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GAUSS DECOMPOSITIONS OF
KAC-MOODY GROUPS

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Dedicated to Eiichi Abe on his 70th birthday

Abstract. We present an axiomatic approach to both a Gauss decomposition of a Kac-Moody group and a Gauss decomposition of the associated Steinberg group. We study also a prescribed version in case of rank 2.

1. AXIOMATIC APPROACH

Here we call $(G,U,T,V,\{\phi_1, \cdots, \phi_n\})$ a triangular system (or a Gauss system) if

1. $G$ is a group, and $U, T, V \leq G$ are subgroups,
2. $\phi_i : SL_2(K) \to G$ is a group homomorphism of $SL_2$ over a field $K$ into $G$ with
(3) $G = \langle U, V_1, \ldots, V_n \rangle$, and $TU_i = U_i T$ for $1 \leq i \leq n$.

(4) There exist the subgroups of $G$ called $U'_i$ and $V'_i$ for $1 \leq i \leq n$ such that

$U = U'_1 U_1 = U_i U'_i$,

$V = V'_1 V_1 = V_i V'_i$,

$V_i U'_i = U'_i V_i$,

$U_i V'_i = V'_i U_i$.

Then we can establish

$G = UVTU$

$= \bigcup_{u \in U} u(VTU)u^{-1}$

(see below). We call this decomposition a Gauss decomposition of $G$.

First, we shall review the situation in the case when $G = SL_2(K), \ n = 1, \ \phi_1 = \text{identity}$ and

$U = U_1, \ T = T_1, \ V = V_1$.

Let us take an element

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(K).$$

Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 & b/a \\ 0 & 1 \end{pmatrix}$$

if $a \neq 0$. Otherwise, $c \neq 0$ and

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 & b/a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b/a \\ 0 & c \end{pmatrix}.$$ 

Hence,

$SL_2(K) = VTVU \cup u(VTVU)u^{-1}$

$= UVTU$

with

$$u = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$ 

Now, we shall return to a general case. We want to confirm $G = UVTU$. Let $X = UVTU \subset G$. Then $UX = X$ and
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Theorem 1. Every triangular system has a Gauss decomposition.

We can find such a system for a standard Kac-Moody group and for a Marcuson-type Kac-Moody group. One can also find a Gauss decomposition for the Steinberg groups associated with Kac-Moody groups. We will discuss them in the next section.

2. KAC-MOODY GROUPS

Let \( A = (a_{ij}) \) be an \( n \times n \) generalized Cartan matrix. Let \( g \) be the Kac-Moody Lie algebra over a field \( C \) defined by \( A \) with the so-called Cartan subalgebra \( \mathfrak{h} \) (cf. [8], [9], [11], [16], [17]). Let \( \Delta \subset \mathfrak{h}^* \) be the root system of \( g \) with respect to \( \mathfrak{h} \) with the fundamental system \( \Pi = \{ \alpha_1, \cdots, \alpha_n \} \). Let \( \Delta^+ \) (resp. \( \Delta^- \)) be the set of positive (resp. negative) roots defined by \( \Pi \), and \( \Delta^r \) the set of real roots. Put \( \Delta^r_{\pm} = \Delta^\pm \cap \Delta^r \). Then we obtain

\[
g = \mathfrak{h} \oplus \bigcup_{\alpha \in \Delta^r} \mathfrak{g}_\alpha \quad \text{(root space decomposition)}
\]
and

\[
g = \mathfrak{g}^- \oplus \mathfrak{h} \oplus \mathfrak{g}^+ \quad \text{(triangular decomposition),}
\]
where \( \mathfrak{g}_\pm = \oplus_{\alpha \in \Delta^\pm} \mathfrak{g}_\alpha \). Let \( M \) be an integrable \( g \)-module, which means that

\[
M = \oplus_{\mu \in \Delta^r} M_\mu,
\]
where \( M_\mu = \{ v \in M \mid hv = \mu(h)v \ (\forall h \in \mathfrak{h}) \} \), and that \( x \) is locally nilpotent on \( M \) for all \( x \in \mathfrak{g}_\alpha \) with \( \alpha \in \Delta^r \). For the set of real roots, \( \Delta^r \), we can choose and fix a Chevalley basis

\[
\{ e_\alpha \mid \alpha \in \Delta^r \}
\]
(cf. [19]). We now suppose that \( M \) has a basis \( \{ v_\gamma \mid \gamma \in \Gamma \} \) whose \( \mathbb{Z} \)-span, \( M_\mathbb{Z} \), is invariant under the action of

\[
\frac{e_\alpha^m}{m!}
\]
for all \( m \geq 0 \). Such basis exists, for example, for the cases of adjoint representations, highest weight integrable representations, lowest weight integrable representations, and some others (see [9], [25] and references therein). Then, for any field \( K \), we put \( M(K) = K \otimes \mathbb{M}_2 \) and define \( x_\alpha(t) \in \text{GL}(M(K)) \) by

\[
x_\alpha(t)(s \otimes v) = \sum_{m=0}^{\infty} t^m s \otimes \frac{\alpha_m}{m!} v.
\]

Let \( G \) be the subgroup of \( \text{GL}(M(K)) \) generated by \( x_\alpha(t) \) for all \( \alpha \in \Delta^r \) and \( t \in K \). We call \( G \) a standard (or elementary) Kac-Moody group (cf. [7], [18], [21], [22], [24], [25], [26]). Let

\[
\begin{align*}
 w_\alpha(t) &= x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t), \\
 h_\alpha(t) &= w_\alpha(t)w_\alpha(-1)
\end{align*}
\]

for \( t \in K^\times \). Then, we put

\[
\begin{align*}
 U &= \langle x_\alpha(a) \mid \alpha \in \Delta^r, a \in K \rangle, \\
 T &= \langle h_\alpha(t) \mid \alpha \in \Delta^r, t \in K^\times \rangle, \\
 V &= \langle x_\alpha(a) \mid \alpha \in \Delta^r, a \in K \rangle,
\end{align*}
\]

Define the maps \( \phi_i \) by

\[
\begin{align*}
 \phi_1 &: \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mapsto x_\alpha(a), \\
 \phi_2 &: \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mapsto h_\alpha(t), \\
 \phi_3 &: \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \mapsto x_{-\alpha}(a),
\end{align*}
\]

These maps are not necessarily injective. The subgroups \( U'_i \) and \( V'_i \) are defined as follows

\[
\begin{align*}
 U'_i &= \langle x_{\alpha_i}(s)x_{\beta}(t)x_{\alpha_i}(-s) \mid s, t \in K, \beta \in \Delta^r \setminus \{\alpha_i\} \rangle, \\
 V'_i &= \langle x_{-\alpha_i}(s)x_{\beta}(t)x_{-\alpha_i}(-s) \mid s, t \in K, \beta \in \Delta^r \setminus \{-\alpha_i\} \rangle.
\end{align*}
\]

Then \( (G, U, T, V; \{\phi_1, \ldots, \phi_3\}) \) is a triangular system. Hence,

\[
G = UV'TU = \bigcup_{u \in U} u(V'TU)u^{-1}.
\]

**Theorem 2.** Every standard Kac-Moody group \( G \) over a field has a Gauss decomposition.

**Remark 1.** Gauss decomposition is tightly connected with the existence of Bruhat decomposition. Since for Kac-Moody groups the existence of Tits system is known (cf. [18], [12], [21]), it is quite natural to look for Gauss
decomposition for these groups, as well as for groups associated with them (cf. [1]).

**Remark 2.** If \( V \) is an integrable highest weight module generated by a maximal vector with a highest weight, then we can construct a bigger group \( \tilde{G} \), called a Marcuson-type Kac-Moody group, than \( G \) (cf. [12], [24]). Namely, \( \tilde{U} \), which is bigger than \( U \), is corresponding to all \( \alpha \in \Delta_+ \). On the other hand, \( \tilde{T}, \tilde{V}, \tilde{\phi}_i \) are the same as \( T, V, \phi_i \). Then we see that \( \{ \tilde{G}, \tilde{U}, \tilde{T}, \tilde{V}, \{ \tilde{\phi}_1, \cdots, \tilde{\phi}_n \} \) is a triangular system. Hence,

\[
\tilde{G} = \tilde{U} \tilde{T} \tilde{U} = \bigcup_{u \in \tilde{U}} u(\tilde{V} \tilde{T} \tilde{U}) u^{-1}.
\]

Therefore, every Marcuson-type Kac-Moody group has a Gauss decomposition.

**Remark 3.** Let \( \text{St}_2(K) \) be the Steinberg group of rank one over a field \( K \) (cf. [15], [22]). Then, it is also easily seen that there is a Gauss decomposition for \( \text{St}_2(K) \) naturally induced from \( \text{SL}_2(K) \). Therefore, we can replace \( \text{SL}_2(K) \) by \( \text{St}_2(K) \) in the condition (2) of a triangular system. Hence, there is a Gauss decomposition of the Steinberg group, \( \tilde{G} = \text{St}(A, K) \), associated with a Kac-Moody group \( G \) (cf. [26]). Namely,

\[
\tilde{G} = \tilde{U} \tilde{T} \tilde{U} = \bigcup_{u \in \tilde{U}} u(\tilde{V} \tilde{T} \tilde{U}) u^{-1}.
\]

where \( \tilde{U}, \tilde{T}, \tilde{V} \) are the corresponding subgroups of \( \tilde{G} \). Therefore, every Steinberg group associated with a standard Kac-Moody group over a field has a Gauss decomposition.

**Remark 4.** If we replace the condition (3) of a triangular system by

\[ (3') \quad G = \langle U, T, V, \cdots, V_n \rangle, \quad TU = UT, \quad VT = TV, \quad \text{and} \quad T U_i = U_i T \quad \text{for} \quad 1 \leq i \leq n. \]

then we can apply our method to a Tits-type Kac-Moody group (cf. [26]). \( \tilde{G} \), whose torus \( \tilde{T} \) is bigger than the group \( T \) above. Hence,

\[
\tilde{G} = \tilde{U} \tilde{T} \tilde{U} = \bigcup_{u \in \tilde{U}} u(\tilde{V} \tilde{T} \tilde{U}) u^{-1},
\]

where \( \tilde{U} \) and \( \tilde{V} \) are the same as \( U \) and \( V \) above. Therefore \( \tilde{G} \) has a Gauss decomposition.

**Remark 5.** The same method can be applied for other groups associated with Kac-Moody data. In particular, we think that it can be used in the framework of the scheme theoretical approach to Kac-Moody groups, introduced by Mathieu [13], [14].

**Remark 6.** Furthermore, axiomatically we can replace the group \( \text{SL}_2(K) \) in the condition (2) of a triangular system by a certain group \( G^* \) with subgroups \( U^*, T^*, V^* \) satisfying
In this case, we must set
\[ \phi_i(U^*) = U_i, \quad \phi_i(T^*) = T_i, \quad \phi_i(V^*) = V_i. \]
Then the same method works.

3. Prescribed version

Let \((G, U, T, V, \{\phi_1, \ldots, \phi_n\})\) be a triangular system. Then, as in Section 1, we obtain
\[ G = \bigcup_{u \in U} u(VTU)u^{-1}. \]
We now take an element \(h^* \in T\). Put
\[ G(h^*) = Z(G) \cup \bigcup_{g \in G} g(Vh^*U)g^{-1}, \]
where \(Z(G)\) is the center of \(G\). Then we want to consider whether \(G = G(h^*)\) or not. If \(G = G(h^*)\) for all \(h^* \in T\), then we say that \(G\) has a Gauss decomposition with prescribed elements in \(T\). This is equivalent to the fact, that for every non central element \(g \in G\) there exists an element \(g_1 \in G\) satisfying \(g_1gg_1^{-1} = vhu\), where \(v \in V, u \in U,\) and \(h\) is a prescribed element from \(T\).

In such a form this decomposition first appeared in the paper \([23]\) for the case of general linear group and then it was studied in detail by Ellers and Gordeev for all split semisimple algebraic groups (=Chevalley groups = finite dimensional Kac-Moody groups), see \([3]\), \([4]\), and for twisted Chevalley groups \([5]\). It turns out that the prescribed Gauss decomposition has various applications and is the main tool for solving remarkable Ore and Thompson conjectures (see \([20]\), \([2]\), \([6]\) and references therein). Here, we will check this in the case when \(G = SL_2(K)\), which is the easiest but important in our discussion later. We choose and fix
\[ h^* = \begin{pmatrix} t & 0 \\ 0 & \frac{1}{t} \end{pmatrix} \in T. \]

Let
\[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in VTU \subset SL_2(K). \]
Then \(a \neq 0\), and
\[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix}. \]
If \(b \neq 0\), then
\[ \begin{pmatrix} 1 & 0 \\ \frac{a-t}{b} & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{t-a}{b} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{c}{t} & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & \frac{1}{t} \end{pmatrix} \begin{pmatrix} 1 & \frac{b}{t} \\ 0 & 1 \end{pmatrix}, \]
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where

\[ c' = \frac{1}{b} \{ ta + td - t^2 - ad + bc \} \]

If \( c \neq 0 \), then

\[
\begin{pmatrix}
1 & \frac{t-a}{c} \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\begin{pmatrix}
1 & \frac{a-t}{c} \\
0 & 1
\end{pmatrix}
= 
\begin{pmatrix}
1 & 0 \\
\frac{c}{t} & 1
\end{pmatrix}
\begin{pmatrix}
t & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & \frac{c}{t} \\
0 & 1
\end{pmatrix},
\]

where

\[ b' = \frac{1}{c} \{ ta + td - t^2 - ad + bc \} \]

If \( b = c = 0, a \neq \pm 1 \), then

\[
\begin{pmatrix}
1 & -1 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
= 
\begin{pmatrix}
a & \frac{a - \frac{1}{a}}{c} \\
0 & \frac{1}{a}
\end{pmatrix},
\]

which arrives at the case of \( b \neq 0 \) above. If \( b = c = 0, a = \pm 1 \), then

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
= 
\begin{pmatrix}
\pm 1 & 0 \\
0 & \pm 1
\end{pmatrix} \in Z(G).
\]

Therefore, we obtain

\[ G = G(h^*) \]

for all \( h^* \in T \) when \( G = SL_2(K) \). Hence, \( SL_2(K) \) has a Gauss decomposition with prescribed elements in \( T \).

4. IN CASE OF RANK 2

Let

\[ A = \begin{pmatrix}
2 & -a \\
-b & 2
\end{pmatrix} \]

be a generalized Cartan matrix with \( ab \geq 4 \). Then, the corresponding Kac-Moody groups are called of type \( A \). Here, we define

\[ m = \max \{ a, b \}. \]

Let

\[ G = G(A, \mathcal{K}) \]

be the family of all standard Kac-Moody groups over a field \( K \) of type \( A \). Then, there is a unique, up to isomorphism, element of \( G \) which dominates all other elements. We fix it, and also we call it \( G \). Put

\[ N = \langle w_\alpha(t) \mid \alpha \in \Delta^e, t \in K^\times \rangle, \]

then \( T \triangleleft N \), and \( W = N/T \) is an infinite dihedral group. For each \( w \in W \), we
can write
\[ wUw^{-1} = \bar{w}U\bar{w}^{-1} \]
if \( \bar{w} \in N \) and \( w = \bar{w} \mod T \). Then
\[ \bigcap_{w \in W} wUw^{-1} = 1 \]
and
\[ \bigcap_{w \in W} wVw^{-1} = 1 \]
(cf. [10], [21]). Therefore, if \( 1 \neq u \in U \) (resp. \( 1 \neq v \in V \)), then there exists \( \bar{w} \in N \) such that
\[ \bar{w}uw^{-1} = u_1u'_1, \quad u_1 \neq 1 \]
(resp. \( \bar{w}uv^{-1} = v_1v'_1, \quad v_1 \neq 1 \))
or
\[ \bar{w}uw^{-1} = u_2u'_2, \quad u_2 \neq 1 \]
(resp. \( \bar{w}uv^{-1} = v_2v'_2, \quad v_2 \neq 1 \)).
where \( u_1 \in U_1, \ u'_1 \in U'_1 \) (resp. \( v_1 \in V_1, \ v'_1 \in V'_1 \)) for \( i = 1, 2 \).

To consider a prescribed version, we will assume, from here on, \( |K| > m+3 \). Let \( h^* = h_{\alpha_1}(t_1)h_{\alpha_2}(t_2^*) \in T \) and fix it. Let
\[ g = vhu \in VTU \]
with \( u \in U, \ h \in T, \ v \in V \). Then we fall into one of the following four cases.

**Case 1:** \( g \) is conjugate to
\[ g' = v'_1v_1h_1h_2u_1u'_1 \]
with \( v'_1 \in V'_1, \ v_1 \in V_1, \ h_1 \in T_1, \ h_2 \in T_2, \ u_1 \in U_1, \ u'_1 \in U'_1 \) and
\( v_1u_1 \neq 1 \).

**Case 2:** \( g \) is conjugate to
\[ g' = v'_2v_2h_1h_2u_2u'_2 \]
with \( v'_2 \in V'_2, \ v_2 \in V_2, \ h_1 \in T_1, \ h_2 \in T_2, \ u_2 \in U_2, \ u'_2 \in U'_2 \) and
\( v_2u_2 \neq 1 \).

**Case 3:** \( g \) is conjugate to
\[ g' = h_1h_2 \]
with \( h_1h_2 \notin Z(G) \).

**Case 4:** \( g \) is just an element of \( Z(G) \).

In Case 3, we obtain
\[ x_{\alpha_i}(1)g'x_{\alpha_i}(-1) = h_1h_2u_i \]
for some \( i = 1, 2 \) with \( 1 \neq u_i \in U_i \), which arrives at Case 1 or Case 2. In Case 2, we will change the numbering of 1 and 2 (and then \( a \) and \( b \) are exchanged), which arrives at Case 1. Therefore, for our purpose, we can assume that Case 1 holds. We choose and fix an element
\[
t' \in K^*
\]
such that
\[
t'^a \neq \bar{t}'^2, \quad t'^2 \neq \bar{t}'^b,
\]
where \( h_2 = h_{\alpha_2}(t_2) \). Then, as in Section 3, we obtain that \( g' \) is conjugate to
\[
g'' = v'h_{\alpha_1}(t')h_2u'
\]
with \( v' \in V \), \( h_2 \in T_2 \), \( u' \in U \). We can rewrite
\[
g'' = v'v_2h_{\alpha_1}(t')h_2u_2u_2'
\]
with \( v_2' \in V'_2 \), \( v_2 \in V_2 \), \( u_2 \in U_2 \), \( u_2' \in U'_2 \). If \( v_2u_2 = 1 \), then \( g'' \) is conjugate to
\[
x_{\alpha_2}(1)g''x_{\alpha_2}(-1) = v''h_{\alpha_1}(t')h_{\alpha_2}(t_2)u_2u_2'
\]
with \( v_2'' \in V'_2 \), \( 1 \neq u_2'' = x_{\alpha_2}(t'^*t_2^{-2} - 1) \in U'_2 \), \( u_2'' \in U'_2 \). Therefore, we can assume
\[
v_2u_2 \neq 1.
\]
Then, also as in Section 3, we see that \( g'' \) is conjugate to
\[
g''' = v''h_{\alpha_1}(t')h_{\alpha_2}(t''_2)u''
\]
with \( v'' \in V \), \( u'' \in U \). We can rewrite again
\[
g''' = v'_1v_1h_{\alpha_1}(t')h_{\alpha_2}(t''_2)u_1u_1'
\]
with \( v'_1 \in V'_1 \), \( v_1 \in V_1 \), \( u_1 \in U_1 \), \( u_1' \in U'_1 \). If \( v_1u_1 = 1 \), then \( g''' \) is conjugate to
\[
x_{\alpha_1}(1)g'''x_{\alpha_1}(-1) = v''h_{\alpha_1}(t')h_{\alpha_2}(t''_2)u''u''_1
\]
with \( v''_1 \in V'_1 \), \( 1 \neq u''_1 = x_{\alpha_1}(t'^{-2}t''_2 - 1) \). \( u''_1 \in U'_1 \). Hence, we can assume
\[
u_1v_1 \neq 1.
\]
Then, again as in Section 3, we obtain that \( g''' \) is conjugate to
\[
g^* = v'''h_{\alpha_1}(t'_1)h_{\alpha_2}(t''_2)u''' = v'''h*u'''
\]
with \( v''' \in V \), \( u''' \in U \). Hence, combining this and Case 4, we see that \( g \) is conjugate to some element \( g^* \in V^*U \) if \( g \notin Z(G) \). Thus,
\[
G(h^*) = G
\]
for all \( h^* \in T \). Therefore, we obtain the following result.

**Theorem 3.** Let \( A = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix} \) be a generalized Cartan matrix with \( ab \geq 4 \). Put \( m = \max \{ a, b \} \). Let \( K \) be a field with \( |K| > m + 3 \). Then every standard Kac-Moody group, \( G \in G(A, K) \), over \( K \) of type \( A \) has a Gauss decomposition with prescribed elements in \( T \).
It remains to consider the same problem for (infinite dimensional) standard Kac-Moody groups of rank $\geq 3$.

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