$$

Clearly, if for any $a$, $M(a,C) < c_a p_a$ then (by Equation (1)) we can omit all members of type $a$ from $C$ and still get a blocking coalition. Thus, we can assume, WLOG, that for all types $a$

$$c_a p_a \leq M(a,C) = \sum_{b \in T, b \neq a} c_a c_b w(a,b) + \frac{c_a(c_a-1)}{2} w(a,a)$$

For types such that $c_a > 0$, i.e. those present in $C$, we may divide both sides by $c_a$, obtaining,

$$p_a \leq \sum_{b \in T, b \neq a} c_b w(a,b) + \frac{c_a - 1}{2} w(a,a) \quad (2)$$

Let $a_0$ be a type present in $C$ for which not all members appear in $C$. Let $i$ be an agent of type $a_0$ not in C, and define $C' = C \cup \{i\}$. Then, combining (1) with (2) we obtain:

$$\begin{aligned} p(C') &= \sum_{a \in T} c_a p_a + p_{a_0} \\ &< \sum_{b \in T, a \neq b} c_a c_b w(a,b) + \sum_{a \in T} \frac{c_a(c_a-1)}{2} w(a,a) \\ &\quad + \sum_{b \in T, b \neq a_0} c_b w(a_0,b) + \frac{c_{a_0} - 1}{2} w(a_0,a_0) \\ &= v(C') \end{aligned}$$

Thus, $C'$ also blocks $p$, as required. □

Thus, in order to check if the core is not empty we need only determine if there is a symmetric imputation that is not blocked by any saturated coalition. To this end we construct a LINEAR PROGRAMMING representation of the problem. The LINEAR PROGRAMMING problem we define shall be a satisfiability problem (rather than an optimization problem). That is, it consists only of a set of constraints, and the question is whether there exists a solution to this set.

The variables of the linear program will represent the symmetric imputation $p$, i.e. a variable $p_a$ for each type $a$. The program is the following:

$$\sum_{a \in T} n_a p_a = v(Ag) \quad (3)$$

$$\forall C \subset T:$$

$$\sum_{a,b \in C, a \neq b} n_a n_b w(a,b) + \sum_{a \in C} \frac{n_a(n_a-1)}{2} w(a,a) \leq \sum_{a \in C} n_a p_a \quad (4)$$

The first constraint (3) ensures that the imputation is efficient. The second set of constraints (4) ensures that there are no saturated coalitions that block this imputation.

The number of variables in the program is $t$ and the number of constraints is $2^t$. Thus, for a fixed $t$, the program can be solved in polynomial time. Furthermore, the run time is FPT in $t$.

## 3.3 In-Core

The problem is defined as follows:

IN THE CORE (IN-CORE)
*Instance:* A cooperative game $\Gamma$ and an imputation $x$.
*Question:* Is $x$ in the core?

The problem was found to be $\mathcal{NP}$-complete in [3].

THEOREM 3.2. *For cooperative game in graphical representation, if the number of agents, $t$, is fixed, then* IN THE CORE *is polynomially tractable.*

PROOF. We prove by providing an algorithm that checks if there is a blocking coalition or not, and takes $O(2^t \cdot |Ag|^t)$ steps.

In order to make sure that an imputation $p$ is in the core, one must make sure that for each possible subset of agents (coalition) the value granted to this coalition by the imputation function is at least the value the coalition's agents can achieve by themselves. Otherwise, this coalition will deviate from the grand coalition and achieve its own value, and therefore will block the imputation.

Thus, it seems that the algorithm must check all possible subsets of agents. However, when checking all possible coalitions, it does not really matter which agent (node) is exactly in the coalition – but only its type. This is due to the fact that given that we have $c_a$ agents of type $a$ in the coalition, then we can always assume that the coalition in question holds the $c_a$ agents of type $a$ that get the smallest allocation from the imputation.

Therefore, the algorithm only needs to consider all possible subgroups of $T$ and in each subgroup - it needs to consider all possible sizes of each $a \in T$. Overall, the number of coalitions to consider is bounded by $2^t \cdot |Ag|^t$. For each such coalition the algorithm will calculate the value the coalition can achieve by itself (polynomial time) and the value it gains from the imputation (again polynomial). If there exists a coalition that can achieve higher value by itself — the imputation is not in the core. Otherwise, the imputation is in the core. □

# 4. CONCISE REPRESENTATION OF SUB-ADDITIVE GAMES

This model was introduced by Conitzer and Sandholm in [2]. Here, we focus on the case of transferrable utilities, defined as follows. We consider a cooperative game defined over a set $Ag$ of agents, with subadditive values for the coalitions. The characteristic function of the game is represented by a set of pairs:

$$W = \{\langle B, v(B)\rangle : B \subseteq Ag\}.$$

The set $W$ provides explicit values for some coalitions. These values form the basis for determining the values of all coalitions as follows. For any coalition $C$, the value of $C$ is the maximum aggregate value it can obtain by partitioning itself into sub-coalitions with explicitly defined values in $W$, i.e.

$$v(C) = \max\{\sum_{i=1}^{r} v(B_i) : (B_1, \ldots, B_r) \text{ is a partition of C,} \\ and: \ \forall i \ \langle B_i, v(B_i)\rangle \in W\}.$$

For this model, Conitzer & Sandholm prove that determining if the core is non-empty is $\mathcal{NP}$-complete. Here we show that when the number of agent types is fixed the problems becomes polynomial. In addition, we consider the problem of computing the Shapley value for this model and show that it is also polynomial for a fixed number of types. The problem of computing the Shapley value was not considered in [2], but was proven hard in many other models.

## 4.1 Determining Strategic Types

For the concise game representation, strategic and representational types need not coincide. As an example, consider the game with four agents, 1,2,3 and 4, and representation:

$$W = \{\langle\{1, 2\}, 1\rangle, \langle\{1, 2, 3\}, 1\rangle\}$$

In this case, the marginal contribution of 3 and 4 to any coalition is 0, so they are strategically equivalent. However, they are described differently. We now show that in the general case it is computationally intractable to determine the strategic type of agents in this representation.

THEOREM 4.1. *Given a subadditive game in concise representation, it is $\mathcal{NP}$–hard to decide if two players are strategically equivalent.*

PROOF. The proof is by reduction from EXACT-3-COVER. Given a universe $U = (u_1, \ldots, u_n)$ ($n$ divisible by 3) and a collection $C = \{\{x_1, y_1, z_1\}, \ldots, \{x_m, y_m, z_m\}\}$ of subsets of $U$, we construct a sub-additive game in concise form $\Gamma$ and two players $i_1, i_2$, such that $i_1$ and $i_2$ are strategically equivalent in $\Gamma$ iff there is an exact-3-cover solution for $(U, C)$. Set $Ag = U \cup \{i_1, i_2\}$. For each triplet $\{x_k, y_k, z_k\} \in C$, let $\langle\{x_k, y_k, z_k\}, 1\rangle \in W$. In addition, $\langle U \cup \{i_1\}, n/3\rangle \in W$. Then, the marginal contribution of $i_2$ is always zero. If there is an exact-3-cover for $U$, then $U$ alone can reach a value of $n/3$ and the marginal contribution of $i_1$ is also zero, and $i_1, i_2$ are strategically equivalent. Otherwise, $i_1$ has a marginal contribution to $U$ while $i_2$ does not, and they are not equivalent. □

This would seem to imply that it is intractable to determine if the number of types is bounded by a constant $t$. This, however, is not the case. We now show for any fixed $t$, it is polynomially tractable to decide if the number of types is at most $t$, and if so, to determine the types of all agents. To this end, we first prove:

LEMMA 4.1. *Let $\Gamma$ be a cooperative game in concise form where agents are of at most $t$ types (strategic or representational). Suppose that the type of each agent is provided. Then, for any fixed $t$, for any coalition $C$, the value of $C$, $v(C)$, can be computed in polynomial time (FPT).*

PROOF. We construct an INTEGER LINEAR PROGRAMMING representation of the problem, as follows. Recall that the input to the problem is a set of pairs $W = \{\langle B, v(B)\rangle : B \subseteq Ag\}$. The integer variables of the program shall be of the form $x_B$, for each $B$ such that $\langle B, v(B)\rangle \in W$. For each agent type $a$ and coalition $B$, let $n(a, B)$ be the number of agents of type $a$ contained in $B$. The program is:

$$Maximize \sum_{B:\langle B,v(B)\rangle\in W} v(B)x_B$$

$$s.t.:$$

$$\forall a \in T \sum_{B:\langle B,v(B)\rangle\in W} x_B n(a, B) = n(a, C)$$

The set of constraints ensure that, in total, the amounts we took from each $B$ form a partition of $C$.

INTEGER LINEAR PROGRAMMING can be calculated in complexity that is exponential only in the number of constraints ($t$) and polynomial in the rest of the input size ([5], page 222). Therefore, for a fixed $t$, we can solve the program in polynomial time, and obtain the value for $v(C)$. Furthermore, the algorithm is also FPT. □

THEOREM 4.2. *Let $\Gamma$ be a cooperative game in concise form. For any fixed $t$, it is polynomially tractable to decide whether the number of strategic types in $\Gamma$ is at most $t$, and if so, to determine the type of each agent.*

PROOF. We determine the equivalence classes inductively, first considering coalitions of size at most 1, then 2, etc. For sizes $s = 1, \ldots, n$, we say that two agents, $i_1$ and $i_2$, are *$s$-equivalent* if for any coalition $C$, of size at most $s - 1$, such that $i_1, i_2 \notin C$, $v(C \cup \{i_1\}) = v(C \cup \{i_2\})$. Thus, $n$-equivalence is strategic equivalence as in Definition 2.1. Also note that $s + 1$-equivalence is a refinement of $s$-equivalence. Thus, the number of $s$-equivalence classes is never greater than the number of strategic types.

For $s = 0$, all agents are equivalent. Suppose the equivalence classes for $s$-equivalence are provided, we determine those for $s+1$-equivalence. Consider two agents $i_1$ and $i_2$ that are $s$-equivalent. They are not $s+1$-equivalent if there exists a coalition $C$ such that $|C| = s$, $i_1, i_2 \notin C$, and $v(C \cup \{i_1\}) \neq v(C \cup \{i_2\})$. By Lemma 2.1 we need not check all coalitions $C$, but rather only the different coalition types. There are at most $s^t = O(n^t)$ different such coalition types. For each such type we do the following. Let $\mathcal{B}_1$ be the collection of sets $B$ containing $i_1$, for which a value is provided in $W$, i.e. $\mathcal{B}_1 = \{B : i_1 \in B \text{ and } \langle B, v(B)\rangle \in W\}$. Then,

$$v(C \cup \{i_1\}) = \max_{B \in \mathcal{B}_1} \{v(B) + v(C \setminus B)\} \qquad (5)$$

(where $v(C \setminus B)$ is not necessarily provided explicitly). Clearly, the size of $\mathcal{B}_1$ is bounded by $|\Gamma|$. If the number of $s$-types is at most $t$, then, by Lemma 4.1, $v(C \setminus B)$ can be computed in polynomial time. Thus, Equation (5) can be computed in polynomial time. Similarly for $v(C \cup \{i_2\})$. Thus $s+1$-equivalence can be determined in polynomial time. There are $n^2$ agents pairs to check. Thus, the $s+1$-equivalence classes can be determined in polynomial time. If ever the number of classes exceeds $t$, we stop and report that the number of strategic types exceeds $t$. Otherwise, for $s = n$, we obtain the strategic types. □

## 4.2 Core Non-Empty

The problem, defined in Section 3.2 was found to be $\mathcal{NP}$-hard for concise representation [2].

Theorem 4.3. *For subadditive cooperative games in concise representation, if the number of types, $t$ (representational or strategic), is fixed, then* Core–Non–Empty *is polynomially tractable (FPT).*

Proof. Conitzer & Sandholm [2] show that if the value of the grand coalition, $v(Ag)$, is provided, then deciding core–non-empty is polynomial. By Lemma 4.1, for a fixed $t$, computing this value is polynomial (FPT). □

## 4.3 Shapley Value

The problem is defined as follows.

Shapley Value (shapley)
*Instance:* A cooperative game $\Gamma$ and agent $i$.
*Question:* What is the Shapley value of agent $i$?

Theorem 4.4. *For subadditive cooperative games in concise representation, if the number of agents types (representational or strategic), $t$, is fixed, then computing the Shapley value of any agent is polynomial.*

Proof. Recall that the Shapley value of an agent $i$ is defined as:

$$\phi(i) = \sum_{C \subseteq Ag \setminus \{i\}} \frac{|C|!(|Ag| - |C| - 1)!}{|Ag|!} (v(C \cup \{i\}) - v(C))$$

In general, this computation can be exponential as one needs to go over all possible coalitions $C$. However, the key observation is that we need only consider the different *coalitions types.* With this observation, we obtain that the Shapley value can be expressed as follows. Let $T = \{a_1, a_2, \ldots, a_t\}$ be the set of types (representational or strategic), and recall that for each type $a$, we denote by $n_a$ the number of agents in $Ag$ of the type. For integers $k_1, k_2, \ldots, k_t$, let $C(k_1, k_2, \ldots, k_t)$ be the coalition type obtained by taking $k_1$ agents of type $a_1$, $k_2$ agents of type $a_2$, etc. For brevity we provide the formula for an agent of type $a_1$. The formula for agents of other types is analogous. The Shapley value for agent $i$ of type $a_1$ can be expressed as:

$$\phi(i) = \sum_{k_1=0}^{n_{a_1}-1} \sum_{k_2=0}^{n_{a_2}} \cdots \sum_{k_t=0}^{n_{a_t}} \binom{n_{a_1}-1}{k_1} \binom{n_{a_2}}{k_2} \cdots \binom{n_{a_t}}{k_t} \frac{|C(k_1,\ldots,k_t)|!(|Ag| - |C(k_1,\ldots,k_t)| - 1)!}{|Ag|!} \cdot (v(C(k_1,\ldots,k_t) \cup \{i\}) - v(C(k_1,\ldots,k_t)))$$

By Lemma 4.1, for a fixed $t$ the value of each summand can be computed in polynomial time. The number of summands in the formula is $< \prod_{k=1}^{t} n_{a_k} = O(n^t)$. Thus, for a fixed $t$, the value of the formula can be computed in polynomial time. (The algorithm is not FPT since $t$ appears in the exponent of $n$.) □

# 5. THE COALITION RESOURCES GAME MODEL

The CRG model introduced in [12], is defined as follows. The model postulates three types of elements: a set of *agents*, $Ag$, a set of *goals*, $G$, and a set of *resources*, $R$. These are related to each other in the following way. Each agent $i$ is associated with a subset of goals $G_i \subseteq G$. Achieving any goal in $g \in G_i$ renders agent $i$ *satisfied.* Goals are *achieved* by having the agents contribute *resources.* Different goals may require different amounts of each resource type. The quantity $\mathbf{req}(g, r)$ denotes the amount of resource $r$ required in order to achieve goal $g$. Each agent, in turn, is *endowed* with certain amounts of some or all of the resources. The quantity $\mathbf{en}(i, r)$ denotes the amount of resource $r$ endowed to agent $i$. It is assumed that both $\mathbf{req}(g, r)$ and $\mathbf{en}(i, r)$ are natural numbers. Each agent aims to become satisfied, while contributing a minimum of its own resources [4]. To this end, it may join with other agents to form coalitions that together achieve their mutual goals.

The CRG model, as described, does not fully confirm to the general cooperative game structure as defined in Section 1.1, as it does not provide a concrete value function for coalitions. Also, the models assumes non-transferrable utilities. We consider this model since it is frequently used for studying agent behavior, on the one hand, and many coalition formation problems have been found to be hard in the model, on the other [12]. Here we show that when the number of agent types is fixed then many of these problems become polynomial.

**Notations.** For a coalition $C$ and a set of goals $G'$, we say that $G'$ *satisfies* $C$ if achieving all goals in $G'$ renders all members of $C$ satisfied, i.e. for each $i \in C$, $G_i \cap G' \neq \emptyset$. We denote by *sat*$(G')$ the coalition of all agents satisfied by $G'$. For a coalition $C$ and a set of goals $G'$, we say that $G'$ is *feasible* for $C$ if the agents of $C$, collectively, have sufficient endowment to achieve all goals of $G'$ simultaneously, specifically, for each $r \in R$, $\sum_{i \in C} \mathbf{en}(i, r) \geq \sum_{g \in G'} \mathbf{req}(g, r)$. Finally, for a coalition $C$, we denote by *sf*$(C)$ the collection of goal-sets that are both feasible for $C$ and satisfy $C$:

$$sf(C) = \{G' \subseteq G : (G' \text{ satisfies } C) \text{and} (G' \text{ is feasible for } C)\}.$$

A coalition $C$ is *successful* if *sf*$(C)$ is non-empty, i.e. it can make all its members satisfied by using contributions from its members alone.

## 5.1 Defining Strategic Types

An noted, the CRG model does not provide a characteristic value function for coalitions. Thus, Definition 2.1 (*strategic equivalence*) does not apply. We now provide an analogous definition for CRG games, similarly based on the marginal contribution of agents.

Recall that each agent's goal is to become satisfied, while contributing a minimum of its resources. Consider two agents $i_1$ and $i_2$. Intuitively, the agents are strategically equivalent if their marginal contribution to any coalition is identical. However, in the CRG model, specifying the coalition alone is not sufficient. We also need to specify how much each agent contributes in each resource. We denote by $\mathbf{con}(i, r)$ the contribution of resource $r$ by agent $i$. A contribution function $\mathbf{con}(\cdot, \cdot)$ is *feasible* if for each resource, each agent contributes at most its endowment of that resource.

DEFINITION 5.1. *Consider a CRG game, and let $i_1$ and $i_2$ be two agents in the game. We say that $i_1$* dominates *$i_2$ if there exists a coalition $C$ and feasible contribution function, $\mathbf{con}(\cdot, \cdot)$, for its members, such that $i_1$ can satisfy $C \cup \{i_2\}$ by adding a contribution $\mathbf{con}(i_1, r), r \in R$, while $i_2$ cannot satisfy $C \cup \{i_2\}$ with an identical or lesser contribution (in all resources), either because it does not have enough endowment, or the contribution will not satisfy $C \cup \{i_2\}$.*

*Agents $i_1$ and $i_2$ are* strategically equivalent *if neither $i_1$ dominates $i_2$ nor $i_2$ dominates $i_1$.*

Similarly to Claim 2.1, it can be shown that strategic equivalence for CRG is indeed an equivalence relation. We call each of the resulting equivalence classes a *strategic type.*

Note that in CRG games, strategic types do not necessarily correspond to representational types. As an example, consider two agents both of which are satisfied by the same set of goals. Assume further that one agent has no endowed resources, and the other agent is endowed with a resource not required by any goal. Then, they are both strategically equivalent, though they are represented differently. Analogously to Lemma 2.1 the performance of a coalition is only determined by its type, not by the actual members.

LEMMA 5.1. *Let $C = \{c_1, \ldots, c_t\}$ and $C' = \{c'_1, \ldots, c'_t\}$ be equivalent coalitions, ordered so that for each $j = 1, \ldots, t$, $c_k$ and $c'_k$ are equivalent (strategically or representatively). Then for any contribution function $\mathbf{con}(\cdot, \cdot)$, such that for any resource $r$ and any $j$, $\mathbf{con}(c_j, r) = \mathbf{con}(c'_j, r)$, $C$ is satisfied iff $C'$ is satisfied.*

The proof is analogous to that of Lemma 2.1 and is omitted.

## 5.2 Determining Strategic Types

We first show that in the general case it is computationally hard to decide strategic equivalence.

THEOREM 5.1. *In CRG games, it is $\mathcal{NP}$-hard to determine if two agents are strategically equivalent.*

PROOF. The proof is again by reduction from EXACT-3-COVER. Consider a universe $U = (u_1, \ldots, u_n)$ ($n$ divisible by 3) and a collection $A = \{a_1, \ldots, a_m\}$ of subsets of $U$ each of size 3, $a_j = \{j_1, j_2, j_3\}$, $j = 1, \ldots, m$. We construct a CRG game as follows. The agent set is $U \cup \{i_1, i_2\}$. For each agent $u_i \in U$ there is a unique resource $r_i$, such that $u_i$ is endowed with one unit of $r_i$. For each $a_j$ we have a unique goal $g(a_j)$, such all and only agents in $a_j$ are satisfied by $g(a_j)$. For $a_j = \{j_1, j_2, j_3\}$ the goal $g(a_j)$ requires one unit of each of the resources $r_{j_1}, r_{j_2}$ and $r_{j_3}$. In addition, there is one additional goal $\hat{g}$, and one additional resource $\hat{r}$. Agent $i_1$ is endowed with one unit of $\hat{r}$. The goal $\hat{g}$ satisfies all agents in $U$ and requires one unit of each resource type (all $r_i$'s and $\hat{r}$). Finally, agent $i_2$ is endowed with no resources and both $i_1$ and $i_2$ are always satisfied.

Clearly, the only possible difference in marginal contribution between agents $i_1$ and $i_2$ could be if $i_1$ contributes its unit of $\hat{r}$ while $i_2$ cannot. This could only possibly benefit the coalition containing all agents of $U$, each of which contributes its single unit of resource ($r_i$ for agent $u_i$). However, if there is an exact-3-cover, then with the same contribution $U$ alone is satisfied without the need for $i_1$'s contribution. This is by each agent $u_i$ contributing to the goal corresponding to the set that covers it in the exact-cover. Thus, the contribution of $i_1$ is non-material and $i_1$ and $i_2$ are equivalent. Conversely, if the agents of $U$ can all be satisfied without the contribution of $i_1$, then this must necessarily be with the goals of the form $g(a_j)$. Since each agent contributes exactly one and each goal requires exactly 3, at most $n/3$ goals can be achieved. Thus, by construction, the achieved goals correspond to an exact-3-cover. We obtain that $i_1$ and $i_2$ are equivalent iff there is an exact-3-cover. □

As we have seen (Section 4.1), this does not preclude the option that it is polynomial to determine the strategic types in the case that the number of types is fixed. At this time, we do not know to prove that this is indeed the case, nor to disprove it. We leave this as an open problem. In the following we assume that the types are somehow known. This can either be representational types, which are easy to determine, or strategic types that have somehow been determined.

## 5.3 Successful Coalition

The Successful Coalition problem was introduced in [11] as the most fundamental question that could be asked on coalitions and proven to be $\mathcal{NP}$-complete in [12].

SUCCESSFUL COALITION (SC)
*Instance:* CRG $\Gamma$, and coalition $C$.
*Question:* Is $C$ successful?

We now show that if the number of types is constant then the problem is polynomial.

CLAIM 5.1. *Let $C$ be a coalition and $G'$ a set of goals. Then, it is polynomially decidable if $G'$ satisfies $C$ and to decide if $G'$ is feasible for $C$.*

PROOF. $G'$ satisfies $C$ iff for each $i \in C$, $G_i \cap G' \neq \emptyset$. $G'$ is feasible for $C$ iff for for each resource $r$, $\sum_{i \in C} \mathbf{en}(i, r) \geq \sum_{g \in G'} \mathbf{req}(g, r)$. Both are easily checkable in polynomial time. □

THEOREM 5.2. *Let $\Gamma$ be CRG game, where agents are of at most $t$ types, and $C \subseteq Ag$ a coalition. Then for any fixed $t$ it is polynomially tractable to decide whether $C$ is a successful coalition.*

PROOF. The key observation is that for any given type, all agents of the type can be satisfied by achieving the same goal. Therefore, it is sufficient to consider goal sets $G'$ of size at most $t$. There are $< (|G| + 1)^t$ such sets. By Claim 5.1 it is polynomial to determine if $G'$ satisfies and is feasible for $C$. Hence, the problem can be decided in time $O(((|G| + 1)^t \cdot poly|\Gamma|)$, which is polynomial for a fixed $t$. □

In a similar way we can prove that the following problems are also polynomial for fixed $t$: Maximal Coalition problem, the Maximal Successful Coalition problem, the Necessary Resource problem, the Strictly Necessary Resource problem, the Successful Coalition with Resource Bounds problem and the $(C, G*, r)$- Optimal problem [12]. All these problems were shown to be hard for the general case.

## 5.4 Existence of Successful Coalition

The problem is defined as follows:

Existence of Successful Coalition of Size $k$ (esck)
*Instance:* A CRG $\Gamma$ and integer $k$.
*Question:* Does there exist a successful coalition of size (exactly) $k$?

The problem was considered in [8], and was found to be $\mathcal{NP}$–hard.

Theorem 5.3. *Let $\Gamma$ a CRG game where agents are of at most $t$ types, and $k \in \mathbb{N}$ an integer. Then for any fixed $t$ it is polynomially tractable to decide whether there exists a successful coalition of size $k$ in $\Gamma$.*

Proof. The key observation is that by Lemma 5.1 we need only consider coalition types. There are at most $k^t = O(n^t)$ different coalition types of size $k$. For each such type we run the algorithm of Theorem 5.2. The entire process is thus polynomial. □

## 5.5 Pareto Efficiency

A goal set $G^*$ is $R$-Pareto efficient with respect to some coalition $C$, iff there is no other goal set in $sf(C)$ which requires less than $G^*$ in all resources. I.e. $\forall G'' \in sf(C)$ :

$$[\exists r_1 \in R : \mathbf{req}(G'', r_1) < \mathbf{req}(G^*, r_1)] \quad \Rightarrow \quad [\exists r_2 \in R : \mathbf{req}(G'', r_2) > \mathbf{req}(G^*, r_2)] \tag{6}$$

Determining if goal set is $R$-Pareto efficient was found to be co-$\mathcal{NP}$-complete in [12].

Theorem 5.4. *Let $\Gamma$ be a game in CRG form, where agents are of at most $t$ types, $C \subseteq Ag$ a coalition, and $G^* \subseteq G$ a goal set. Then for any fixed $t$ it is polynomially tractable to decide whether $G^*$ is the $R$-pareto efficient for $C$.*

Proof. Here again we need only check goals sets of size at most $t$. For each such set it is easy to check if Equation (6) holds, and by Claim 5.1 polynomial to check if in $sf(C)$. □

# 6. CONCLUSIONS AND FUTURE WORK

In this paper we re-examined the complexity of coalition formation problems in cooperative games in light of the notion of agents types. We introduced two notions of types, *strategic types* and *representational types*, and showed that they do not necessarily correspond. Furthermore, we showed that for some representations, e.g. CRG and concise representation of subadditive games, it is computationally hard to determine strategic types. We also showed that if the number of types is constant, then many previously known intractable problems become polynomial. This phenomena was established for several different models: graphical representation, CRG and concise subadditive representation. A recurring underlying theme behind these results is that when the number of agent types is constant, while the overall number of coalitions is exponential, the number of coalitions to consider is only polynomial. The breadth of different models and problems for which we could establish the results gives us reason to believe that similar results also hold for other models and related problems.

From a practical point of view, the results suggest that when faced with a coalition formation problem one should consider not only the number of agents, but also their types. If the number of types is small, it may well be that the problem is tractable. Thus, in some settings, coalition problems in the cooperative models are perhaps easier than previously presumed.

There are many avenues for future work. First and foremost, one should examine additional coalition problems and additional models. It would interesting to find cases for which the problems remain hard even with a constant number of types. More generally, we believe that the notion of agent types is relevant for other problems as well. It would be interesting to see other problem domains where intractable problems become tractable when the number of agent types is constant.

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