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EXCEPTIONAL FUNCTIONS WANDERING ON THE SPHERE AND NORMAL FAMILIES

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Abstract

We extend Carathéodory's generalization of Montel's fundamental normality test to "wandering" exceptional functions (i.e. depending on the respective function in the family under consideration), and we give a corresponding result on shared functions. Furthermore we prove that if we have a family of pairs (a, b) of functions meromorphic in a domain such that a and b uniformly "stay away from each other", then the families of the functions a resp. b are normal. The proofs are based on a "simultaneous rescaling" version of Zalcman's lemma.

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1. An extension of Carathéodory's normality result

The probably most essential and fundamental[1] result in the theory of normal families is Montel's Theorem which says that a family of functions meromorphic in a domain in the complex plane $\mathbb{C}$ that omit three distinct fixed values in $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is normal. There are two natural directions to generalize this result:

(1) Instead of fixed exceptional values, one may consider exceptional values depending on the respective function in the family under consideration. Of course, in this context one can hope for normality only under additional assumptions on these exceptional values: It does not suffice that the exceptional values are distinct; they have to be kind of "uniformly distinct". The respective version of the FNT is due to Carathéodory [2, p. 202].

Theorem A: *Let $\mathcal{F}$ be a family of meromorphic functions on a domain D. Suppose there exists an $\varepsilon > 0$ such that each $f \in \mathcal{F}$ omits three distinct values $a_f, b_f, c_f \in \overline{\mathbb{C}}$ satisfying*

$$\min \{\sigma(a_f, b_f), \sigma(a_f, c_f), \sigma(b_f, c_f)\} \geq \varepsilon,$$

where σ denotes the spherical metric on $\overline{\mathbb{C}}$. Then $\mathcal{F}$ is normal in D.

(2) Instead of exceptional values one may consider exceptional functions with disjoint graphs, i.e. omitting each other.

The case of meromorphic exceptional functions is almost trivial: If a, b, c are meromorphic functions on a domain D omitting each other and if each $f \in \mathcal{F}$ omits a, b and c, then we consider the family $\mathcal{G}$ of the cross ratio functions

$$z \mapsto \frac{f(z) - a(z)}{f(z) - b(z)} \cdot \frac{c(z) - b(z)}{c(z) - a(z)}$$

all of which omit the values 0, 1 and ∞ in D. By the FNT we obtain the normality of $\mathcal{G}$, hence of $\mathcal{F}$. J. Chang, M. Fang and L. Zalcman [3] have shown that the condition that a, b, c omit each other can be skipped: It suffices to assume that they are distinct meromorphic functions in D.

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1 Schiff [13] calls it the "Fundamental Normality Test" (FNT).

An analogous normality result even holds for exceptional functions $a, b, c : D \longrightarrow \overline{\mathbb{C}}$ which are merely continuous (with respect to the spherical metric on $\overline{\mathbb{C}}$) and which have disjoint graphs as Bargmann, Bonk, Hinkkanen and Martin [1] have shown.

In this paper we combine these two directions of generalization by considering meromorphic exceptional functions which depend on the respective function in the family under consideration, i.e. kind of "wandering" exceptional functions. We denote the class of meromorphic functions $f : D \longrightarrow \overline{\mathbb{C}}$ in an arbitrary domain D by $\mathcal{M}(D)$. However, without loss of generality, it suffices to deal with functions defined on the open unit disk $\mathbb{D} := \{z \in \mathbb{C} \, : \, |z| < 1\}$. Then we have the following result.

Theorem 1: *Let $\mathcal{F}$ be a family of functions meromorphic in $\mathbb{D}$ and $\varepsilon > 0$. Assume that for each $f \in \mathcal{F}$ there exist functions $a_f, b_f, c_f \in \mathcal{M}(\mathbb{D}) \cup \{\infty\}$ such that f omits the functions a_f, b_f, c_f in $\mathbb{D}$ and*

$$\min \{\sigma(a_f(z), b_f(z)), \sigma(a_f(z), c_f(z)), \sigma(b_f(z), c_f(z))\} \geq \varepsilon \tag{1.1}$$

for all $z \in \mathbb{D}$. Then $\mathcal{F}$ is a normal family.

Of course, here it is crucial that we use the spherical metric. Obviously, this result is wrong for the euclidean metric since one could simply choose $a_f := f+1$, $b_f := f+2$, $c_f := f+3$ for any f. A similar comment also applies to several of the results below.

Theorem 1 no longer holds for exceptional functions which are merely continuous on $\overline{\mathbb{C}}$ as the following counterexample shows.

Example. Let $f_n(z) := e^{nz}$, $a_n :\equiv 0$, $b_n :\equiv \infty$, $c_n(z) := -e^{in\,\mathrm{Im}(z)}$ for all n. Then each f_n omits the continuous functions a_n, b_n and c_n and

$$\sigma(a_n(z), b_n(z)) = \frac{\pi}{2}, \qquad \sigma(a_n(z), c_n(z)) = \sigma(b_n(z), c_n(z)) = \frac{\pi}{4}$$

for all $n \in \mathbb{N}$, $z \in \mathbb{D}$, but $(f_n)_n$ is not normal in $\mathbb{D}$.

So there seems to be no "natural" way to extend the result of Bargmann, Bonk, Hinkkanen and Martin to "wandering" exceptional functions. In [7], P. Lappan gave such an extension, however under the assumption that the exceptional functions a_f, b_f, c_f are already known to form convergent sequences whose limit functions have disjoint graphs (hence, by compactness, satisfy (1.1) for a suitable $\varepsilon > 0$). This a priori convergence assumption is not required in the situation of Theorem 1 – it follows from the other assumptions. This is the content of the following result, from which Theorem 1 is an easy consequence and which we hope to be of interest for its own.

THEOREM 2: *Let $\mathcal{G} \subseteq (\mathcal{M}(\mathbb{D}))^2$ be a family of pairs of meromorphic functions in $\mathbb{D}$ and $\varepsilon > 0$. Assume that*

$$\sigma(a(z), b(z)) \geq \varepsilon \qquad \text{for all } (a,b) \in \mathcal{G} \text{ and all } z \in \mathbb{D}. \tag{1.2}$$

Then the families $\{a \mid (a,b) \in \mathcal{G}\}$ and $\{b \mid (a,b) \in \mathcal{G}\}$ are normal in $\mathbb{D}$.

In other words, if we have a family of pairs (a, b) of functions meromorphic in $\mathbb{D}$ such that a and b uniformly "stay away from each other", then the families of the functions a resp. b are normal[2]. Obviously, this can also be reformulated in the following way.

COROLLARY 3: *Let $(f_n)_n$ be a sequence of meromorphic functions in $\mathbb{D}$ all of whose subsequences are* **not** *normal. Then for each $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that*

$$\inf_{z\in\mathbb{D}} \sigma(f_n(z), g(z)) < \varepsilon \qquad \text{for all } g \in \mathcal{M}(\mathbb{D}) \text{ and all } n \geq N.$$

In this context, we shall briefly mention a few results on so-called unavoidable functions due to W. Hayman / L. Rubel / C.-C. Yang [6, 12], P. Lappan [9] and Z. Slodkowski [15]. Though they go into a slightly different direction, it might be illustrative to compare them with the results above. Here we say that two continuous functions $a, b : D \longrightarrow \overline{\mathbb{C}}$ **avoid** each other if $a(z) \neq b(z)$ for all $z \in D$, and we call a family $\mathcal{G}$ of continuous functions $D \longrightarrow \overline{\mathbb{C}}$ an **unavoidable** family if there is no meromorphic function in D which avoids all $g \in \mathcal{G}$.

THEOREM B: (a) [12, 6] *Let $D \subseteq \mathbb{C}$ be a domain. Then for each pair of functions $g, h \in \mathcal{M}(D)$ there exists a function $f \in \mathcal{M}(D)$ that avoids both g and h. On the other hand, there exists a function φ analytic in D such that $\{\varphi, -\varphi, \infty\}$ is an unavoidable family.*

[2] To avoid misconceptions, we'd like to point out that it is not assumed that one has a family $\mathcal{F} \subseteq \mathcal{M}(\mathbb{D})$ such that a and b satisfy (1.2) for all $a, b \in \mathcal{F}$. This, of course, would be an almost trivial problem. We just assume that to each function a there belongs some function b such that (1.2) holds.

So the minimal cardinality of an unavoidable family of **meromorphic** *functions is three.*

(b) [9] *There exists a continuous function* $\varphi : \mathbb{D} \longrightarrow \overline{\mathbb{C}}$ *such that no meromorphic function in* $\mathbb{D}$ *avoids* φ. *There exists a continuous function* $\Psi : \mathbb{C} \longrightarrow \overline{\mathbb{C}}$ *such that no meromorphic function in* $\mathbb{C}$ *avoids* Ψ.

So the minimal cardinality of an unavoidable family of **continuous** *functions is one, at least with respect to* $\mathbb{D}$ *and* $\mathbb{C}$.

(c) [8, Theorem 2] *Let* $g_1, g_2, g_3 \in \mathcal{M}(\mathbb{C}) \cup \{\infty\}$ *be given. If* g_1, g_2, g_3 *avoid each other, then there are uncountably many meromorphic functions in* $\mathbb{C}$ *which avoid* g_1, g_2, g_3.

(d) [15] *(cf.* [6, Theorem 3]*) If* $g_1, \dots, g_n$ *are meromorphic functions in* $\mathbb{D}$ *that avoid each other, then for any* $\lambda \notin \{g_1(0), \dots, g_n(0)\}$ *there exists a function* $g_{n+1} \in \mathcal{M}(\mathbb{D})$ *which avoids* $g_1, \dots, g_n$ *and satisfies* $g_{n+1}(0) = \lambda$.

One might wonder why the second result in (a) does not contradict (c). The reason is that the function φ in (a) has zeros, hence φ and $-\varphi$ do not avoid each other.

2. A corresponding result for shared functions

One can ask for a value sharing result corresponding to Theorem 1 which involves "wandering" shared functions, i.e. shared functions that depend on the function $f \in \mathcal{F}$. A result in this direction was stated by A.P. Singh and A. Singh in [14] (see also [5] for a corrected proof). Here, we use the notation

$$\overline{E}_f(a) := \{z \in \mathbb{D} \ : \ f(z) = a\}\,.$$

Theorem C: *Let* $\mathcal{F}$ *be a family of functions meromorphic in* $\mathbb{D}$. *Assume that for each* $f \in \mathcal{F}$ *there exist* $a_f, b_f, c_f \in \mathbb{C} \setminus \{0\}$ *such that* $a_f b_f / c_f^2 = M$ *for some constant* M,

$$\min\{\sigma(a_f, b_f), \sigma(b_f, c_f), \sigma(a_f, c_f)\} \geq \varepsilon \tag{2.1}$$

for some $\varepsilon > 0$ *and* $\overline{E}_f(0) = \overline{E}_{f'}(a_f)$ *and* $\overline{E}_f(c_f) = \overline{E}_{f'}(b_f)$. *Furthermore, assume that* $M = ab/c^2$ *where* a, b, c *are distinct and that the elements of*

$\overline{E}_f(c_f)$ and $\overline{E}_f(0)$ are the only solutions of the equations

$$f'(z) = \frac{a_f b}{a} \cdot \left(1 - \left(\frac{1}{c_f} - \frac{a}{ca_f}\right) \cdot f(z)\right)^2$$

and

$$f'(z) = a_f \cdot \left(1 - \left(\frac{1}{c_f} - \frac{a}{ca_f}\right) \cdot f(z)\right)^2$$

respectively. Then $\mathcal{F}$ is normal in $\mathbb{D}$.

However, this result is restricted to shared *values* (i.e. constant functions), and it requires making several quite technical assumptions. On the other hand, there are counterexamples restricting the possibilities to weaken the assumptions in Theorem C, even if a_f, b_f, c_f are constant. In particular, the condition that the elements of $\overline{E}_f(c_f)$ and $\overline{E}_f(0)$ are the only solutions of certain equations (which looks rather arbitrary, not to say artificial at first sight) cannot be skipped as we have seen in [5, example 1.3].

In the following we prove a more natural result which is to a certain extent a combination of Theorem 1 and Theorem C. While the assumption of Theorem C essentially contains four value sharing conditions, our result gets by with "only" three (and more symmetric) conditions of that kind.

But first, we define what exactly we mean by "sharing a function". The most common definition is the following one: Let f, g be two non-constant meromorphic functions in a domain $D \subseteq \mathbb{C}$, and let A be a function meromorphic in D. We say that f and g *share A IM* (ignoring multiplicities) provided that $f - A$ and $g - A$ have the same zeros in D. Furthermore, we say that f and g share the value ∞ IM if f and g have the same poles ignoring multiplicities.

THEOREM 4: *Let $\mathcal{F}$ be a family of functions meromorphic in $\mathbb{D}$ and $\varepsilon > 0$. Assume that for each $f \in \mathcal{F}$ there exist functions $a_1^{(f)}, a_2^{(f)}, a_3^{(f)} \in \mathcal{M}(\mathbb{D}) \cup \{\infty\}$ which do not have common poles with f such that f and f' share the functions $a_1^{(f)}, a_2^{(f)}, a_3^{(f)}$ and such that*

$$(2.2) \quad \sigma\left(a_j^{(f)}(z), a_k^{(f)}(z)\right) \geq \varepsilon \quad \text{for all } z \in \mathbb{D}, \text{ all } j,k \in \{1,2,3\} \text{ with } j \neq k.$$

Then $\mathcal{F}$ is normal in $\mathbb{D}$.

Here, if one of the functions $a_j^{(f)}$ is the constant ∞, the assumption on the poles implies that f is analytic. This assumption ensures that none of the value sharing conditions reduces to the statement that f and f' share ∞ (which doesn't contain any information since f and f' have the same poles anyway).

An easy example shows that the number of three shared functions in Theorem 4 is best possible.

Example. We set

$$f_n(z) := e^{nz}, \quad a_{1,n}(z) := 0, \qquad a_{2,n}(z) := ne^n$$

for all n. Then f_n shares the (constant) functions $a_{1,n}$ and $a_{2,n}$ with f'_n (in fact, f_n and f'_n omit these functions), $\sigma(a_{1,n}(z), a_{2,n}(z)) \geq \frac{\pi}{4}$ for all n and all $z \in \mathbb{D}$, but $(f_n)_n$ is not normal in the unit disk.

3. Simultaneous rescaling in Zalcman's lemma

The most important tool in our proofs is the following well-known rescaling lemma due to L. Zalcman [16] (see also [10, Lemma 4.1]). Here, $f^\# := \frac{|f'|}{1+|f|^2}$ denotes the spherical derivative of a meromorphic function f.

Lemma 5: **(Zalcman's Lemma)** *Let $\mathcal{F}$ be a family of functions meromorphic in $\mathbb{D}$. Then $\mathcal{F}$ is not normal at $z_0 \in \mathbb{D}$ if and only if there exist sequences $(f_n)_n \subseteq \mathcal{F}$, $(z_n)_n \subseteq \mathbb{D}$ and $(\varrho_n)_n \subseteq]0,1[$ such that* $\lim_{n\to\infty} \varrho_n = 0$, $\lim_{n\to\infty} z_n = z_0$ *and such that the sequence $(g_n)_n$ defined by*

$$g_n(\zeta) := f_n(z_n + \varrho_n \zeta)$$

converges locally uniformly in $\mathbb{C}$ (with respect to the spherical metric) to a non-constant function g meromorphic in $\mathbb{C}$ which satisfies $g^\#(\zeta) \leq g^\#(0) = 1$ for all $\zeta \in \mathbb{C}$.

In the proof of Theorem 2 it is convenient to use a corollary to Zalcman's lemma which allows some kind of simultaneous rescaling. To simplify notations, for $p \in \mathbb{N}$ and $j = 1, \ldots, p$ we introduce the projections

$$\pi_j : (\mathcal{M}(\mathbb{D}))^p \longrightarrow \mathcal{M}(\mathbb{D}), \quad \pi_j(f_1, \ldots, f_p) := f_j \quad \text{for } (f_1, \ldots, f_p) \in (\mathcal{M}(\mathbb{D}))^p.$$

Lemma 6: **(Simultaneous rescaling version of Zalcman's Lemma)** *Let p be a natural number and $\mathcal{F} \subseteq (\mathcal{M}(\mathbb{D}))^p$. Assume that there exists $j_0 \in \{1, \ldots, p\}$ such that the family $\pi_{j_0}(\mathcal{F})$ is not normal at $z_0 \in \mathbb{D}$. Then there* *exist sequences* $(f_n)_n = ((f_{1,n},\dots,f_{p,n}))_n \subseteq \mathcal{F}$, $(z_n)_n \subseteq \mathbb{D}$ *and* $(\varrho_n)_n \subseteq]0;1[$ *such that* $\lim_{n\to\infty} z_n = z_0$, $\lim_{n\to\infty}\varrho_n = 0$ *and such that for all* $j = 1,\dots,p$ *the sequences* $(g_{j,n})_n$ *defined by*

$$g_{j,n}(\zeta) := f_{j,n}(z_n + \varrho_n\zeta)$$

converge to functions $g_j \in \mathcal{M}(\mathbb{C}) \cup \{\infty\}$ *locally uniformly in* $\mathbb{C}$ *(with respect to the spherical metric) where at least one of the functions* $g_1,\dots,g_p$ *is not constant.*

We give two proofs of this result, one by iterated application of Zalcman's lemma and one by slightly modifying its original proof.

Proof 1. We prove this by induction. The case $p = 1$ is just Zalcman's Lemma.

Now we fix some $p \ge 2$ and assume that the assertion is true for $p-1$ instead of p. We take a family $\mathcal{F}$ as in the statement of the theorem such that $\pi_{j_0}(\mathcal{F})$ is not normal at $z_0 \in \mathbb{D}$ for some $j_0 \in \{1,\dots,p\}$. Without loss of generality we may assume $1 \le j_0 \le p-1$. Then by the induction hypothesis there exist sequences $(f_n)_n = ((f_{1,n},\dots,f_{p,n}))_n \subseteq \mathcal{F}$, $(z_n^*)_n \subseteq \mathbb{D}$ and $(\varrho_n^*)_n \subseteq]0;1[$ such that $\lim_{n\to\infty} z_n^* = z_0$, $\lim_{n\to\infty}\varrho_n^* = 0$ and such that for $j = 1,\dots,p-1$ the sequences $(G_{j,n})_n$, where

$$G_{j,n}(w) := f_{j,n}(z_n^* + \varrho_n^* w),$$

converge locally uniformly in $\mathbb{C}$ to functions $G_1,\dots,G_{p-1} \in \mathcal{M}(\mathbb{C})\cup\{\infty\}$ where at least one of the functions $G_1,\dots,G_{p-1}$ is not constant.

Now we consider the sequence of the functions $G_{p,n}(w) := f_{p,n}(z_n^* + \varrho_n^* w)$. If this sequence is normal in $\mathbb{C}$, we are done since turning to an appropriate subsequence gives the assertion.

If $(G_{p,n})_n$ is not normal in $\mathbb{C}$, then by Zalcman's Lemma we can find a subsequence (which without loss of generality we also denote by $(G_{p,n})_n$) and sequences $(w_n)_n \subseteq \mathbb{C}$ and $(\tau_n)_n \subseteq]0;1[$ such that $(w_n)_n$ converges to some point $w_0 \in \mathbb{C}$ (at which $(G_{p,n})_n$ is not normal), $\lim_{n\to\infty}\tau_n = 0$ and such that the sequence of the functions $g_{p,n}(\zeta) := G_{p,n}(w_n + \tau_n\zeta)$ converges to some non-constant function $g_p \in \mathcal{M}(\mathbb{C})$ locally uniformly in $\mathbb{C}$. Now we set

$$z_n := z_n^* + \varrho_n^* w_n, \quad \varrho_n := \varrho_n^*\tau_n \in]0,1[, \quad g_{j,n}(\zeta) := G_{j,n}(w_n + \tau_n\zeta) = f_{j,n}(z_n + \varrho_n\zeta).$$

Then we obtain $\lim_{n\to\infty} z_n = z_0$, $\lim_{n\to\infty} \varrho_n = 0$, and for $j = 1, \ldots, p-1$ the sequence $(g_{j,n})_n$ converges to the constant function $g_j(\zeta) :\equiv G_j(w_0)$ locally uniformly in $\mathbb{C}$ while $(g_{p,n})_n$ converges to g_p locally uniformly in $\mathbb{C}$. This shows the assertion. ■

Proof 2. Without loss of generality we may assume that $z_0 = 0$. By Marty's theorem there exist sequences $(f_n)_n \subseteq \mathcal{F}$ and $(z_n^*)_n \subseteq \mathbb{D}$ such that $\lim_{n\to\infty} z_n^* = 0$ and $\lim_{n\to\infty} f_{j_0,n}^{\#}(z_n^*) = \infty$. We define

$$r_n := \left(f_{j_0,n}^{\#}(z_n^*)\right)^{-\frac{1}{2}} + 2 \cdot |z_n^*|.$$

Then we have $\lim_{n\to\infty} r_n = 0$ and $\frac{|z_n^*|}{r_n} \le \frac{1}{2}$ for all n. Furthermore, we define

$$M_n := \max_{|z|\le r_n} \left(1 - \frac{|z|^2}{r_n^2}\right) \cdot \left(f_{1,n}^{\#}(z) + \cdots + f_{p,n}^{\#}(z)\right).$$

Then we have

$$M_n = \left(1 - \frac{|z_n|^2}{r_n^2}\right) \cdot \left(f_{1,n}^{\#}(z_n) + \cdots + f_{p,n}^{\#}(z_n)\right)$$

for certain z_n with $|z_n| < r_n$. From

$$r_n \cdot M_n \ge \left(1 - \frac{|z_n^*|^2}{r_n^2}\right) \cdot \left(f_{j_0,n}^{\#}(z_n^*)\right)^{\frac{1}{2}} \ge \frac{3}{4} \cdot \left(f_{j_0,n}^{\#}(z_n^*)\right)^{\frac{1}{2}}$$

we see $\lim_{n\to\infty} r_n M_n = \infty$. We define

$$\varrho_n := \frac{r_n^2 - |z_n|^2}{r_n^2 M_n} \qquad \text{and} \qquad R_n := \frac{r_n - |z_n|}{\varrho_n}.$$

Then

$$\frac{\varrho_n}{r_n - |z_n|} = \frac{r_n + |z_n|}{r_n^2 M_n} \le \frac{2}{r_n M_n} \longrightarrow 0 \quad (n \to \infty).$$

So we have $\lim_{n\to\infty} \varrho_n = 0$ and $\lim_{n\to\infty} R_n = \infty$. The functions

$$g_{j,n}(\zeta) := f_{j,n}(z_n + \varrho_n \zeta)$$

are meromorphic in $|\zeta| < R_n$ and satisfy

$$\begin{aligned} g_{j,n}^{\#}(\zeta) &= \varrho_n \cdot f_{j,n}^{\#}(z_n + \varrho_n \zeta) \le \frac{\varrho_n M_n}{1 - \frac{1}{r_n^2} \cdot |z_n + \varrho_n \zeta|^2} \\ &= \frac{r_n^2 - |z_n|^2}{r_n^2 - |z_n + \varrho_n \zeta|^2} \le \frac{r_n - |z_n|}{r_n - |z_n| - \varrho_n |\zeta|} \le \frac{1}{1 - \frac{R}{R_n}} < 2 \end{aligned}$$

for $|\zeta| \le R < \frac{1}{2} \cdot R_n$. So by Marty's theorem each sequence $(g_{j,n})_n$ is normal in $|\zeta| < R$ for every $R > 0$. Therefore, we may assume that $(g_{j,n})_n$ converges locally uniformly in $\mathbb{C}$ to some $g_j \in \mathcal{M}(\mathbb{C}) \cup \{\infty\}$ for every $j = 1, \ldots, p$. From

$$g_{1,n}^{\#}(0)+\cdots+g_{p,n}^{\#}(0) = \varrho_n \cdot \left(f_{1,n}^{\#}(z_n) + \cdots + f_{p,n}^{\#}(z_n) \right) = \varrho_n \cdot M_n \cdot \frac{r_n^2}{r_n^2 - |z_n|^2} = 1$$

we finally obtain $g_1^{\#}(0)+\cdots+g_p^{\#}(0) = 1$, i.e. $g_j^{\#}(0) > 0$ for some $j \in \{1, \ldots, p\}$ which means that not all of the g_j are constant. ■

One major disadvantage of this Lemma is the fact that it does not give any control which of the limit functions g_j are non-constant. (Of course, only those g_j can be non-constant for which $\pi_j(\mathcal{F})$ is not normal.) Proving a stronger version of this Lemma where one can prescribe which g_j is non-constant would be a great leap for the theory of normal families. In many possible applications it would even suffice if one could exclude the case $g_j \equiv \infty$. On the other hand, in general one cannot expect that one can construct several non-constant limit functions by simultaneous rescaling. The deeper reason for this is the fact that if $(f_n)_n$ and $(g_n)_n$ are sequences in $\mathcal{M}(\mathbb{D})$ which are not normal at $z_0 \in \mathbb{D}$, then one cannot conclude that there exists a sequence $(z_n)_n$ in $\mathbb{D}$ with $\lim_{n\to\infty} z_n = z_0$ such that both $(f_n^{\#}(z_n))_n$ and $(g_n^{\#}(z_n))_n$ are unbounded as the following counterexample illustrates.

Counterexample. Consider the functions $f_n(z) := nz + \sqrt{n}$ and $g_n(z) := -nz + \sqrt{n}$. Then

$$f_n^{\#}(z) = \frac{n}{1 + n|1 + \sqrt{n}z|^2}, \qquad g_n^{\#}(z) = \frac{n}{1 + n|1 - \sqrt{n}z|^2},$$

so

$$f_n^{\#}\left(-\frac{1}{\sqrt{n}}\right) = g_n^{\#}\left(\frac{1}{\sqrt{n}}\right) = n$$

for all n. Hence by Marty's theorem both sequences $(f_n)_n$ and $(g_n)_n$ are not normal at the origin.

On the other hand, $f_n^{\#}(z_n) \to \infty$ implies $1 + \sqrt{n}z_n \to 0$ while $g_n^{\#}(z_n) \to \infty$ implies $1 - \sqrt{n}z_n \to 0$ for $n \to \infty$. Obviously, it is impossible to satisfy both conditions simultaneously. If $f_n(z_n + \varrho_n\zeta) = nz_n + \varrho_n n\zeta + \sqrt{n}$ (with z_n, ϱ_n as in Zalcman's Lemma) tends to some non-constant limit function, then $g_n(z_n + \varrho_n\zeta) = -f_n(z_n + \varrho_n\zeta) + 2\sqrt{n}$ tends to ∞ (and vice versa).

4. Proofs

Proof of Theorem 2. If $\pi_1(\mathcal{G}) = \{a \mid (a,b) \in \mathcal{G}\}$ or $\pi_2(\mathcal{G}) = \{b \mid (a,b) \in \mathcal{G}\}$ was not normal, then by the simultaneous rescaling version of Zalcman's Lemma (Lemma 6) one could construct functions $a, b \in \mathcal{M}(\mathbb{C}) \cup \{\infty\}$ which satisfy

$$\sigma(a(z), b(z)) \geq \varepsilon$$

for all $z \in \mathbb{C}$ such that at least one of the functions a and b is not constant. Without loss of generality we may assume that a is not constant.

If $b \equiv \infty$ then we deduce that a is bounded, hence constant by Liouville's theorem, a contradiction. If $b \not\equiv \infty$, then we can conclude that

$$|a(z) - b(z)| \geq \sigma(a(z), b(z)) \geq \varepsilon$$

for all $z \in \mathbb{C}$ which implies that $a - b$ is constant. Denoting the chordal metric on $\overline{\mathbb{C}}$ by χ, from

$$\begin{aligned} \left(1 + |a(z)|^2\right) \cdot \left(1 + |b(z)|^2\right) &= \frac{|a(z) - b(z)|^2}{(\chi(a(z), b(z)))^2} \\ &\leq \frac{|a(z) - b(z)|^2}{\left(\frac{2}{\pi}\sigma(a(z), b(z))\right)^2} \leq \frac{\pi^2}{4\varepsilon^2} \cdot |a(z) - b(z)|^2 \end{aligned}$$

for all $z \in \mathbb{C}$ we see that a and b themselves are bounded, hence constant. This is again a contradiction. ■

Proof of Theorem 1. This follows easily from Theorem 2 and Lappan's result on continuous "wandering" exceptional functions mentioned on p. 4, but to point out the elementary nature of our reasoning we give a direct proof.

We assume that $\mathcal{F}$ is not normal in $\mathbb{D}$, say at $z_0 \in \mathbb{D}$. Then by Lemma 6 there exist sequences $(f_n)_n \in \mathcal{F}$, $(a_n)_n, (b_n)_n, (c_n)_n \subseteq \mathcal{M}(\mathbb{D}) \cup \{\infty\}$, $(z_n)_n \in \mathbb{D}$ and $(\varrho_n)_n \in]0;1[$ such that $\lim_{n\to\infty} z_n = z_0$, $\lim_{n\to\infty} \varrho_n = 0$, f_n omits the functions a_n, b_n, c_n,

$$\min\{\sigma(a_n(z), b_n(z)), \sigma(a_n(z), c_n(z)), \sigma(b_n(z), c_n(z))\} \geq \varepsilon$$

for all $z \in \mathbb{D}$ and all n and such that the sequences $(g_n)_n$, $(A_n)_n$, $(B_n)_n$ and $(C_n)_n$ defined by

$$g_n(\zeta) := f_n(z_n + \varrho_n\zeta),$$

$$A_n(\zeta) := a_n(z_n + \varrho_n\zeta), \qquad B_n(\zeta) := b_n(z_n + \varrho_n\zeta), \qquad C_n(\zeta) := c_n(z_n + \varrho_n\zeta)$$

converge locally uniformly in $\mathbb{C}$ to functions $g, A, B, C \in \mathcal{M}(\mathbb{C}) \cup \{\infty\}$, resp., not all of which are constant. Now Theorem 2 ensures that $(a_n)_n$, $(b_n)_n$ and $(c_n)_n$ are normal. This forces A, B and C to be constant. Therefore, g is not constant. By Hurwitz's theorem, g omits the three distinct constants A, B and C. This contradicts Picard's theorem. ∎

In the proof of Theorem 4 we require the following well-known result due to J. Clunie and W. K. Hayman [4].

LEMMA 7: *Let f be an entire function with bounded spherical derivative. Then the order of f is at most one.*

In the following considerations it is important to distinguish between convergence with respect to the spherical and with respect to the euclidean metric. We write "locally σ-uniformly" when the convergence is locally uniform with respect to the spherical metric and just "locally uniformly" if it is locally uniform with respect to the euclidean metric.

Proof of Theorem 4. By (2.2) and Theorem 2, the family $\left\{a_j^{(f)} \mid f \in \mathcal{F}\right\}$ is normal for $j = 1, 2, 3$.

We fix some $z_0 \in \mathbb{D}$ and show the normality of $\mathcal{F}$ at z_0.

Let a sequence $\mathcal{L} \subseteq \mathcal{F}$ be given. Then there exists a subsequence $(f_n)_n \subseteq \mathcal{L}$ such that for each $j = 1, 2, 3$ the sequence $\left(a_j^{(f_n)}\right)_n$ converges to some $A_j \in \mathcal{M}(\mathbb{D}) \cup \{\infty\}$ locally σ-uniformly in $\mathbb{D}$. From (2.2) we obtain $A_j(z) \neq A_k(z)$ for all $z \in \mathbb{D}$ and all $j \neq k$. In particular, $A_j(z_0) = \infty$ holds for at most one $j \in \{1, 2, 3\}$. Without loss of generality we may assume $A_1(z_0), A_2(z_0) \in \mathbb{C}$.

For the sake of abbreviation, we set $a_{j,n} := a_j^{(f_n)}$.

If $(f_n)_n$ is not normal at z_0, by Zalcman's Lemma we may assume (after turning to an appropriate subsequence) that there exist sequences $(z_n)_n \subseteq \mathbb{D}$ and $(\varrho_n)_n \subseteq]0, 1[$ such that $\lim_{n\to\infty} \varrho_n = 0$, $\lim_{n\to\infty} z_n = z_0$ and such that the sequence $(g_n)_n$ where

$$g_n(\zeta) := f_n(z_n + \varrho_n \zeta)$$

converges locally σ-uniformly in $\mathbb{C}$ to some non-constant meromorphic function g of bounded spherical derivative.

For $j = 1, 2$ and $n \to \infty$ we have $a_{j,n}(z_n + \varrho_n \zeta) \longrightarrow A_j(z_0) \in \mathbb{C}$ locally uniformly on $\mathbb{C}$,

(4.1)

$$f_n(z_n + \varrho_n\zeta) - a_{j,n}(z_n + \varrho_n\zeta) \longrightarrow g(\zeta) - A_j(z_0) \quad \text{locally } \sigma\text{-uniformly on } \mathbb{C}$$

and, by Weierstraß' theorem,

(4.2)

$$\varrho_n \left(f_n'(z_n + \varrho_n\zeta) - a_{j,n}(z_n + \varrho_n\zeta)\right) \longrightarrow g'(\zeta) \quad \text{locally uniformly on } \mathbb{C} \setminus P_g,$$

where P_g is the set of poles of g in $\mathbb{C}$. From (4.1), (4.2) and the value sharing assumption of the theorem we conclude by Hurwitz's theorem that $g(\zeta) = A_j(z_0)$ implies $g'(\zeta) = 0$ for $j = 1, 2$. In particular, this excludes the possibility that g is a polynomial of degree one. Hence g' is non-constant, and so again by (4.1) and (4.2) and the value sharing assumption we obtain that also the opposite implication holds, i.e. that $g'(\zeta) = 0$ implies $g(\zeta) = A_j(z_0)$. So we have

$$g(\zeta) = A_j(z_0) \quad \Longleftrightarrow \quad g'(\zeta) = 0 \qquad \text{for } j = 1, 2.$$

But this means that g omits the values $A_1(z_0)$ and $A_2(z_0)$ (and that g' omits the value 0). We conclude that $1/(g - A_1(z_0))$ is an entire function of bounded spherical derivative, hence it has order at most 1 by Lemma 7, and so has g itself. From this we deduce that g has the form

$$g(z) = A_1(z_0) + \frac{(A_2 - A_1)(z_0)}{e^{cz+d} + 1} \tag{4.3}$$

with certain $c, d \in \mathbb{C}$, $c \neq 0$.

Case 1: $A_3(z_0) \in \mathbb{C}$

Then with the same reasoning as above (interchanging the roles of $A_2(z_0)$ and $A_3(z_0)$) we obtain that g also omits the value $A_3(z_0)$. Since $A_1(z_0), A_2(z_0), A_3(z_0)$ are distinct, Picard's theorem yields that g is constant, a contradiction.

Case 2: $A_3(z_0) = \infty$

Assume that ζ_0 is a pole of g. Since $\left(\frac{1}{g_n(\zeta)} - \frac{1}{a_{3,n}(z_n+\varrho_n\zeta)}\right)_n$ tends to $\frac{1}{g}$ uniformly (with respect to the euclidean metric) in a certain neighborhood U of ζ_0 (chosen in such a way that g has no zeros in $\overline{U}$), by Hurwitz's theorem there exists a sequence $(\zeta_n)_{n\geq N}$ tending to ζ_0 such that

$$0 = \frac{1}{g_n(\zeta_n)} - \frac{1}{a_{3,n}(z_n + \varrho_n\zeta_n)} = \frac{1}{f_n(z_n + \varrho_n\zeta_n)} - \frac{1}{a_{3,n}(z_n + \varrho_n\zeta_n)}$$

for all $n \geq N$. Now by our assumption f and $a_{3,n}$ do not have common poles. Therefore we can conclude that $f_n(z_n + \varrho_n\zeta_n)$ and $a_{3,n}(z_n + \varrho_n\zeta_n)$ are in $\mathbb{C}$ and that $z_n + \varrho_n\zeta_n$ is a zero of $f_n - a_{3,n}$. This yields $f_n'(z_n + \varrho_n\zeta_n) = a_{3,n}(z_n + \varrho_n\zeta_n)$ for all $n \geq N$ by our value sharing assumption, hence

$$g_n^{\#}(\zeta_n) = \varrho_n f^{\#}(z_n + \varrho_n\zeta_n) = \varrho_n \cdot \frac{|a_{3,n}(z_n + \varrho_n\zeta_n)|}{1 + |a_{3,n}(z_n + \varrho_n\zeta_n)|^2} \leq \varrho_n \quad \overset{n\to\infty}{\longrightarrow} \quad 0.$$

From this we conclude that $g^{\#}(\zeta_0) = 0$ which means that g has a multiple pole at ζ_0.

So we have shown that g has only multiple poles. However, this contradicts (4.3). This shows that $\mathcal{F}$ is normal at z_0. ■

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